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On Riemann-Hilbert Problems and new Soliton equations

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PLAN

- The inverse scattering method
- RHP with canonical normalization
- Jets of order k
- RHP, Reductions and Kac-Moody algebras
- New N-wave equations – $k \geq 2$
- mKdV equations related to simple Lie algebras
- Conclusions and open questions

Based on:

- V. S. Gerdjikov, D. J. Kaup. *Reductions of* 3×3 *polynomial bundles and new types of integrable* 3*–wave interactions.* In Nonlinear evolution equations: integrability and spectral methods, Ed. A. P. Fordy, A. Degasperis, M. Lakshmanan, Manchester University Press, (1981), p. 373–380

- V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012).

- V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Pliska Stud. Math. Bulgar. **21**, 201–216 (2012).

- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

- V. S. Gerdjikov, A. B. Yanovski *On soliton equations with* $\mathbb{Z}_h$

and $\mathbb{D}_h$ reductions: conservation laws and generating operators. J. Geom. Symmetry Phys. **31**, 57–92 (2013).

- V. S. Gerdjikov, A B Yanovski. Riemann-Hilbert Problems, families of commuting operators and soliton equations Journal of Physics: Conference Series **482** (2014) 012017 doi:10.1088/1742-6596/482/1/012017

The inverse scattering method

The inverse scattering method for the N-wave equations – Zakharov, Shabat, Manakov (1973).

Lax representation:

$$
\begin{aligned}
&[L, M] \equiv 0, \\
&L\psi \equiv i\frac{\partial \psi}{\partial x} + (U_1(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \\
&M\psi \equiv i\frac{\partial \psi}{\partial t} + (V_1(x,t) - \lambda K)\psi(x,t,\lambda) = 0,
\end{aligned} \tag{1}
$$

where J, K – constant diagonal matrices.

$$
\begin{aligned}
&\lambda^2 \qquad \text{a)} \qquad [J, K] = 0, \\
&\lambda \qquad \text{b)} \qquad [U_1, K] + [J, V_1] = 0, \\
&\lambda^0 \qquad \text{c)} \qquad iV_{1,x} - iU_{1,t} + [U_1, V_1] = 0.
\end{aligned} \tag{2}
$$

Eq. a) is satisfied identically.

Eq. b) is satisfied identically if:

$$U_1(x,t) = [J, Q_1(x,t)], \qquad V_1(x,t) = [K, Q_1(x,t)],$$

Then eq. c) becomes the N-wave equation:

$$i\left[J, \frac{\partial Q_1}{\partial t}\right] - i\left[K, \frac{\partial Q_1}{\partial x}\right] + [[K, Q_1], [J, Q_1]] = 0.$$

Simplest non-trivial case:

$$N = 3, \qquad \mathfrak{g} \simeq sl(3), \qquad Q_1(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ u_1^* & 0 & u_2 \\ u_3^* & u_2^* & 0 \end{pmatrix}.$$

Then the 3-wave equations take the form:

$$\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0,$$
$$\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0,$$
$$\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

Solving Nonlinear Cauchy problems by the Inverse scattering method

Find solution to the N-wave eqs. such that

$$Q_1(x, t = 0) = q_0(x).$$

$$\begin{array}{ccccc} q_0 \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow q(x,t) \\ & \mathrm{I}\Big\downarrow & & \Big\uparrow\mathrm{III} & \\ & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & \end{array} \qquad (3)$$

Step I: Given $Q_1(x, t = 0) = q_0(x)$ construct the scattering matrix $T(\lambda, 0)$.

Jost solutions:

$$L\phi(x,\lambda) = 0, \qquad \lim_{x\to -\infty} \phi(x,\lambda) e^{i\lambda J x} = \mathbb{1},$$

$$L\psi(x,\lambda) = 0, \qquad \lim_{x\to \infty} \psi(x,\lambda) e^{i\lambda J x} = \mathbb{1},$$

$$T(\lambda,0) = \psi^{-1}(x,\lambda)\phi(x,\lambda).$$

Step II: From the Lax representation there follows:

$$i\frac{\partial T}{\partial t} - \lambda[K, T(\lambda,t)] = 0,$$

i.e.

$$T(\lambda,t) = e^{-i\lambda K t} T(\lambda,0) e^{i\lambda K t}.$$

Step III: Given $T(\lambda,t)$ construct the potential $Q_1(x,t)$ for $t > 0$.
For $\mathfrak{g} \simeq sl(2)$ – GLM eq. – Volterra type integral equations
For higher rank simple Lie algebras – GLM eq. become Fredholm type integral equations, very complicated. But it can be reduced to Riemann-Hilbert problem.

Important: All steps reduce to **linear** integral equations.
Thus the nonlinear Cauchy problem reduces to a sequence of three **linear Cauchy problems**; each has unique solution!

The fundamental analytic solutions and Riemann-Hilbert problem

Shabat (1974) – introduced the fundamental analytic solutions of L.

$$\chi^{+}(x,t,\lambda) = \phi(x,t,\lambda)S^{+}(\lambda,t) = \psi(x,t,\lambda)T^{-}(\lambda,t)D^{+}(\lambda), \qquad \lambda \in \mathbb{C}_{+},$$

$$\chi^{-}(x,t,\lambda) = \phi(x,t,\lambda)S^{-}(\lambda,t) = \psi(x,t,\lambda)T^{+}(\lambda,t)D^{-}(\lambda), \qquad \lambda \in \mathbb{C}_{-},$$

where $S^{+}(\lambda,t), T^{+}(\lambda,t)$ – upper-triangular matrices
$S^{-}(\lambda,t), T^{-}(\lambda,t)$ – lower-triangular matrices
$D^{+}(\lambda), D^{-}(\lambda)$ – diagonal matrices
Gauss decomposition of $T(\lambda,t)$:

$$T(\lambda,t) = T^{-}(\lambda,t)D^{+}(\lambda)\hat{S}^{+}(\lambda,t) = T^{+}(\lambda,t)D^{-}(\lambda)\hat{S}^{-}(\lambda,t).$$

Then

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda)G_0(\lambda,t), \qquad \lambda \in \mathbb{R}, \qquad G_0(\lambda,t) = \hat{S}^-(\lambda,t)S^+(\lambda,t).$$

Introduce

$$\xi^+(x,t,\lambda) = \chi^+(x,t,\lambda)e^{i\lambda Jx}, \qquad \xi^-(x,t,\lambda) = \chi^-(x,t,\lambda)e^{i\lambda Jx},$$

Then $\xi^\pm(x,t,\lambda)$ are FAS of the linear problem:

$$i\frac{\partial \xi^\pm}{\partial x} + U_1(x,t)\xi^\pm(x,t,\lambda) - \lambda[J,\xi^\pm(x,t,\lambda)] = 0,$$

that satisfy RHP:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda \in \mathbb{R},$$

$$i\frac{\partial G}{\partial x} - \lambda[J,G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda[K,G(x,t,\lambda)] = 0,$$

Canonical normalization

$$\lim_{\lambda\to\infty} \xi^\pm(x,t,\lambda) = \mathbb{1}.$$

Theorem 1 (Zakharov-Shabat). *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables x and t as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\tilde{L}\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda[J,\xi^{\pm}(x,t,\lambda)] = 0,$$
$$\tilde{M}\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda[K,\xi^{\pm}(x,t,\lambda)] = 0.$$

Proof. Introduce the functions:

$$g^{\pm}(x,t,\lambda) = i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda\xi^{\pm}(x,t,\lambda)J\hat{\xi}^{\pm}(x,t,\lambda),$$
$$p^{\pm}(x,t,\lambda) = i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda\xi^{\pm}(x,t,\lambda)K\hat{\xi}^{\pm}(x,t,\lambda),$$

and using

$$i\frac{\partial G}{\partial x} - \lambda[J,G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda[K,G(x,t,\lambda)] = 0.$$

prove that

$$g^+(x,t,\lambda) = g^-(x,t,\lambda), \qquad p^+(x,t,\lambda) = p^-(x,t,\lambda),$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g^+(x,t,\lambda) = \lambda J, \qquad \lim_{\lambda\to\infty} p^+(x,t,\lambda) = \lambda K.$$

and make use of Liouville theorem to get

$$\begin{aligned} g^+(x,t,\lambda) &= g^-(x,t,\lambda) = \lambda J - U_1(x,t), \\ p^+(x,t,\lambda) &= p^-(x,t,\lambda) = \lambda K - V_1(x,t). \end{aligned}$$

We shall see below that the coefficients $U_1(x,t)$ and $V_1(x,t)$ can be expressed in terms of the asymptotic coefficients Q_s of $\xi^\pm(x,t,\lambda)$.

Now remember the definition of $g^+(x,t,\lambda)$

$$\begin{aligned} g^\pm(x,t,\lambda) &= i\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,t,\lambda) + \lambda\xi^\pm(x,t,\lambda)J\hat{\xi}^\pm(x,t,\lambda) \\ &= \lambda J - U_l(x,t), \end{aligned}$$

Multiply both sides by $\xi^{\pm}(x,t,\lambda)$ and move all the terms to the left:

$$i\frac{\partial \xi^{\pm}}{\partial x} + U_l(x,t)\xi^{\pm}(x,t,\lambda) - \lambda[J,\xi^{\pm}(x,t,\lambda)] = 0,$$

i.e. $\tilde{L}\xi^{\pm}(x,t,\lambda) = 0$ or $L\chi^{\pm}(x,t,\lambda) = 0$. □

Zakharov-Shabat dressing method and soliton solutions

Starting from a regular solution $\chi_0^{\pm}(x,t,\lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x,t)$ construct new singular solutions $\chi_1^{\pm}(x,t,\lambda)$ of L with a potential $Q_{(1)}(x,t)$ with two pole singularities located at prescribed positions $\lambda_1^{\pm} \in \mathbb{C}_{\pm}$; the reduction $Q = Q^{\dagger}$ ensures that $\lambda_1^{-} = (\lambda_1^{+})^*$. It is related to the regular one by a dressing factor $u(x,t,\lambda)$

$$\chi_1^{\pm}(x,t,\lambda) = u(x,\lambda)\chi_0^{\pm}(x,t,\lambda)u_{-}^{-1}(\lambda). \qquad u_{-}(\lambda) = \lim_{x\to-\infty} u(x,\lambda) \quad (4)$$

Note that $u(x,\lambda)$ must satisfy

$$i\partial_x u + [J, Q_{(1)}(x)]u - u[J, Q_{(0)}(x)] - \lambda[J, u(x,\lambda)] = 0, \qquad (5)$$

and the normalization condition $\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}$.

The construction of $u(x,\lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ-dependence:

$$u(x,\lambda) = \mathbb{1} + (c(\lambda) - 1)P(x,t), \qquad c(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \tag{6}$$

where $P(x,t)$ is a projector

$$P(x,t) = \frac{|n_1(x,t)\rangle\langle n_1^\dagger(x,t)|}{\langle n_1^\dagger(x,t)|n_1(x,t)\rangle}, \qquad |n_1(x,t)\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{0,1}\rangle. \tag{7}$$

Taking the limit $\lambda \to \infty$ in eq. (5) we get that

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^- - \lambda_1^+)[J, P(x,t)].$$

ISM as generalized Fourier transform

Based on the Wronskian relations

$$\rho_{ij}^\pm(\lambda,t) = [\![Q_1(x,t), e_{ji}^\pm(x,t,\lambda)]\!], \qquad [\![X,Y]\!] = \int_{-\infty}^{\infty} \mathrm{tr}\,(X,[J,Y])\,,$$

$$e^{\pm}_{ji}(x,t,\lambda) = \pi_J \chi^{\pm}(x,t,\lambda) E_{ij} \hat{\chi}^{\pm}(x,t,\lambda), \qquad \pi_J X = \mathrm{ad}\,_J^{-1} \mathrm{ad}\,_J X.$$

But the 'squared' solutions satisfy completeness relation! So every function, including $Q_1(x,t)$ allows expansion

$$\begin{aligned} Q_1(x,t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{i<j} \left(\rho^+_{ij} e^+_{ij}(x,t,\lambda) - \rho^-_{ji} e^-_{ji}(x,t,\lambda) \right) \\ &\quad + \sum_{a=1}^{N} \mathrm{Res}\, \ldots \end{aligned} \tag{8}$$

Hamiltonian hierarchies of N-wave equations

The Lie bracket on $\mathfrak{g}$ induces Poisson structure on the co-adjoint orbit passing through J.

The functions $D^{\pm}(\lambda)$ are t-independent and generate an infinite number of integrals of motion in involution.

$$\Omega_0 = [\![\mathrm{ad}\,_J^{-1} \delta Q_1 \mathop{\wedge}_{,} \mathrm{ad}\,_J^{-1} \delta Q_1]\!],$$

$$\Omega_p = [\![\mathrm{ad}\,_J^{-1} \delta Q_1 \mathop{\wedge}_{,} \Lambda^p \mathrm{ad}\,_J^{-1} \delta Q_1]\!],$$

where Λ is the recursion operator:

$$\Lambda e_{ij}^{+}(x,t,\lambda) = \lambda e_{ij}^{+}(x,t,\lambda).$$

see VSG, P. Kulish (1981) and VSG, Yanovski, Vilasi. *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods* Lecture Notes in Physics **748**, Springer Verlag, Berlin, Heidelberg, New York (2008).

Generalizations to polynomial Lax operators

$$
\begin{aligned}
&[L, M] \equiv 0, \\
&\quad L\psi \equiv i\frac{\partial\psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\
&\quad M\psi \equiv i\frac{\partial\psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0,
\end{aligned} \tag{9}
$$

where J, K – constant diagonal matrices.

λ^4 a) $[J, K] = 0,$ λ^3 b) $[U_1, K] + [J, V_1] = 0,$

λ^2 c) $[U_1, V_1] - [U_2, K] - [J, V_2] = 0.$

Eqs. a)–c) must be satisfied identically if

$$
U_1(x,t) = [J, Q_1(x,t)], \qquad V_1(x,t) = [K, Q_1(x,t)],
$$

$$
U_2 = [J, Q_2] - \frac{1}{2}\mathrm{ad}\,_{Q_1}^2 J, \qquad U_2 = [K, Q_2] - \frac{1}{2}\mathrm{ad}\,_{Q_1}^2 K.
$$

Thus we obtain NLEE the generalization of the N-wave equation:

$$\begin{aligned} &\lambda^1 \qquad \text{d)} \qquad iV_{1,x} - iU_{1,t} + [U_2, V_1] + [U_1, V_2] = 0, \\ &\lambda^0 \qquad \text{e)} \qquad iV_{1,x} - iU_{1,t} + [U_2, V_1] + [U_1, V_2] = 0. \end{aligned} \tag{10}$$

for the functions $Q_1(x,t)$ and $Q_2(x,t)$.

Note: Going to higher powers λ^k makes more complicated

1. the problem of correct parametrizing
2. Wronskian relations, 'squared' solutions, recursion operators
3. The potential functions of L and M

$$U(x,t,\lambda) = U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J, \qquad V(x,t,\lambda) = V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K,$$

can be viewed as elements of a Kac-Moody algebra $\mathfrak{g}_{\rm KM}$.

4. Hamiltonian properties are on the co-adjoint orbits of the $\mathfrak{g}_{\rm KM}$.

RHP with canonical normalization

$$\xi^{+}(x,t,\lambda) = \xi^{-}(x,t,\lambda)G(x,t,\lambda), \qquad \lambda^{k} \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^{+}(x,t,\lambda) = \mathbb{1},$$

$\xi^{\pm}(x,t,\lambda) \in \mathfrak{G}$

Consider particular type of dependence $G(x,t,\lambda)$:

$$i\frac{\partial G}{\partial x} - \lambda^{k}[J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^{k}[K, G(x,t,\lambda)] = 0.$$

where $J \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

where all $Q_k(x,t) \in \mathfrak{g}$ and $Q(x,t,\lambda) \in \mathfrak{g}_{\rm KM}$. However,

$$\mathcal{J}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)J\hat{\xi}^{\pm}(x,t,\lambda), \qquad \mathcal{K}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)K\hat{\xi}^{\pm}(x,t,\lambda),$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. If in addition K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}(x,t,\lambda), \mathcal{K}(x,t,\lambda)] = 0.$$

Generalized Zakharov-Shabat theorem

Theorem 2. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables x and t as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\begin{aligned}
\tilde{L}\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0,\\
\tilde{M}\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [K, \xi^{\pm}(x,t,\lambda)] = 0.
\end{aligned}$$

Proof. Introduce the functions:

$$\begin{aligned}
g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda),\\
p^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda),
\end{aligned}$$

and using

$$i\frac{\partial G}{\partial x} - \lambda^k[J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k[K, G(x,t,\lambda)] = 0.$$

prove that

$$g^+(x,t,\lambda) = g^-(x,t,\lambda), \qquad p^+(x,t,\lambda) = p^-(x,t,\lambda),$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g^+(x,t,\lambda) = \lambda^k J, \qquad \lim_{\lambda\to\infty} p^+(x,t,\lambda) = \lambda^k K.$$

and make use of Liouville theorem to get

$$g^+(x,t,\lambda) = g^-(x,t,\lambda) = \lambda^k J - \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l},$$

$$p^+(x,t,\lambda) = p^-(x,t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(x,t)\lambda^{k-l}.$$

We shall see below that the coefficients $U_l(x,t)$ and $V_l(x,t)$ can be expressed in terms of the asymptotic coefficients Q_s of $\xi^{\pm}(x,t,\lambda)$.

Now remember the definition of $g^{+}(x,t,\lambda)$

$$\begin{aligned} g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda) \\ &= \lambda^k J - \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \end{aligned}$$

Multiply both sides by $\xi^{\pm}(x,t,\lambda)$ and move all the terms to the left:

$$i\frac{\partial \xi^{\pm}}{\partial x} + \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}\xi^{\pm}(x,t,\lambda) - \lambda^k[J,\xi^{\pm}(x,t,\lambda)] = 0,$$

i.e. $\tilde{L}\xi^{\pm}(x,t,\lambda) = 0$ and $L\chi^{\pm}(x,t,\lambda) = 0$ where $\chi^{\pm}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda)e^{-i\lambda^k Jx}$.

□

Lemma 1. *The operators L and M commute*

$$[L, M] = 0,$$

i.e. the following set of equations hold:

$$i\frac{\partial U}{\partial t} - i\frac{\partial V}{\partial x} + [U(x,t,\lambda) - \lambda^k J, V(x,t,\lambda) - \lambda^k K] = 0.$$

where

$$U(x,t,\lambda) = \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \qquad V(x,t,\lambda) = \sum_{l=0}^{k} V_l(x,t)\lambda^{k-l}.$$

Jets of order k

How to parametrize $U(x,t,\lambda)$ and $V(x,t,\lambda)$?
Use:

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

and consider the jets of order k of $\mathcal{J}_+(x,\lambda)$ and $\mathcal{K}_+(x,\lambda)$:

$$\mathcal{J}_+(x,t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) J_l \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k J - U(x,t,\lambda),$$

$$\mathcal{K}_+(x,t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k K - V(x,t,\lambda).$$

Express $U(x) \in \mathfrak{g}$ in terms of $Q_s(x)$:

$$\begin{aligned}
\mathcal{J}(x,t,\lambda) &\equiv = \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda) \\
&= J + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k J, \\
\mathcal{K}(x,t,\lambda) &\equiv \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda) \\
&= K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k K, \\
\mathrm{ad}\,_Q Z &= [Q,Z], \qquad \mathrm{ad}\,_Q^2 Z = [Q,[Q,Z]], \quad \ldots
\end{aligned}$$

and therefore for U_l we get:

$$U_1(x,t) = -\operatorname{ad}_{Q_1} J, \qquad U_2(x,t) = -\operatorname{ad}_{Q_2} J - \frac{1}{2}\operatorname{ad}_{Q_1}^2 J$$
$$U_3(x,t) = -\operatorname{ad}_{Q_3} J - \frac{1}{2}\left(\operatorname{ad}_{Q_2}\operatorname{ad}_{Q_1} + \operatorname{ad}_{Q_1}\operatorname{ad}_{Q_2}\right) J - \frac{1}{6}\operatorname{ad}_{Q_1}^3 J.$$

and similar expressions for $V_l(x,t)$ with J replaced by K.

Reductions of polynomial bundles

Using $\mathcal{J}_+(x,t,\lambda)$ and $\mathcal{K}_+(x,t,\lambda)$ we end up with a set of NLEE for the coefficients $Q_1(x,t), Q_2(x,t), \dots, Q_k(x,t)$. Too many functions, too complicated equations.

They can be simplified by using Mikhailov's reduction group:
$\mathbb{Z}_2$**-reductions (involutions)**:

$$\text{a)} \qquad A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^-(x,t,\lambda), \qquad AQ^\dagger(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda),$$
$$\text{b)} \qquad B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^-(x,t,\lambda), \qquad BQ^*(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda),$$
$$\text{c)} \qquad C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^-(x,t,\lambda), \qquad CQ^\dagger(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda),$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$.

$\mathbb{Z}_N$**-reductions**:

$$D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda),$$

where $\omega^h = 1$ and $D^h = \mathbb{1}$.

If D is the Coxeter element of $\mathfrak{g}$ then $Q(x,t,\lambda)$ belongs to the corresponding $\mathfrak{g}_{\rm KM}$ of height 1.

If D is the Coxeter element of $\mathfrak{g}$ composed by $V-$ an external automorphism of $\mathfrak{g}$ then $Q(x,t,\lambda)$ belongs to the corresponding $\mathfrak{g}_{\rm KM}$ of height 2 or 3.

On N-wave equations – $k = 1$

Lax representation involves two Lax operators linear in λ:

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[K, \xi^{\pm}(x,t,\lambda)] = 0.$$

The corresponding equations take the form:

$$i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J,Q],[K,Q(x,t)]] = 0$$

$$Q(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad \begin{array}{r} J = \mathrm{diag}\,(a_1, a_2, a_3), \\ K = \mathrm{diag}\,(b_1, b_2, b_3), \end{array}$$

Then the 3-wave equations take the form:

$$\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0,$$
$$\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0,$$
$$\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

New 3-wave equations – $k \geq 2$

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(x,t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$

Fix up $k=2$. Then the Lax pair becomes

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)]\right) + \lambda[J, Q_1],$$
$$V \equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)]\right) + \lambda[K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \operatorname{diag}(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix},$$

and we obtain new type of integrable 3-wave equations:

$$
\begin{aligned}
&i(a_1-a_2)\frac{\partial u_1}{\partial t}-i(b_1-b_2)\frac{\partial u_1}{\partial x}+\epsilon\kappa u_2^* u_3+\epsilon\frac{\kappa(a_1-a_2)}{(a_1-a_3)}u_1|u_2|^2=0,\\
&i(a_2-a_3)\frac{\partial u_2}{\partial t}-i(b_2-b_3)\frac{\partial u_2}{\partial x}+\epsilon\kappa u_1^* u_3-\epsilon\frac{\kappa(a_2-a_3)}{(a_1-a_3)}|u_1|^2u_2=0,\\
&i(a_1-a_3)\frac{\partial u_3}{\partial t}-i(b_1-b_3)\frac{\partial u_3}{\partial x}-\frac{i\kappa}{a_1-a_3}\frac{\partial(u_1u_2)}{\partial x}\\
&\quad+\epsilon\kappa\left(\frac{a_1-a_2}{a_1-a_3}|u_1|^2+\frac{a_2-a_3}{a_1-a_3}|u_2|^2\right)u_1u_2+\epsilon\kappa u_3(|u_1|^2-|u_2|^2)=0,
\end{aligned}
$$

where

$$
\kappa=a_1(b_2-b_3)-a_2(b_1-b_3)+a_3(b_1-b_2),\qquad u_3=w_3+\frac{2a_2-a_1-a_3}{2(a_1-a_3)}u_1u_2.
$$

The diagonal terms in the Lax representation are λ-independent.

Two of them read:

$$
\begin{aligned}
&i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,\\
&i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,
\end{aligned}
$$

These relations are satisfied identically as a consequence of the NLEE.

New types of 4-wave interactions

The Lax pair for these new equations will be provided by:

$$
\begin{aligned}
L\psi &= i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,\\
M\psi &= i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0,
\end{aligned}
$$

where $U_j(x,t)$ and $V_j(x,t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$U_1(x,t) = [J, Q_1(x,t)], \qquad U_2(x,t) = [J, Q_2(x,t)] - \frac{1}{2}\operatorname{ad}_{Q_1}^2 J,$$
$$V_1(x,t) = [K, Q_1(x,t)], \qquad V_2(x,t) = [K, Q_2(x,t)] - \frac{1}{2}\operatorname{ad}_{Q_1}^2 K.$$

Here $\operatorname{ad}_{Q_1} X \equiv [Q_1(x,t), X]$.

Assume $Q_1(x,t)$ and $Q_2(x,t)$ to be generic elements of $so(5)$:

$$Q_1(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha^1 E_\alpha + p_\alpha^1 E_{-\alpha}) + r_1^1 H_{e_1} + r_2^1 H_{e_2},$$
$$Q_2(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha^2 E_\alpha + p_\alpha^2 E_{-\alpha}) + r_1^2 H_{e_1} + r_2^2 H_{e_2},$$
$$J = a_1 H_{e_1} + a_2 H_{e_2} = \operatorname{diag}(a_1, a_2, 0, -a_2, -a_1),$$
$$K = b_1 H_{e_1} + b_2 H_{e_2} = \operatorname{diag}(b_1, b_2, 0, -b_2, -b_1),$$

Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction

$$B_0 U(x,t,\epsilon\lambda^*)^\dagger B_0^{-1} = U(x,t,\lambda), \qquad B_0 = \mathrm{diag}\,(1,\epsilon,1,\epsilon,1), \quad \epsilon^2 = 1.$$

As a result:

$$B_0(\chi^+(x,t,\epsilon\lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t,\lambda))^{-1}, \qquad B_0(T(t,\epsilon\lambda^*))^\dagger B_0^{-1} = (T(t,\lambda))^{-1},$$

which provide $p^1_\alpha = \epsilon(q^1_\alpha)^*$, $p^2_\alpha = \epsilon(q^2_\alpha)^*$. Then the Lax representation will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements q^1_α and q^2_α. Additional $\mathbb{Z}_2$ reduction condition

$$D\xi^\pm(x,t,-\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \qquad DQ(x,t,-\lambda)\hat{D} = Q(x,t,\lambda),$$
$$D = \mathrm{diag}\,(1,-1,1,-1,1)$$

$$Q_1(x,t) = \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix},$$

$$Q_2(x,t) = \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix},$$

$$J = a_1 H_{e_1} + a_2 H_{e_2} = \operatorname{diag}(a_1, a_2, 0, -a_2, -a_1),$$

$$K = b_1 H_{e_1} + b_2 H_{e_2} = \operatorname{diag}(b_1, b_2, 0, -b_2, -b_1),$$

Combining both reductions for the matrix elements of $Q_j(x,t)$ we have:

$$v_1 = \epsilon u_1^*, \qquad v_2 = \epsilon u_2^*, \qquad v_3 = \epsilon u_3^*, \qquad v_4 = u_4^*,$$

The commutativity condition for the Lax pair

$$i\left(\frac{\partial V_2}{\partial x}+\lambda\frac{\partial V_1}{\partial x}\right)-i\left(\frac{\partial U_2}{\partial t}+\lambda\frac{\partial U_1}{\partial t}\right)+[U_2+\lambda U_1-\lambda^2 J, V_2+\lambda V_1-\lambda^2 K]=0$$

must hold identically with respect to λ. The terms proportional to λ^4, λ^3 and λ^2 vanish identically. The term proportional to λ and the λ-independent term vanish provided Q_i satisfy the NLEE:

$$\begin{aligned}
&i\frac{\partial V_1}{\partial x}-i\frac{\partial U_1}{\partial t}+[U_2,V_1]+[U_1,V_1]=0,\\
&i\frac{\partial V_2}{\partial x}-i\frac{\partial U_2}{\partial t}+[U_2,V_2]=0.
\end{aligned}$$

In components the corresponding NLEE:

$$
\begin{aligned}
& -2i(a_1 - a_2)\frac{\partial u_1}{\partial t} + 2i(b_1 - b_2)\frac{\partial u_1}{\partial x} + \kappa\epsilon u_2^*(\epsilon u_2^* u_3 - u_1u_2 - 2u_4) = 0, \\
& -2ia_2\frac{\partial u_2}{\partial t} + 2ib_2\frac{\partial u_2}{\partial x} - \kappa(u_2\epsilon(|u_3|^2 - |u_1|^2) + 2u_3u_4^* + 2\epsilon u_1^* u_4) = 0, \\
& -2i(a_1 + a_2)\frac{\partial u_3}{\partial t} + 2i(b_1 + b_2)\frac{\partial u_3}{\partial x} + \kappa u_2(\epsilon u_2^* u_3 - u_1u_2 + 2u_4) = 0, \\
& -2ia_1\frac{\partial u_4}{\partial t} + 2ib_1\frac{\partial u_4}{\partial x} + i\frac{\partial}{\partial t}\left(-(2a_2 - a_1)u_1u_2 + (2a_2 + a_1)\epsilon u_2^* u_3\right) \\
& + i(2b_2 - b_1)\frac{\partial(u_1u_2)}{\partial x} - i(2b_2 + b_1)\epsilon\frac{\partial(u_2^*u_3)}{\partial x} - \kappa\Big(2\epsilon u_4(|u_1|^2 - |u_3|^2) \\
& + \epsilon u_1u_2(|u_1|^2 + 3|u_3|^2) - u_3u_2^*(3|u_1|^2 + |u_3|^2)\Big) = 0.
\end{aligned}
$$

NLS and MKdV-type equations with $sl(n)$-series

Drinfeld, Sokolov (1981).

$$
\begin{aligned}
L\psi &\equiv i\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi = 0, \\
M\psi &\equiv i\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi = \psi C(\lambda),
\end{aligned}
$$

For the case of $\mathbb{Z}_N$-reduction (Mikhailov (1981)):

$$
C_1 U(x,t,\lambda) C_1^{-1} = U(x,t,\omega\lambda), \quad C_1 V(x,t,\lambda) C_1^{-1} = V(x,t,\omega\lambda),
$$

where $C_1^N = \mathbb{1}$ is a Coxeter automorphism of the algebra $\mathfrak{sl}(N,\mathbb{C})$ and $\omega = \exp(2\pi i/N)$.

Let $\mathfrak{g} \simeq \mathfrak{sl}(N,\mathbb{C})$ and the group of reduction is $\mathbb{Z}_N$. The class of relevant NLEE may be considered as generalizations of the derivative

NLS equations

$$i\frac{\partial\psi_k}{\partial t}+\gamma\frac{\partial}{\partial x}\left(\cot\left(\frac{\pi k}{N}\right)\cdot\psi_{k,x}+i\sum_{p=1}^{N-1}\psi_p\psi_{k-p}\right)=0,$$

$k=1,2,\ldots,N-1$, where γ is a constant and the index $k-p$ should be understood modulus N and $\psi_0=\psi_N=0$.

The automorphism Ad_{C_1} ($\mathrm{Ad}_{C_1}(Y)\equiv C_1YC_1^{-1}$ for every Y from $\mathfrak{g}$) defines a grading in the Lie algebra

$$\mathfrak{sl}(N,\mathbb{C})=\bigoplus_{k=0}^{N-1}\mathfrak{g}^{(k)},$$

$$J^{(k)}=\sum_{j=1}^{N}\omega^{kj}E_{j,j+s},\qquad C^{-1}J^{(k)}C=\omega^{-k}J^{(k)}.$$

where $(E_{j,s})_{q,r}=\delta_{jq}\delta_{sr}$. Obviously

$$\left[J^{(k)},J_l^{(m)}\right]=\left(\omega^{ms}-\omega^{kl}\right)J_{s+l}^{(k+m)}.$$

Examples of DNLS-type equations

If $N = 5$ we can apply the involution: $\psi_0 = \psi_5 = 0$, $\psi_1 = \psi_4^*$, $\psi_2 = \psi_3^*$, i.e., we have only two independent complex-valued fields and

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} + \gamma\mathrm{cotan}\,\frac{\pi}{5}\frac{\partial^2\psi_1}{\partial x^2} + i\gamma\frac{\partial}{\partial x}\left(2\psi_2\psi_1^* + (\psi_2^*)^2\right) = 0,\\ i\frac{\partial\psi_2}{\partial t} + \gamma\mathrm{cotan}\,\frac{2\pi}{5}\frac{\partial^2\psi_2}{\partial x^2} + i\gamma\frac{\partial}{\partial x}\left(2\psi_1^*\psi_2^* + (\psi_1)^2\right) = 0, \end{aligned} \tag{11}$$

For $N = 6$ and $\psi_1 = \psi_5^*$, $\psi_2 = \psi_4^*$, $\psi_3 = \psi_3^*$, so we have a system for two complex-valued fields ψ_1 and ψ_2 and the real field ψ_3:

$$\begin{aligned} i\frac{\partial\psi_1}{\partial t} + \gamma\mathrm{cotan}\,\frac{\pi}{6}\frac{\partial^2\psi_1}{\partial x^2} + 2i\gamma\frac{\partial}{\partial x}\left(\psi_1^*\psi_2 + \psi_2^*\psi_3\right) = 0,\\ i\frac{\partial\psi_2}{\partial t} + \gamma\mathrm{cotan}\,\frac{2\pi}{6}\frac{\partial^2\psi_2}{\partial x^2} + i\gamma\frac{\partial}{\partial x}\left(\psi_1^2 + 2\psi_1^*\psi_3 + (\psi_2^*)^2\right) = 0,\\ \frac{\partial\psi_3}{\partial t} + 2\gamma\frac{\partial}{\partial x}\left(\psi_1\psi_2 + \psi_1^*\psi_2^*\right) = 0, \end{aligned} \tag{12}$$

Examples of MKdV-type equations

Next choose $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as follows:

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \quad Q(x,t) = \sum_{j=1}^{N-1} \psi_j(x,t) J_j^{(0)}, \quad J = aJ_0^{(1)}$$

$$V(x,t,\lambda) = V_3(x,t) + \lambda V_2(x,t) + \lambda^2 V_1(x,t) - \lambda^3 K,$$

where

$$V_1(x,t) = \sum_{k=1}^{N} v_k^1(x,t) J_k^{(2)}, \qquad V_2(x,t) = \sum_{l=1}^{N} v_l^2(x,t) J_l^{(1)},$$

$$V_3(x,t) = \sum_{j=1}^{N-1} v_j^3(x,t) J_j^{(0)}, \qquad K = bJ_0^{(3)}.$$

The constants a and b determine the dispersion law of the MKdV eqs.

The next step is to request that $[L,M]=0$ identically with respect to λ.

$$v_k^1(x,t) = \frac{b}{a}(\omega^{2k} + \omega^k + 1)\psi_k, \qquad k = 1, \dots, N-1,$$

and $v_N^1 = C(t)$ with $C(t)$ - arbitrary function of time. For

$$v_l^2(x,t) = \frac{b}{a^2} \sum_{j+k=l}^{N-1} \frac{\omega^{2l} + \omega^{2j+k} - \omega^k - 1}{1-\omega^l} \psi_j \psi_k$$
$$+ i\frac{b}{a^2}\left(\frac{\omega^{2l}+\omega^l+1}{1-\omega^l}\right)\frac{\partial \psi_l}{\partial x} - \frac{C}{a}(\omega^l+1)\psi_l,$$

for $l = 1, \ldots, N-1$ and

$$v_N^2 = -\frac{b}{a^2} \sum_{j+l=0}^{N-1} \left(\cos\frac{2\pi j}{N} + \frac{1}{2}\right) \psi_j \psi_l + D(t),$$

with D(t) - another arbitrary function of time. And for

$$v_j^3 = \frac{b}{a^3} \cot\left(\frac{\pi j}{N}\right) \sum_{k+l=j}^{N-1} \frac{\partial}{\partial x}(\psi_k \psi_l) + \frac{C}{a^2} \sum_{m+l=j}^{N-1} (\psi_m \psi_l)$$
$$+ \frac{b}{2a^3} \sum_{k+l=j}^{N-1} \frac{\cos\frac{\pi(k-l)}{N}}{\sin\frac{\pi j}{N}} \frac{\partial}{\partial x}(\psi_k \psi_l) - \frac{D}{a}\psi_j$$

$$
\begin{aligned}
&+\frac{b}{a^3}\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}(\psi_i\psi_k\psi_m)+\frac{3b}{2a^3}\sum_{l+m=j}^{N-1}\cot\left(\frac{\pi l}{N}\right)\frac{\partial\psi_l}{\partial x}\psi_m \\
&+\frac{b}{a^3}\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\sin\frac{\pi(j-2k)}{N}-\sin\frac{\pi(j-2m)}{N}}{\sin\frac{\pi j}{N}}(\psi_i\psi_k\psi_m) \\
&-\frac{b}{4a^3}\cot\left(\frac{\pi j}{N}\right)\sum_{l+m=j}^{N-1}\frac{\partial}{\partial x}(\psi_l\psi_m)+\frac{C}{a^2}\cot\left(\frac{\pi j}{N}\right)\frac{\partial\psi_j}{\partial x} \\
&-\frac{b}{2a^3}\sum_{l+m=j}^{N-1}\frac{\cos\frac{\pi(l-m)}{N}}{\sin\frac{\pi j}{N}}\frac{\partial}{\partial x}(\psi_l\psi_m)+\frac{b}{a^3}\left(\cot^2\frac{\pi j}{N}-\frac{1}{4\sin^2\frac{\pi j}{N}}\right)\frac{\partial^2\psi_j}{\partial x^2} \\
&+\frac{b}{a^3}\sum_{k=1}^{N-1}\left(\cos\frac{2\pi k}{N}+\frac{1}{2}\right)(\psi_k\psi_{N-k}\psi_j)
\end{aligned}
$$

where j is running from 1 to N-1. We choose $C(t)=0$ and $D(t)=0$.

In the end we get the following system of mKdV equations:

$$\alpha\frac{\partial\psi_j}{\partial t} = \left(\cot^2\frac{\pi j}{N} - \frac{1}{4\sin^2\frac{\pi j}{N}}\right)\frac{\partial^3\psi_j}{\partial x^3} + \sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\partial}{\partial x}(\psi_i\psi_k\psi_m)$$

$$+\sum_{l+m=j}^{N-1}\sum_{i+k=l}^{N-1}\frac{\sin\frac{\pi(j-2k)}{N} - \sin\frac{\pi(j-2m)}{N}}{\sin\frac{\pi j}{N}}\frac{\partial}{\partial x}(\psi_i\psi_k\psi_m)$$

$$+\sum_{k=1}^{N-1}\left(\cos\frac{2\pi k}{N} + \frac{1}{2}\right)\frac{\partial}{\partial x}(\psi_k\psi_{N-k}\psi_j) + \frac{3}{4}\cot\left(\frac{\pi j}{N}\right)\sum_{k+l=j}^{N-1}\frac{\partial^2}{\partial x^2}(\psi_k\psi_l)$$

$$+\frac{3}{4}\sum_{k+l=j}^{N-1}\frac{\partial}{\partial x}\left(\cot\left(\frac{\pi l}{N}\right)\frac{\partial\psi_l}{\partial x}\psi_k + \cot\left(\frac{\pi k}{N}\right)\frac{\partial\psi_k}{\partial x}\psi_l\right)$$

where $\alpha = a^3/b$.

In the case of $\mathfrak{sl}(2,\mathbb{C})$ algebra we obtain the well-known MKdV equation

$$\alpha\frac{\partial\psi_1}{\partial t} = -\frac{1}{4}\frac{\partial^3\psi_1}{\partial x^3} - \frac{1}{2}\frac{\partial}{\partial x}(\psi_1^3).$$

In the case of $\mathfrak{sl}(3,\mathbb{C})$ algebra we have the system of trivial equations $\partial_t\psi_1 = 0$ and $\partial_t\psi_2 = 0$. In the case of $\mathfrak{sl}(4,\mathbb{C})$ algebra we find:

$$\alpha\frac{\partial\psi_1}{\partial t} = \frac{1}{2}\frac{\partial^3\psi_1}{\partial x^3} + \frac{3}{2}\frac{\partial}{\partial x}\left(\frac{\partial\psi_2}{\partial x}\psi_3\right) + \frac{3}{2}\frac{\partial}{\partial x}(\psi_1\psi_2^2) + \frac{\partial}{\partial x}(\psi_3^3),$$

$$\alpha\frac{\partial\psi_2}{\partial t} = -\frac{1}{4}\frac{\partial^3\psi_2}{\partial x^3} + \frac{3}{4}\frac{\partial^2}{\partial x^2}\left(\psi_1^2\right) - \frac{3}{4}\frac{\partial^2}{\partial x^2}\left(\psi_3^2\right) + 3\frac{\partial}{\partial x}(\psi_1\psi_2\psi_3) - \frac{1}{2}\frac{\partial}{\partial x}(\psi_2^3), \quad (13)$$

$$\alpha\frac{\partial\psi_3}{\partial t} = \frac{1}{2}\frac{\partial^3\psi_3}{\partial x^3} - \frac{3}{2}\frac{\partial}{\partial x}\left(\psi_1\frac{\partial\psi_2}{\partial x}\right) + \frac{3}{2}\frac{\partial}{\partial x}(\psi_2^2\psi_3) + \frac{\partial}{\partial x}(\psi_1^3).$$

If we apply case a) we get the same set of MKdV equations with ψ_1, ψ_2 and ψ_3 purely real functions.

In the case b) we put $\psi_1 = -\psi_3^* = u$ and $\psi_2 = -\psi_2^* = iv$ and get:

$$\alpha \frac{\partial v}{\partial t} = -\frac{1}{4}\frac{\partial^3 v}{\partial x^3} + \frac{3}{4i}\frac{\partial^2}{\partial x^2}\left(u^2 - u^{*,2}\right) - 3\frac{\partial}{\partial x}(|u|^2 v) + \frac{1}{2}\frac{\partial}{\partial x}(v^3),$$

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - i\frac{3}{2}\frac{\partial}{\partial x}\left(u^* \frac{\partial v}{\partial x}\right) - \frac{3}{2}\frac{\partial}{\partial x}(uv^2) - \frac{\partial}{\partial x}((u^*)^3),$$

where u is a complex function, but v is a purely real function.
In the case c):

$$\alpha \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - \frac{\partial}{\partial x}(u^3),$$

where u is a complex function, we recover the MKdV equation. In the case of $\mathfrak{sl}(6, \mathbb{C})$ algebra with $\mathbb{D}_6$-reduction in the case c) we find

$$\alpha \frac{\partial u}{\partial t} = 2\frac{\partial^3 u}{\partial x^3} - 2\sqrt{3}\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) - 6\frac{\partial}{\partial x}(uv^2),$$

$$\alpha \frac{\partial v}{\partial t} = \sqrt{3}\frac{\partial^2}{\partial x^2}\left(u^2\right) - 6\frac{\partial}{\partial x}(u^2 v),$$

where u and v are complex functions.

MKdV and $so(8)$

Normally with each simple Lie algebra one can associate just one MKdV eq. The only exception is $so(8)$ which allows a one-parameter family of MKdV equations. The reason is that only $so(8)$ has 3 as a double exponent!

$$
\begin{aligned}
\partial_t q_1 = 2a\left[\partial_x^3 q_1 - \sqrt{3}\partial_x(q_1\partial_x q_2)\right] - \sqrt{3}\left[(3a+b)\partial_x(q_4\partial_x q_3) + (3a-b)\partial_x(q_3\partial_x q_4)\right] \\
- 3\partial_x\left[q_1\left(2aq_2^2 + (a-b)q_3^2 + (a+b)q_4^2\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\partial_t q_2 = \sqrt{3}a\partial_x^2 q_1^2 + \frac{\sqrt{3}}{2}(a+b)\partial_x^2 q_3^2 + \frac{\sqrt{3}}{2}(a-b)\partial_x^2 q_4^2 \\
- 3\partial_x\left[q_2\left(2aq_1^2 + (a+b)q_3^2 + (a-b)q_4^2\right)\right],
\end{aligned}
$$

$$\partial_t q_3 = -(a+b)\left[\partial_x^3 q_3 - \sqrt{3}\partial(q_3\partial_x q_2)\right] - \sqrt{3}\left[(3a+b)\partial_x(q_4\partial_x q_1) + 2b\partial_x(q_1\partial_x q_4)\right]$$
$$+ 3\partial_x\left[q_3\left(2aq_4^2 + (a-b)q_1^2 + (a+b)q_2^2\right)\right],$$

$$\partial_t q_4 = -(a-b)\left[\partial_x^3 q_4 - \sqrt{3}\partial_x(q_4\partial_x q_2)\right] - \sqrt{3}\left[(3a-b)\partial_x(q_3\partial_x q_1) - 2b\partial_x(q_1\partial_x q_3)\right]$$
$$+ 3\partial_x\left[q_4\left(2aq_3^2 + (a-b)q_2^2 + (a+b)q_1^2\right)\right].$$

Conclusions and open questions

- More classes of new integrable equations: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power k of the polynomials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ and iv) different reductions of U and V.

- These new NLEE must be Hamiltonian. View the jets $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Apply Zakharov-Shabat dressing method for constructing their N-soliton solutions and study their interactions.

- 'Squared' solutions, Recursion operators, Hamiltonian hierarchies

- Apply the above methods to twisted Kac-Moody algebras – work in progress

Thank you for your attention!