Encoding graphs into quantum states: an axiomatic approach

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Outline

Setting the scene

Motivation, notations, approaches

From graphs to quantum states, the axiomatic way

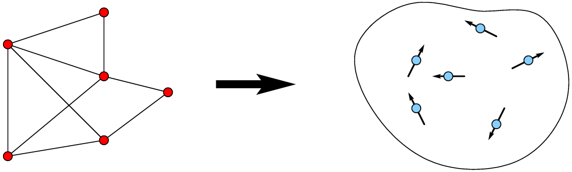
A fresh look

Three axioms

Examples

Conclusions

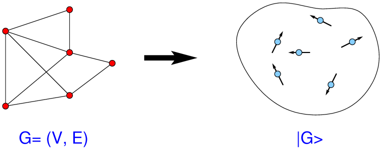
The main question



G= (V, E) |G>

Given a graph $G = (V, E)$, how do we map it to a quantum state $|G\rangle$?

Motivation



- going beyond graph states
  many interesting states (W_n, $D_{n,k}$) are not graph states
  Can we define new families of entangled states based on graphs?

- solving graph problems using the “quantum power”
  hard problems: (sub)graph isomorphism, 3COL etc
  Can we derive properties of G from the associated $|G\rangle$? graph invariants?

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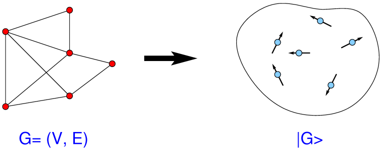

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Notations and background

a quick overview

graph: $G = (V, E)$

- **order** of G: $n = |V|$

- **adjacency** matrix: $A(G): \quad A_{ij} = 1 \quad \text{iff} \quad (i,j) \in E$, 0 otherwise

- **simple graph**: undirected, no loops, no multiedges: $A_{ij} = A_{ji}$

- graph **isomorphism**:

$$G \simeq G' \Leftrightarrow A(G') = P\, A(G)\, P^{-1}, \quad P \in \mathcal{S}_n$$

Several approaches

given $G = (V, E)$, how to: (i) find $\mathcal{H}$
(ii) find $|G\rangle \in \mathcal{H}$

- map V to qubits: graph states
- map E to qubits: Kitaev-like
- other: $\frac{1}{d_G} L(G) = \frac{\Delta(G) - A(G)}{2|E|} =: \rho(G)$

Braunstein *et al.* quant-ph/0406165

random walks ...

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How big $\mathcal{H}$ needs to be?

a simple counting

$G = (V, E)$, $n = |V|$, $L = |E|$
we have

$$0 \leq L \leq n(n-1)/2$$

$\#$ graphs with n vertices: $2^{n(n-1)/2}$ not all distinct!

$\#$ graphs with n vertices, L edges: $\binom{n(n-1)/2}{L}$

need a bigger $\mathcal{H}$ than n-dim

Reframing the question

What properties we want for the map $G \to |G\rangle$?

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What properties we want for the map $G \rightarrow |G\rangle$?

New approach: start with Axioms $\rightarrow$ derive properties

How to choose the Axioms?

Three axioms: A1

tensor product

- A1. Tensor product:

$$|G_1 \uplus G_2\rangle = |G_1\rangle \otimes |G_2\rangle$$

Corrolary 1 (empty graph): $|E_n\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$

Corrolary 2: Given $G = (V, E)$, to each $i \in V$ we associate $\mathcal{H}_i$, $\mathcal{H} = \bigotimes_i \mathcal{H}_i$

vertices $\mapsto$ *qudits*

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Three axioms: A2

isomorphism

- A2. Graph isomorphism:

$$G_1 \simeq G_2 \quad \Rightarrow \quad \rho_2 = P \rho_1 P^{-1}$$

$\rho_{1,2} = |G_{1,2}\rangle\langle G_{1,2}|$, $P \in \mathcal{S}_n$, well-defined

Corrollary 3: If $P \in \mathcal{S}_n$ automorphism of G, then $[P, \rho] = 0$

Proposition: Given $G = (V, E)$, $|G\rangle \in \mathcal{H}$, with $\mathcal{H} = \mathcal{H}_1^{\otimes n}$

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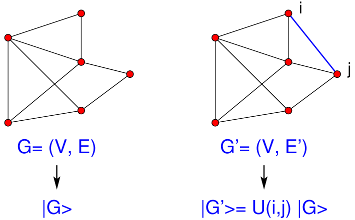
Three axioms: A3

edge operator

- A3. Universal edge operator:

$$|G'\rangle = U(i,j)|G\rangle$$



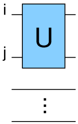
$$E' = E \cup \{(i,j)\}$$

U does not depend on G, G'

Consistency: properties of U

1. local action on $\mathcal{H}_{ij} = \mathcal{H}_i \otimes \mathcal{H}_j$

$$U(i,j) = U \otimes I^{\otimes n-2}$$



2. unoriented edge $U(i,j) = U(j,i)$

$$[S, U] = 0$$

3. edge commutativity

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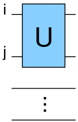

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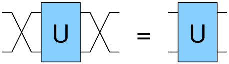

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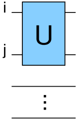

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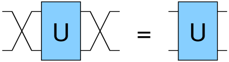

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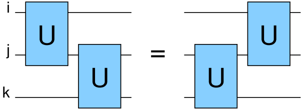


To summarize

The theory is characterized by a triplet

$$(\mathcal{H}_1, |\phi\rangle, U)$$

where:

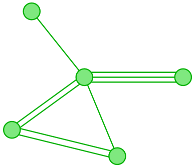
- $\mathcal{H}_1$: Hilbert space at each vertex
- $|\phi\rangle \in \mathcal{H}_1$: initial state
- $U \in \mathcal{L}(\mathcal{H}_1^{\otimes 2})$: local edge operator

$$|G\rangle = \prod_{i,j \in E(G)} U(i,j)\, |\phi\rangle^{\otimes n}$$

Examples I

Qudit graph states

- $\mathcal{H}_1 = \mathbb{C}^d$ $\qquad$ $d = 2$ graph states
- $|\phi\rangle = |+\rangle_d = \frac{1}{\sqrt{d}} \sum_i |i\rangle$
- $U = CZ_d = \sum_{j,k} \omega^{jk} |jk\rangle\langle jk|$
  $\omega = e^{2\pi i/d}$



Raussendorf et al., PRA **68**, 022312 (2003)

Looi et al., PRA **78**, 042303 (2008)

Examples II

Gaussian states, CV cluster states

vertex = qumode (HO)

- $\mathcal{H}_1 = span\{|s\rangle_p\}$ $\quad\quad$ $\mathbf{p}|s\rangle_p = s|s\rangle_p$
- $|\phi\rangle = |0\rangle_p$ (CV cluster states)
  $|\phi\rangle = |\beta\rangle$ (Gaussian states)
- $U = C_{ij}(\kappa) = e^{i\kappa q_i q_j}$

$$|G\rangle = e^{\frac{i}{2}\kappa \sum_{i,j} A(G)_{ij} q_i q_j} |\phi\rangle^{\otimes n}$$

Menicucci et al., PRA **83**, 042335 (2011)

Regular graphs

- so far: $\mathcal{H}_1 = \mathbb{C}^d$

  no structure at each vertex

- regular graphs: each vertex order g

$$\mathcal{H}_1 = (\mathbb{C}^d)^g$$

add structure at the vertex

Examples III

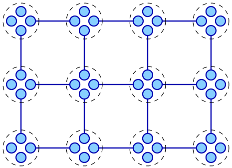
Projected entangled pair states (PEPS)

- $\mathcal{H}_1 = (\mathbb{C}^d)^g$
- $|\phi\rangle = |+\rangle_d^{\otimes g}$
- $U \sim \Pi^{(k)} V$

$V := d^{-3/2} \sum_{i,j,k,l} \omega^{(i-j)(k-l)} |ij\rangle\langle kl|$

$|\Phi_d\rangle = V|+\rangle_d^{\otimes 2} = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$

$g = 4$



$$|+\rangle_d \xrightarrow{V} |\Phi_d\rangle \xrightarrow{\Pi^{(k)}} PEPS$$

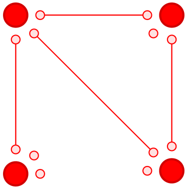
Verstraete et al., PRL **96**, 220601 (2006)

Examples IV

Quantum random networks (QRN)

- $\mathcal{H}_1 = (\mathbb{C}^2)^{n-1}$
- $|\phi\rangle = |0\rangle^{\otimes n-1}$
- $U = |\Phi^+\rangle\langle\Phi^+| V$

$$V := \sqrt{1-\tfrac{p}{2}}\, I^{\otimes 2} + \tfrac{i}{2}\sqrt{\tfrac{p}{2}}\, (X-Y)^{\otimes 2}$$
$$|\Omega\rangle = V|00\rangle = \sqrt{1-\tfrac{p}{2}}|00\rangle + \sqrt{\tfrac{p}{2}}|11\rangle$$



Perseguers et al., Nature Phys. **6**, 539 (2010)

Solutions: $d = 2$

- $U = \mathrm{diag}(a, b, b, c)$

  (a) U unitary $\Rightarrow$ graph states (CZ): $a = b = 1, c = -1$

  (b) U projector $\Rightarrow$ Bell projector (parity gate):
  $P_0 = \mathrm{diag}(1, 0, 0, 1), \quad P_1 = \mathrm{diag}(0, 1, 1, 0)$

- $U = aI^{\otimes 2} + b(T \otimes I + I \otimes T) + cT \otimes T$

  $T = \begin{bmatrix} 0 & 1 \\ \gamma & -\alpha \end{bmatrix}$

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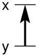
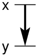
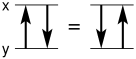
Extension: directed graphs

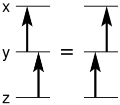
oriented edge operator : $V =$ 

$\Rightarrow V' := SVS =$ 

- $[V, V'] = 0$



- $[V \otimes I, I \otimes V] = 0$



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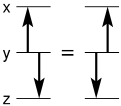
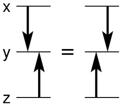


- $[V' \otimes I, I \otimes V] = 0$

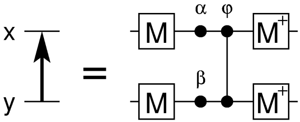


Directed graphs: a solution

Any soln. V for directed graphs provides soln. U for undirected graphs

$$U(x, y) = V(x, y)\,V'(x, y)$$

Example:



Finale I

overview

- axiomatic way

  unifying framework for several states: graph, Gaussian/CV cluster, PEPS, QRN

- modular approach

  replace axioms with stronger/weaker ones $\Rightarrow$ different class of states

- generalizes graph/cluster states

  oriented graphs, weighted graphs

R.I., T. Spiller, Phys. Rev. A **85**, 062313 (2012)

Finale II

open questions

- future work: general solution for qudit $\mathbb{C}^d$
- what class of quantum states can be described by A1-A3?
- what graph-theoretical problems can be solved?
- define new graph invariants?

R.I., T. Spiller, Phys. Rev. A **85**, 062313 (2012)

Thank you!