

University of Warsaw
Faculty of Physics

Jakub Filipek

Student no. 420873

The Gauging of Groupoidal Symmetries of Sigma Models

Master's thesis
in PHYSICS

Supervisor:

Dr. Rafał Suszek

Department of Mathematical Methods in Physics

Warsaw, August 2025

Abstract

We give an overview of the theory of principaloid bundles with connection developed by T. Strobl and R.R. Suszek. These are to be understood as models of the configuration bundle and the gauge field of a field theory with a rigid Lie-groupoidal symmetry gauged (in the sense of Cartan and Ehresmann). Using their formalism, we construct the minimal coupling scheme for tensorial sigma models with groupoidal symmetry. We show that in order to allow for non-zero-dimensional orbits in the presence of a metric term, one usually needs to non-trivially reduce the structure group of the principaloid bundle. To gauge sigma models with a non-tensorial topological coupling, we study the stack structure of degree-2 Cheeger-Simons differential characters, and we construct an isomorphism between local cohomological data of a character and the character descended onto the configuration bundle of the gauged sigma model, identified with the Godement quotient of the principaloid bundle with respect to the Lie groupoid action. For multiplicative differential forms ρ on a Lie groupoid $\mathcal{G}$, appearing naturally in the gauging beyond the minimal coupling scheme, we define ρ -equivariant differential characters on $\mathcal{G}$. Finally, we conjecture that under an appropriate reduction of the structure group of the principaloid bundle and given a ρ -equivariant differential character of an arbitrary degree, there exists a canonical gauged sigma model, the construction of which generalizes the minimal coupling procedure. We prove the assertion for degree-2 ρ -equivariant differential characters. Our constructive proof yields a novel definition (generalizing those formulated by Gawędzki, Szuszek and Waldorf in degree-3 for the specific Lie-group action groupoid) of a 1-dimensional field theory with a Lie-groupoidal symmetry gauge in a quantum-mechanical consistent manner.

Keywords

sigma models, gauge theory, Lie groupoids, field theory, minimal coupling, differential characters

Thesis domain (Socrates-Erasmus subject area codes)

13.2 Physics

11.1 Mathematics

Subject classification

58H05 Pseudogroups and differentiable groupoids

81T13 Yang-Mills and other gauge theories in quantum field theory

53C80 Applications of global differential geometry to the sciences

55N35 Other homology theories in algebraic topology

Tytuł pracy w języku polskim

Teoria cechowania symetrii grupoidalnych modeli sigma

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Introduction

Sigma models with a generically non-tensorial topological term arise for extended distributions of probe charge of arbitrary dimension. In the simplest 1-dimensional case, modeling the evolution of a point charge in external gravitational and electromagnetic fields, such a term is given by the line holonomy of the $U(1)$ bundle that geometrizes the latter field. Two-dimensional sigma models abound in string theory and in the effective field theory of slow spinons in a quantum spin-chain [AH87]. One should also mention their relation to the so-called emergent spectral non-commutative geometry [FG94]. The gauging of 2-dimensional sigma models with group symmetry and the Dirac-Feynman amplitude given by a Cheeger-Simons differential character [CS85] was developed by K. Gawędzki, R. R. Suszek and K. Waldorf in [GSW10] and [GSW13]. They operated in the formalism of $U(1)$ -bundle gerbes [Mur96; MS00], whose generalized holonomies are in a bijection with $U(1)$ -valued differential characters [Bry07, 1.5.]. This thesis might be viewed as a natural extension of their results to a broader class of symmetries – namely those modeled by a smooth fibered action of a Lie groupoid on its object manifold, chosen as the target space of the sigma model. Lie groupoids are much better suited to model symmetries of physical systems, in particular, for bounded configuration fibers – see [Wei96]. In this thesis, instead of studying groupoid-equivariant bundle gerbes, we avoid extensive geometrization by adapting a differential-topological approach. To this end, we invoke the axiomatic description of differential characters introduced by J. Simons and D. Sullivan in [SiSu07], and utilize the cohomological model constructed by M. J. Hopkins and I. M. Singer in [HS05]. Using the cochain complex $DC_s^\bullet(M, \mathbb{R}/\mathbb{Z})$ from [HS05], E. Lerman and A. Malkin proved in [LM08a] that the cocycle category of $DC_2^\bullet(\cdot, \mathbb{R}/\mathbb{Z})$ forms a stack over **Man**. This, then is an approach which conforms with the interpretation of the results of [GSW10] in terms of gauge-symmetry defects, worked out by I. Runkel and R. R. Suszek [RS09], in which the 2-stack structure on the bicategory of $U(1)$ -gerbes is implicitly used. In particular, the stack structure gives the necessary and sufficient conditions for a degree-2 differential character on an open cover of a manifold to descend. A candidate for the target space of a sigma model with groupoidal symmetry gauged was proposed by T. Strobl and R. R. Suszek in [SS25]. They introduced the concept of a *principaloid bundle* $\mathcal{P}$, which is essentially, for a fixed Lie groupoid $\mathcal{G}$, a $\text{Mor } \mathcal{G}$ fiber-bundle with the group $\mathbb{B}(\mathcal{G})$ of bisections of $\mathcal{G}$ serving as its structure group, the latter acting on the fibers via canonical left action. It was showed [SS25] that every principaloid bundle canonically induces another fiber-bundle over the same base, denoted by $\mathcal{F}$, whose typical fiber is $\text{Ob } \mathcal{G}$, and which admits the action of the group of gauge transformations of $\mathcal{P}$ (i.e., the vertical automorphisms of $\mathcal{P}$). Crucially, for the case of the action groupoid $G \times_\lambda M$, corresponding to a smooth action λ of a Lie group on a smooth manifold M , under a certain restriction of the structure group of a $G \times_\lambda M$ -principaloid bundle, the induced bundle is the bundle $P \times_\lambda M$ associated to the principal G -bundle P . This is known to be a model for the target space of a sigma model with the group symmetry gauged. Our approach rests on the interpretation of the gauging as an effective descent of a field theory to the space of orbits of the rigid symmetry [GSW13].

Following the universal Cartan construction of a homotopy quotient [Car50], we seek smooth realizations of the latter over the spacetime of the field theory of interest. Conceptually, one should consider the whole collection of such models—one for each isoclass of $\mathcal{P}$ —in order to capture all degrees of freedom of possible field configurations and gauge transformations.

Generalizing principal connections on principal G -bundles, one can define *compatible connections* on principaloid bundles, which descend to $\mathcal{F} \simeq \mathcal{P}/\mathcal{G}$ in the sense of Sh. Kobayashi and K. Nomizu [KN69]. Consequently, the covariant derivative is defined, and we make use of it to provide a gauging scheme for tensorial couplings. In particular, this schema may be applied to the metric term. However, it follows from the smooth structure of a Lie groupoid with a metric object manifold $(\text{Ob } \mathcal{G}, g)$ that whenever there exists a global bisection through each arrow of $\mathcal{G}$ (such groupoids are called $\mathbb{B}$ -complete), and if every bisection induces an isometry of $\text{Ob } \mathcal{G}$ through the target map of $\mathcal{G}$, the $\mathcal{G}$ -orbits on $\text{Ob } \mathcal{G}$ are 0-dimensional. This reveals the necessity of a reduction of the structure group of a principaloid bundle in most field-theoretic applications.

A non-tensorial coupling in a sigma model is determined by a differential character f on the target space. Hence, in order to construct the Dirac-Feynman amplitude for the gauged model, it suffices to give a gauge-invariant differential character on $\mathcal{F}$, under the constraint that for a trivial character, fixed by a $\mathbb{B}(\mathcal{G})$ -invariant global primitive of the character's curvature, the gauging scheme specialize to the minimal coupling scheme. The constructive results of [GSW10], put on display a key role in descent of multiplicative extensions of the integral de Rham class of curvatures of the differential characters to classes in the Bott-Schulman-Stasheff complex for the symmetry groupoid [BSS76]. Hence, when looking for non-trivial gaugings (for characters which *do not* descend directly) we seek to imitate the idea of ρ -augmented descent, worked out in [GSW10]. Given a multiplicative form ρ on $\mathcal{G}$ (see [CSS12] for references), we define ρ -pre-equivariant and ρ -equivariant differential characters. The ρ -pre-equivariant characters ones are those, whose first cohomological obstruction to equivariance vanishes. We conjecture, and prove for degree-2, that given a ρ -pre-equivariant differential character f and upon reducing the structure group of a principaloid bundle $\mathcal{P}$ to ρ -holonomic bisections [CSS12], we can define a differential character on the cover of the total space of $\mathcal{F}$ canonically induced by a trivializing cover of its base, in such a way that on each element of the cover the corresponding pullback of f is corrected by a trivial character, and so that the restrictions of local characters agree on double overlaps. Moreover, each of the local characters is gauge-invariant. However, this does not solve the problem of gauging just yet, since the differential character functor $\hat{H}^k(\cdot, \mathbb{R}/\mathbb{Z})$ fails to be a sheaf. We characterize the obstruction to the descent of the character on the cover to $\mathcal{F}$. We show that if the character descends, then the resultant Dirac-Feynman amplitude is gauge-invariant, and the scheme reduces to the minimal coupling for a trivial $\mathbb{B}(\mathcal{G})$ -invariant f . Finally, we conjecture, and prove for degree-2, that given a ρ -equivariant differential character f , that is, whenever a secondary cohomological obstruction vanishes, the local characters descend to $\mathcal{F}$ in a canonical way. We anticipate that the structure of proofs for arbitrary degree $n \in \mathbb{N}_{>0}$ of f is analogous, albeit significantly more complex. The author hopes to prove the general statements in his future works.

The structure of this work is the following: In Chapter 1, we introduce the category $\mathcal{G}\text{-Man}$ ($\text{Man-}\mathcal{G}$) of left (right) $\mathcal{G}$ -manifolds for a fixed Lie groupoid $\mathcal{G}$. In Chapter 2, we give an overview of the theory of principaloid and principal $\mathcal{G}$ -bundles, describing their topological and smooth structure in Section 2.1, and the theory of compatible connections in Section 2.2. In Chapter 3, we present the gauging scheme for $\mathbb{B}(\mathcal{G})$ -invariant tensorial couplings in sigma

models with groupoidal symmetries. Chapter 4 is concerned with the metric term for $\mathbb{B}$ -complete structure groupoids, and demonstrates the necessity to reduce the structure group of the principaloid bundle used for the gauging. In Chapter 5, we study the gauging of groupoidal symmetries in sigma models with non-tensorial topological couplings. In Section 5.1, we give an overview of Cheeger-Simons differential characters and prove the effective descent of degree-2 characters over **Man**. In Section 5.2, we define ρ -pre-equivariant and ρ -equivariant differential characters for a multiplicative form ρ , as well as the corresponding subgroup of $\mathbb{B}(\mathcal{G})$ which admits the gauging. Finally, in Section 5.3, we state the fundamental theorems and conjectures about the existence of the gauged sigma models, and give proofs in the case of degree-2. [GSW13]

Chapter 1

Lie Groupoids and Other Prerequisites

We assume every manifold and every map between manifolds to be smooth. In the case of infinite-dimensional manifolds (for example, the group of bisections $\mathbb{B}(\mathcal{G})$) refer to [KM97, Ch. VI] for the relevant notion of smoothness. We denote by $C_\bullet$ the graded abelian group of *smooth* singular chains, and by $Z_\bullet$, $B_\bullet$ its subgroups of smooth singular cycles and smooth singular boundaries, respectively. We make use of the fact that there is a chain-homotopy equivalence between the complex of smooth singular chains and the complex of continuous singular chains [Lee00, Ch. 18]. For prerequisites in homological algebra and algebraic topology, refer to Appendix A.

Definition 1.0.1. A *groupoid* is a category in which all morphisms are isomorphisms. A group can be identified with a groupoid $\mathbf{BG}$ with just one object. A morphism between two groupoids $\mathcal{G}$ and $\mathcal{H}$ is any functor $F : \mathcal{G} \rightarrow \mathcal{H}$. Therefore, groupoids form a full subcategory $\mathbf{Grpd}$ of $\mathbf{Cat}$.

In this work, we assume all groupoids to be small categories.

Example 1.0.2. Let S be a set. A *pair groupoid* $\mathbf{Pair}(S)$ comes with the set of objects $\mathbf{Ob} \mathbf{Pair}(S) = S$, the set of morphisms $\mathbf{Mor} \mathbf{Pair}(S) = S \times S$ such that for any $a, b, c \in S$, the arrow (a, b) connects b to a , and the composition law is the following:

$$(a, b) \circ (b, c) = (a, c). \quad (1.1)$$

The identities and inverses are now obvious. It is clear that in the category $\mathbf{Grpd}/X$, whose objects are groupoids $\mathcal{G}$ with $\mathbf{Ob} \mathcal{G} = X$ and morphisms are functors $F : \mathcal{G} \rightarrow \mathcal{H}$ satisfying $F|_{\mathbf{Ob} \mathcal{G}} = \mathrm{id}_{\mathbf{Ob} \mathcal{G}}$, the pair groupoid is the terminal object.

Definition 1.0.3. Let S be a set and let $\mathcal{G}$ be a groupoid. A fiberwise left (right) action of $\mathcal{G}$ on S is a pair (μ, Λ) where $\mu \in \mathbf{Set}(S, \mathbf{Ob} \mathcal{G})$ and $\Lambda : \mathcal{G} \rightarrow \mathbf{Set}$ ($\Lambda : \mathcal{G}^{\mathrm{op}} \rightarrow \mathbf{Set}$) satisfy

$$\forall A \in \mathbf{Ob} \mathcal{G} : \Lambda(A) = \mu^{-1}(A). \quad (1.2)$$

It is clear from the definition that whenever $\mathcal{G}(A, B) \neq \emptyset$, we have $\mu^{-1}(A) \simeq \mu^{-1}(B)$.

Example 1.0.4. Let S be a set and let $\mathbf{Pair}(S)$ be the associated pair groupoid. There is a canonical left action of $\mathbf{Pair}(S)$ on S given by $\mu = \mathrm{id}_S$ and $\Lambda(a, b)(b) = a$ for every $a, b \in S$. $\mathbf{Pair}(S)$ can also act canonically on $\mathbf{Mor} \mathbf{Pair}(S) = S \times S$. The corresponding map $\mu = \mathrm{pr}_1$, and for every $a, b, c \in S$ the functor Λ is defined by

$$\Lambda(a, b)(b, c) = (a, c). \quad (1.3)$$

In order to topologize or geometrize groupoids, we need an alternative way to present a category. For any small category $\mathcal{C}$, we define functions that assign to each morphism its domain and codomain. We call the first one the *source map* $s : \text{Mor } \mathcal{C} \rightarrow \text{Ob } \mathcal{C}$, and the second one the *target map* $t : \text{Mor } \mathcal{C} \rightarrow \text{Ob } \mathcal{C}$. Moreover, there is the multiplication map, which is the composition

$$m : \text{Mor } \mathcal{C} \times_t \text{Mor } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}, \quad (1.4)$$

and the identity map $\text{Id} : \text{Ob } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}$.

In a groupoid $\mathcal{G}$, we have one more structure map – the inverse

$$i : \text{Mor } \mathcal{G} \rightarrow \text{Mor } \mathcal{G}. \quad (1.5)$$

The structure maps satisfy the axioms of a groupoid. Therefore, we can think of a groupoid as a diagram in **Set** together with the axioms. In other words—a groupoid object in **Set**. We now have the following useful generalizations.

Definition 1.0.5. A groupoid $\mathcal{G}$ is called a *topological* groupoid if both $\text{Ob } \mathcal{G}$ and $\text{Mor } \mathcal{G}$ are topological spaces and all structure maps are continuous. In other words, a topological groupoid is a groupoid object in **Top**.

Definition 1.0.6. A groupoid $\mathcal{G}$ is called a *Lie* groupoid if both $\text{Ob } \mathcal{G}$ and $\text{Mor } \mathcal{G}$ are smooth manifolds, all structure maps are smooth, and s, t are submersions.

Similarly, we can put additional structure on groupoid morphisms. A morphism between topological/Lie groupoids $\mathcal{G}, \mathcal{H}$ is a functor $F : \mathcal{G} \rightarrow \mathcal{H}$ such that both $F|_{\text{Ob } \mathcal{G}}$ and $F|_{\text{Mor } \mathcal{G}}$ are continuous/smooth.

Example 1.0.7. An important example of a Lie groupoid is the *action groupoid* $G \times_\lambda M$ associated to a smooth action λ of a Lie group G on a manifold M . It is constructed as follows:

- $\text{Ob } G \times_\lambda M = M$,
- $\text{Mor } G \times_\lambda M = G \times M$,
- $s = \text{pr}_2, t = \lambda$,
- $(g, \lambda(h, m)) \circ (h, m) = (gh, m)$,
- $\text{Id}(m) = (e, m)$,
- $i(g, m) = (g^{-1}, \lambda(g, m))$.

Remark 1.0.8. Let **ActGrpd** denote the full subcategory of **Grpd** consisting of action groupoids. For an action groupoid $G \times_\lambda M$, there exists a canonical functor

$$F_\lambda : G \times_\lambda M \rightarrow BG : (g, m) \mapsto g. \quad (1.6)$$

In particular, we can think BG as of an action groupoid with $M = \text{pt}$. This way, we can define a category **ActGrpd**/ BG whose objects are functors F_λ , and whose morphisms are commutative diagrams

$$\begin{array}{ccc} G \times_{\lambda_1} M & \xrightarrow{\quad} & G \times_{\lambda_2} N \\ & \searrow F_{\lambda_1} & \swarrow F_{\lambda_2} \\ & BG. & \end{array} \quad (1.7)$$

in **ActGrpd**. It is an easy check to see that G -equivariant maps $\text{Hom}_G(M, N)$ are in a natural bijection with $(\mathbf{ActGrpd}/BG)(G \times_{\lambda_1} M, G \times_{\lambda_2} N)$.

Definition 1.0.9. We call a left action (μ, Λ) of a Lie groupoid (topological groupoid) $\mathcal{G}$ on a manifold M (topological space X) *smooth* (*continuous*), whenever μ is smooth (continuous) and

$$\text{Mor } \mathcal{G} \times_{s \times \mu} M \ni (g, m) \mapsto \Lambda(g)(m) \in M \quad (1.8)$$

is smooth (continuous). For a right smooth (continuous) action, replace s by t in 1.8. A $\mathcal{G}$ -manifold is a triple (M, μ, Λ) , where M is a manifold, and (μ, Λ) is a smooth action of $\mathcal{G}$ on M .

Definition 1.0.10. A map φ between two $\mathcal{G}$ -manifolds (M_1, μ_1, Λ_1) and (M_2, μ_2, Λ_2) is called $\mathcal{G}$ -equivariant if the collection $\{\varphi|_{\Lambda(a)}\}_{a \in \text{Ob } \mathcal{G}}$ forms a natural transformation between Λ_1 and Λ_2 . In particular,

$$\forall f \in \text{Mor } \mathcal{G} \ \forall x \in M_1 : \varphi(\Lambda_1(f)(x)) = \Lambda_2(f)(\varphi(x)). \quad (1.9)$$

Clearly, the above constructions compose a category $\mathcal{G}\text{-}\mathbf{Man}$ of left $\mathcal{G}$ -manifolds and $\mathcal{G}$ -equivariant maps. Analogously, we define the category $\mathbf{Man}\text{-}\mathcal{G}$ of right $\mathcal{G}$ -manifolds. We will write $\text{Hom}_{\mathcal{G}}$ to denote $\text{Hom}_{\mathbf{Man}\text{-}\mathcal{G}}$ or $\text{Hom}_{\mathcal{G}\text{-}\mathbf{Man}}$, depending on the context.

Chapter 2

Principaloid and Principal $\mathcal{G}$ -Bundles

2.1. Classification

Definition 2.1.1. A *fiber-bundle object* in the category $\mathbf{Man}\text{-}\mathcal{G}$ is a $\mathcal{G}$ -equivariant surjective submersion $\pi : E \twoheadrightarrow B$ such that there exists an open cover

$$\mathcal{O} = \bigsqcup_{i \in I} O_i \quad j_i : O_i \hookrightarrow B \in \text{Hom}_{\mathcal{G}}(O_i, B), \quad (2.1)$$

and the following commutative diagram in $\mathbf{Man}\text{-}\mathcal{G}$:

$$\begin{array}{ccccc} \mathcal{O} \times F & \xleftarrow{\simeq} & \mathcal{O} \times_B E & \xrightarrow{\quad} & E \\ & & \downarrow \lrcorner & & \downarrow \pi \\ & & \mathcal{O} & \xrightarrow{j} & B. \end{array} \quad (2.2)$$

We call the $\mathcal{G}$ -module F the *typical fiber* of π , and the isomorphism $\tau : \mathcal{O} \times_B E \rightarrow \mathcal{O} \times F$ a *local trivialization* of π .

Definition 2.1.2 ([SS25]). Given a Lie groupoid $\mathcal{G}$, a *principaloid $\mathcal{G}$ -bundle* is a fiber-bundle object $\pi_{\mathcal{P}} : \mathcal{P} \twoheadrightarrow B$ in the category $\mathbf{Man}\text{-}\mathcal{G}$ such that the typical fiber of $\pi_{\mathcal{P}}$ is $\text{Mor } \mathcal{G}$ with the canonical right action

$$\mu = s, \quad \Lambda_{\mathcal{G}}(f)(g) = m(g, f) = g \circ f, \quad (2.3)$$

and B is trivial as a right $\mathcal{G}$ -module. In particular, $\pi_{\mathcal{P}}$ is $\mathcal{G}$ -invariant. It also follows that, for any trivializing cover, the transition maps are $\mathcal{G}$ -equivariant. A morphism between two principaloid $\mathcal{G}$ -bundles $\pi_{\mathcal{P}_A} : \mathcal{P}_A \rightarrow B_A$, $A \in \{1, 2\}$ is a bundle morphism in the category of right $\mathcal{G}$ -spaces. Thus, principaloid $\mathcal{G}$ -bundles and their morphisms form a full subcategory of fiber-bundle objects in the category of right $\mathcal{G}$ -spaces. We will continue to refer to principaloid $\mathcal{G}$ -bundles and principaloid bundles interchangeably, whenever it does not cause confusion.

Definition 2.1.3. Given a Lie groupoid $\mathcal{G}$, we say that a section $\beta : \text{Ob } \mathcal{G} \rightarrow \text{Mor } \mathcal{G}$ of the source map s is a (global) *bisection* of $\mathcal{G}$ if $t \circ \beta \in \text{Diff}(\text{Ob } \mathcal{G})$.

Proposition 2.1.4. *The set of bisections of $\mathcal{G}$ forms a group, denoted by $\mathbb{B}(\mathcal{G})$, with the following structure maps:*

- $(\beta_1 \cdot \beta_2)(x) = \beta_1(t(\beta_2(x))) \circ \beta_2(x),$
- $\text{Id}(x) = \text{id}_x,$
- $\beta^{-1}(x) = \beta((t \circ \beta)^{-1}(x))^{-1}.$

Lemma 2.1.5. *The left (right) action (μ, Λ) of a groupoid $\mathcal{G}$ on a manifold M induces the left (right) action of $\mathbb{B}(\mathcal{G})$ on M . The left induced action is given by*

$$L_\beta(x) := \beta \triangleright x \equiv \Lambda(\beta(\mu(x)))(x), \quad (2.4)$$

and the right induced action is given by

$$R_\beta(x) := x \triangleleft \beta \equiv \Lambda\left((\beta^{-1}(\mu(x)))^{-1}\right)(x). \quad (2.5)$$

Proof. The right-hand side of 2.4 is well-defined, since

$$s(\beta(\mu(x))) = \mu(x). \quad (2.6)$$

Obviously, for any $x \in M$ we have $\text{Id} \triangleright x = x$. Take $\beta_1, \beta_2 \in \mathbb{B}(\mathcal{G})$. Using Proposition 2.1.4, we calculate

$$\begin{aligned} (\beta_1 \cdot \beta_2) \triangleright x &= \Lambda\left(\beta_1(t(\beta_2(\mu(x)))) \circ \beta_2(\mu(x))\right)(x) \\ &= \left(\Lambda\left(\beta_1(t(\beta_2(\mu(x))))\right) \circ \Lambda(\beta_2(\mu(x)))\right)(x) = \beta_1 \triangleright (\beta_2 \triangleright x). \end{aligned} \quad (2.7)$$

A similar proof for the right action can be found in [SS25, 2.20.]. $\square$

Remark 2.1.6. Consider the canonical left and right actions of $\mathcal{G}$ on $\text{Mor } \mathcal{G}$. They induce left and right actions of $\mathbb{B}(\mathcal{G})$, and the *adjoint* action, which we denote by C . The latter is defined by

$$\forall \beta \in \mathbb{B}(\mathcal{G}) \forall g \in \text{Mor } \mathcal{G} : C_\beta(g) = \beta \triangleright g \triangleleft \beta^{-1}. \quad (2.8)$$

By construction, it is a left action of $\mathbb{B}(\mathcal{G})$ on $\text{Mor } \mathcal{G}$.

Later in this chapter, we will state several results about principal and principaloid $\mathcal{G}$ -bundles without providing proofs. However, they can be found in the cited papers.

Theorem 2.1.7 ([SS25, 2.14.]). *Consider the canonical right action of $\mathcal{G}$ on $\text{Mor } \mathcal{G}$. Let $\text{Diff}_{\mathcal{G}}(\text{Mor } \mathcal{G}) := \text{End}_{\mathcal{G}}(\text{Mor } \mathcal{G}) \cap \text{Diff}(\text{Mor } \mathcal{G})$ denote the group of right $\mathcal{G}$ -equivariant diffeomorphisms of $\text{Mor } \mathcal{G}$. Then, the commutant $\text{Diff}_{\mathcal{G}}(\text{Mor } \mathcal{G})'$ in $\text{Diff}(\text{Mor } \mathcal{G})$ coincides with the group of left actions of $\mathbb{B}(\mathcal{G})$ induced by the canonical left action of $\mathcal{G}$ on $\text{Mor } \mathcal{G}$.*

Corollary 2.1.8. *Any principaloid $\mathcal{G}$ -bundle admits a model*

$$\mathcal{P} \simeq \bigsqcup_{i \in I} (O_i \times \text{Mor } \mathcal{G}) / \sim_{L_{\beta_{ij}}} \ni [(b, g, i)] \xrightarrow{\pi_{\mathcal{P}}} b,$$

expressed in terms of a transition 1-cocycle $(\beta_{ij} \in C^\infty(O_{ij}, \mathbb{B}(\mathcal{G})) : i, j \in I)$ associated with an open cover $\{O_i : i \in I\}$ of the base B .

Remark 2.1.9. Corollary 2.1.8 manifestly implies that $\mathcal{P}$ is a bundle associated to a principal $\mathbb{B}(\mathcal{G})$ -bundle. However, note that just as $\mathbb{B}(\mathcal{G}) \hookrightarrow (\text{Mor } \mathcal{G})^{\text{Ob } \mathcal{G}}$, the restriction of $L : \mathbb{B}(\mathcal{G}) \times \text{Mor } \mathcal{G} \rightarrow \text{Mor } \mathcal{G}$ to the identity submanifold $\text{Id}(\text{Ob } \mathcal{G})$ gives the inclusion

$$\mathbb{B}(\mathcal{G}) \hookrightarrow (\text{Mor } \mathcal{G})^{\text{Id}(\text{Ob } \mathcal{G})}. \quad (2.9)$$

This is a highly non-generic property for associated bundles, which implies one can recover the transition cocycle β_{ij} of the original principal bundle from its realization $L_{\beta_{ij}}$.

Definition 2.1.10 ([MM03]). Given a Lie groupoid $\mathcal{G}$, a right *principal $\mathcal{G}$ -bundle* is a $\mathcal{G}$ -equivariant surjective submersion between two right $\mathcal{G}$ -manifolds $\pi : E \rightarrow B$ such that

- B is trivial as a $\mathcal{G}$ -manifold,
- the map $E \times_{\mu \times_t \text{Mor } \mathcal{G}} E \ni (e, f) \mapsto (e, \Lambda(f)(e)) \in E \times_{\pi \times \pi} E$ is a diffeomorphism, where (μ, Λ) is the right $\mathcal{G}$ action on E .

We say that the action (μ, Λ) on E is *principal*.

Remark 2.1.11. A principal $\mathcal{G}$ -bundle is not necessarily a fiber-bundle. Every such bundle is locally trivial (in the sense of) [MM03, 5.7] with the local model being the *unital bundle*

$$t : \text{Mor } \mathcal{G} \rightarrow \text{Ob } \mathcal{G}. \quad (2.10)$$

The t -fibers might well be non-diffeomorphic. For example, consider an action Lie groupoid $G \times_{\lambda} M$ and its full subgroupoid obtained by restricting the object manifold M to its proper submanifold M' such that some G -orbits are fully contained in M' , and some only partially so. The t -fibers are isomorphic to submanifolds in G whose action on points in M reproduces intersections of orbits with M' .

Theorem 2.1.12 ([SS25, 2.16.]). *Every principaloid $\mathcal{G}$ -bundle $\pi_{\mathcal{P}} : \mathcal{P} \rightarrow B$ canonically induces a fiber-bundle $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow B$ with a model*

$$\mathcal{F} \simeq \bigsqcup_{i \in I} (O_i \times \text{Ob } \mathcal{G}) / \sim_{t \circ \beta_{ij}},$$

written in terms of a trivializing cover $\{O_i\}_{i \in I}$ of B for $\mathcal{P}$, and the corresponding transition 1-cocycle $(\beta_{ij} : i, j \in I)$ of $\mathcal{P}$. Moreover, there is a bundle map

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathcal{D}} & \mathcal{F} \\ \pi_{\mathcal{P}} \searrow & & \swarrow \pi_{\mathcal{F}} \\ & B, & \end{array} \quad (2.11)$$

which is locally modeled on $t : \text{Mor } \mathcal{G} \rightarrow \text{Ob } \mathcal{G}$. Importantly, the map $\mathcal{P} \xrightarrow{\mathcal{D}} \mathcal{F}$ carries a canonical structure of a right principal $\mathcal{G}$ -bundle object in the category of fiber-bundles over B .

Remark 2.1.13. It should be noted that, by The Godement Criterion (Theorem A.0.25), there exists a smooth structure on the space of orbits

$$\mathcal{P}/\mathcal{G} := \left\{ \Lambda(t^{-1}(\mu(p)))(p) : p \in \mathcal{P} \right\}, \quad (2.12)$$

with respect to which the quotient map

$$\mathcal{P} \ni p \mapsto \Lambda(t^{-1}(\mu(p)))(p) \in \mathcal{P}/\mathcal{G} \quad (2.13)$$

is a submersion. Clearly, the bundle $\mathcal{F}$ is diffeomorphic to $\mathcal{P}/\mathcal{G}$.

Example 2.1.14. If $\mathcal{G} \simeq BG$ is a groupoid over a point, then $\mathbb{B}(\mathcal{G}) \simeq G$, and a principaloid $\mathcal{G}$ -bundle reduces to a standard principal G -bundle. Similarly, as $\mathcal{F} \simeq B$, so does the induced principal groupoid bundle.

Example 2.1.15. Let $\mathcal{G} = G \times_\lambda M$ be the action groupoid (Example 1.0.7) associated with an action λ of a Lie group G on a manifold M . Consider the embedding $G \xrightarrow{\iota} \mathbb{B}(\mathcal{G}) : g \mapsto \beta_g$, where $\beta_g(m) = (g, m)$. Let $\mathbb{B}_0 := \iota(G)$, and let $\mathcal{P}_0$ be a principaloid $\mathcal{G}$ -bundle with the transition cocycle taking values in $\mathbb{B}_0 \subseteq \mathbb{B}(\mathcal{G})$. Clearly, the cocycle factors as $\beta_{ij} = \iota \circ \beta'_{ij}$, where $\beta'_{ij} : O_{ij} \rightarrow G$. Then $\mathcal{P}_0 \simeq P \times M$, where P is a principal G -bundle with the transition cocycle β'_{ij} . Importantly, the induced fiber-bundle satisfies $\mathcal{F} \simeq P \times_\lambda M$, with $\mathcal{D} : P \times M \rightarrow P \times_\lambda M$ being the quotient map.

Keeping in mind that we are seeking a candidate for the set of matter fields of a gauged field theory with groupoidal symmetry, we note that Example 2.1.15 suggests that going from a group to groupoidal symmetry, one should replace $\Gamma(P \times_\lambda M)$ by $\Gamma(\mathcal{F})$. A deeper understanding of this fact comes from a careful study of Cartan's construction of the homotopy quotient of a group/groupoid action (see, e.g. [Tu20]), which, however, lies beyond the scope of the present thesis.

2.2. Connective structures

Definition 2.2.1 ([SS25]). A *compatible connection* on a principaloid $\mathcal{G}$ -bundle $\mathcal{P}$ is an Ehresmann connection $T\mathcal{P} \simeq V\mathcal{P} \oplus H\mathcal{P}$, with the vertical subbundle $V\mathcal{P} := \ker T\pi_{\mathcal{P}}$, and the horizontal subbundle $H\mathcal{P}$ satisfying $H\mathcal{P} \subseteq \ker T\mu$ and $\mathcal{G}$ -invariant. The invariance means that, for any $g \in \text{Mor } \mathcal{G}$ and any $p \in \mu^{-1}(t(g))$, we have

$$T_p(\Lambda(g))(H_p\mathcal{P}) = H_{\Lambda(g)(p)}\mathcal{P}. \quad (2.14)$$

Note that as $T\Lambda(g) : T(\mu^{-1}(t(g))) \rightarrow T(\mu^{-1}(s(g)))$ and $T\mu^{-1}(b) = \ker T\mu|_{\mu^{-1}(b)}$, the restriction of $H\mathcal{P}$ to $\ker T\mu$ is necessary. Since μ is modeled on s in a local trivialization $\mathcal{P}\tau_i$, the kernel $\ker T\mu$ becomes $\text{pr}_1^*TO_i \oplus \text{pr}_2^*\ker Ts$, which makes the restriction meaningful from the point of view of the definition of a horizontal distribution.

Definition 2.2.2 ([SS25]). Let $\mathcal{P}$ be a principaloid $\mathcal{G}$ -bundle. A *compatible connection 1-form* on $\mathcal{P}$ is a bundle morphism $\mathcal{A} : T\mathcal{P} \rightarrow V\mathcal{P}$ satisfying:

- $\mathcal{A}|_{V\mathcal{P}} = \text{id}_{V\mathcal{P}}$,
- $T\mu \circ \mathcal{A} = T\mu$,
- $\forall g \in \text{Mor } \mathcal{G} : \mathcal{A} \circ T(\Lambda(g)) = T(\Lambda(g)) \circ \mathcal{A}$.

Proposition 2.2.3 ([SS25, 3.4.]). *Compatible connections on a principaloid $\mathcal{G}$ -bundle $\mathcal{P}$ are in a one-to-one correspondence with compatible connection 1-forms on $\mathcal{P}$.*

Corollary 2.2.4. *On every principaloid $\mathcal{G}$ -bundle, there exists a compatible connection. Given an open cover $\{O_i : i \in I\}$ of B , the corresponding local trivialisations $\{\mathcal{P}\tau_i : i \in I\}$, and a smooth partition of unity $\{h_i : i \in I\}$, the corresponding compatible connection 1-form is given by*

$$\mathcal{A} := \sum_{i \in I} (\pi_{\mathcal{P}}^* h_i) \mathcal{A}_i, \quad (2.15)$$

where

$$\mathcal{A}_i := T\mathcal{P}\tau_i^{-1} \circ j_2 \circ \text{pr}_2 \circ T\mathcal{P}\tau_i, \quad (2.16)$$

with $\text{pr}_2 : T(O_i \times \text{Mor } \mathcal{G}) \rightarrow T(\text{Mor } \mathcal{G})$ and $j_2 : T(\text{Mor } \mathcal{G}) \hookrightarrow T(O_i \times \text{Mor } \mathcal{G}) \simeq TO_i \oplus T(\text{Mor } \mathcal{G})$.

Theorem 2.2.5 ([SS25, 3.7.]). *A compatible connection on a principaloid $\mathcal{G}$ -bundle $\mathcal{P}$ canonically induces an Ehresmann connection on the induced bundle $\mathcal{F}$. It is defined as*

$$\mathrm{H}\mathcal{F} := \mathrm{TD}(\mathrm{H}\mathcal{P}). \quad (2.17)$$

Definition 2.2.6 ([SS25]). The induced connection on $\mathcal{F}$ is uniquely determined by a bundle morphism $\mathcal{A}_{\mathcal{F}} : \mathrm{T}\mathcal{F} \rightarrow \mathrm{V}\mathcal{F}$ satisfying $\mathrm{TD}(\mathrm{H}\mathcal{P}) = \ker \mathcal{A}_{\mathcal{F}}$. We call this morphism the *shadow connection 1-form*.

Theorem 2.2.7 ([SS25, 4.18.]). *There is a canonical group homomorphism*

$$\mathcal{F}_* : \mathrm{Aut}(\mathcal{P}) \rightarrow \mathrm{Aut}(\mathcal{F}) \quad (2.18)$$

satisfying $\forall \Phi \in \mathrm{Aut}(\mathcal{P}) : \mathcal{D} \circ \Phi = \mathcal{F}_(\Phi) \circ \mathcal{D}$.*

Definition 2.2.8. Vertical automorphisms (i.e., those covering id_B) $\mathrm{Aut}_{\mathrm{vert}}(\mathcal{P})$ form a subgroup of $\mathrm{Aut}(\mathcal{P})$ that we call the *group of gauge transformations* and denote by $\mathrm{Gauge}(\mathcal{P})$.

Definition 2.2.9 ([SS25]). Let $\mathcal{P}$ be a principaloid bundle with compatible connection 1-form $\mathcal{A}$. Given $\Phi \in \mathrm{Aut}(\mathcal{P})$, we denote

$$\mathcal{A}^{\Phi} := \mathrm{T}\Phi \circ \mathcal{A} \circ (\mathrm{T}\Phi)^{-1}. \quad (2.19)$$

Let $\mathbf{core}(\mathbf{PBun}^{\nabla}(B))$ be the maximal groupoid inside the category of principaloid bundles with compatible connection over a fixed base B . Then, the maps

$$\begin{array}{ccc} (\mathcal{P}_1, \mathcal{A}) & \xrightarrow{\Phi} & (\mathcal{P}_2, \mathcal{A}^{\Phi}) \\ \downarrow \pi_{\mathcal{P}_1} & & \downarrow \pi_{\mathcal{P}_2} \\ B & \xrightarrow{\psi} & B \end{array} \quad (2.20)$$

form morphisms in $\mathbf{core}(\mathbf{PBun}^{\nabla}(B))$. Whenever $\Phi \in \mathrm{Gauge}(\mathcal{P})$, the morphism $\mathcal{A}^{\Phi}$ is called the *gauge transform* of $\mathcal{A}$ induced by Φ .

Thus, Theorem 2.2.7 provides us with the mechanism of induction of distinguished (gauge) automorphisms of the shadow bundle (and so, also of gauge transformations of its sections) from those of $\mathcal{P}$.

Definition 2.2.10 ([SS25]). The *covariant derivative* of a section $\varphi \in \Gamma(\mathcal{F})$ relative to a compatible connection 1-form $\mathcal{A}$ on $\mathcal{P}$ is the $\mathbb{R}$ -linear mapping

$$\nabla^{\mathcal{A}}\varphi : \Gamma(\mathrm{T}\Sigma) \rightarrow \Gamma(\mathrm{V}\mathcal{F}) : \mathcal{V} \mapsto (\mathcal{A}_{\mathcal{F}} \circ \mathrm{T}\varphi)(\mathcal{V}) =: \nabla_{\mathcal{V}}^{\mathcal{A}}\varphi. \quad (2.21)$$

Proposition 2.2.11 ([SS25, 4.34.]). *For a section $\varphi \in \Gamma(\mathcal{F})$ and a gauge transformation $\Phi \in \mathrm{Gauge}(\mathcal{P})$, let φ^{Φ} denote the corresponding gauge transform of φ :*

$$\varphi^{\Phi} := \mathcal{F}_*(\Phi) \circ \varphi. \quad (2.22)$$

The covariant derivative of φ relative to a compatible connection 1-form $\mathcal{A}$ on $\mathcal{P}$ transforms covariantly as

$$\nabla^{\mathcal{A}^{\Phi}}(\varphi^{\Phi}) = \mathrm{T}(\mathcal{F}_*(\Phi)) \circ \nabla^{\mathcal{A}}\varphi. \quad (2.23)$$

Proposition 2.2.12 ([SS25]). A gauge transformation $\Phi \in \text{Gauge}(\mathcal{P})$ is locally presented by a family of smooth maps

$$\gamma_i : O_i \rightarrow \mathbb{B}(\mathcal{G}), \quad i \in I \quad (2.24)$$

as

$$\Phi|_{\pi_{\mathcal{P}}^{-1}(O_i)} : \pi_{\mathcal{P}}^{-1}(O_i) \xrightarrow{\cong} \pi_{\mathcal{P}}^{-1}(O_i) : \mathcal{P}\tau_i^{-1}(b, g) \mapsto \mathcal{P}\tau_i^{-1}(b, L_{\gamma_i(b)}(g)). \quad (2.25)$$

The maps are subject to the following gluing conditions over O_{ij} :

$$\gamma_i|_{O_{ij}} = (\beta_{ij} \cdot \gamma_j \cdot \beta_{ji})|_{O_{ij}}. \quad (2.26)$$

Moreover, there is a one-to-one correspondence between such families and gauge transformations.

Proof. The existence of (γ_i) that represent Φ locally follows from Theorem 2.1.7 and the fact that every gauge transformation is right $\mathcal{G}$ -equivariant. The gluing condition 2.26 translates to the ordinary gluing condition of the $\Phi|_{\pi_{\mathcal{P}}^{-1}(O_i)}$'s on double intersections $\pi_{\mathcal{P}}^{-1}(O_{ij})$. $\square$

Definition 2.2.13. Let M be a smooth manifold. A real *Lie algebroid* over M (of rank $N \in \mathbb{N}_+$) is a tuple $(\mathcal{E}, M, \mathbb{R}^N, \pi_{\mathcal{E}}, \rho_{\mathcal{E}}, [\cdot, \cdot]_{\mathcal{E}})$ composed of

- a vector bundle $\pi_{\mathcal{E}} : \mathcal{E} \rightarrow M$ with typical fiber $\mathbb{R}^N$,
- a vector bundle morphism

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\rho_{\mathcal{E}}} & TM \\ \pi_{\mathcal{E}} \searrow & & \swarrow \pi_{TM} \\ & M, & \end{array} \quad (2.27)$$

called the *anchor*,

- a Lie bracket $[\cdot, \cdot]_{\mathcal{E}} : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$ satisfying the *Leibniz rule*

$$\forall e_1, e_2 \in \Gamma(\mathcal{E}) \forall f \in C^\infty(M, \mathbb{R}) : [e_1, f \triangleright e_2]_{\mathcal{E}} = f \triangleright [e_1, e_2]_{\mathcal{E}} + \rho_{\mathcal{E}}(e_1)(f) \triangleright e_2, \quad (2.28)$$

where $\triangleright$ is the canonical action of $C^\infty(M)$ on $\Gamma(\mathcal{E})$.

Example 2.2.14. The *tangent Lie algebroid* of M is the canonical structure of a Lie algebroid on the tangent bundle $\pi_{TM} : TM \rightarrow M$, that is $(TM, M, \mathbb{R}^{\dim M}, \pi_{TM}, \text{id}_{TM}, [\cdot, \cdot]_{TM})$.

Example 2.2.15. Associated with an action groupoid $G \times_{\lambda} M$ is the corresponding *action algebroid*:

$$\begin{array}{ccc} \mathfrak{g} \times M & \xrightarrow{\lambda_*} & TM \\ \text{pr}_2 \searrow & & \swarrow \pi_{TM} \\ & M, & \end{array} \quad (2.29)$$

where λ_* is the fundamental vector field of λ . The Lie bracket is uniquely determined by the Lie bracket on $\mathfrak{g}$ and the Leibniz rule.

Definition 2.2.16. The set of *right-invariant vector fields* on a Lie groupoid $\mathcal{G}$ is defined as

$$\Gamma(\mathrm{T}(\mathrm{Mor} \mathcal{G}))_{\mathrm{R}} := \{\mathcal{V} \in \Gamma(\ker \mathrm{T}s) : \forall g \in \mathrm{Mor} \mathcal{G} \forall h \in s^{-1}(t(g)) : \mathrm{T}_h r_g(\mathcal{V}(h)) = \mathcal{V}(h \circ g)\}. \quad (2.30)$$

The (*right*) *tangent algebroid* of $\mathcal{G}$ is the pullback bundle

$$\pi_{E_{\mathrm{R}}} : E_{\mathrm{R}} := \mathrm{Id}^* \ker \mathrm{T}s = \mathrm{Ob} \mathcal{G} \times_{\mathrm{Id} \times \pi_{\mathrm{T}\mathcal{G}}} \ker \mathrm{T}s \xrightarrow{\mathrm{pr}_1} \mathrm{Ob} \mathcal{G} \quad (2.31)$$

with the anchor

$$\rho_{E_{\mathrm{R}}} \equiv \mathrm{T}t \circ \mathrm{pr}_2, \quad (2.32)$$

and the Lie bracket

$$[\cdot, \cdot]_{E_{\mathrm{R}}} \equiv \iota_{\mathrm{R}}^{-1} \circ [\cdot, \cdot]_{\mathrm{T}\mathcal{G}} \circ (\iota_{\mathrm{R}} \times \iota_{\mathrm{R}}) \quad (2.33)$$

induced by the canonical $\mathbb{R}$ -linear isomorphism

$$\iota_{\mathrm{R}} : \Gamma(E_{\mathrm{R}}) \rightarrow \Gamma(\mathrm{T}(\mathrm{Mor} \mathcal{G}))_{\mathrm{R}} : (\mathrm{id}_{\mathrm{Ob} \mathcal{G}}, \sigma) \mapsto (g \mapsto \mathrm{T}_{\mathrm{Id}(t(g))} r_g(\sigma(t(g)))), \quad (2.34)$$

with the inverse

$$\iota_{\mathrm{R}}^{-1}(\mathcal{V}) = (x \mapsto (x, \mathcal{V}(\mathrm{Id}(x))). \quad (2.35)$$

Remark 2.2.17. The adjoint action of the group of bisections $\mathbb{B}(\mathcal{G})$ on $\mathrm{Mor} \mathcal{G}$, defined in Remark 2.1.6, induces the map $\beta \mapsto \mathcal{C}_{\beta}$, where

$$\mathcal{C}_{\beta} := \mathrm{id}_M \times \mathrm{TC}_{\beta}|_{\ker \mathrm{T}s}|_{\mathrm{Id}(M)}. \quad (2.36)$$

The latter is well-defined since $\mathrm{TC}_{\beta}(\ker \mathrm{T}s) \subseteq \ker \mathrm{T}s$ for any $\beta \in \mathbb{B}(\mathcal{G})$, and due to the identity

$$\mathcal{C}_{\beta} \circ \mathrm{Id} = \mathrm{Id} \circ t \circ \beta. \quad (2.37)$$

The map 2.36 clearly induces the map on sections

$$\mathcal{C}_{\beta} : \alpha \mapsto \mathrm{Id}^* \mathrm{TC}_{\beta}(\alpha) \in \mathrm{End}_{\mathbb{R}}(\Gamma(E_{\mathrm{R}})), \quad (2.38)$$

which we should denote by the same symbol.

Definition 2.2.18 ([FS14]). The *right-invariant Maurer-Cartan form* on $\mathcal{G}$ is the bundle morphism

$$\begin{array}{ccc} \ker \mathrm{T}s & \xrightarrow{\theta_{\mathrm{R}}} & E_{\mathrm{R}} \\ \pi_{\mathrm{T}(\mathrm{Mor} \mathcal{G})}|_{\ker \mathrm{T}s} \downarrow & & \downarrow \pi_{E_{\mathrm{R}}} \\ \mathrm{Mor} \mathcal{G} & \xrightarrow{t} & \mathrm{Ob} \mathcal{G}, \end{array} \quad (2.39)$$

defined by

$$\forall g \in \mathrm{Mor} \mathcal{G} \forall w \in (\ker \mathrm{T}s)_g : \theta_{\mathrm{R}}(w) \equiv (t(g), \mathrm{T}_g r_{g^{-1}}(w)). \quad (2.40)$$

Equivalently, we can think of the right-invariant Maurer-Cartan form as a section of

$$(\ker \mathrm{T}s)^* \otimes t^* E_{\mathrm{R}} \rightarrow \mathrm{Mor} \mathcal{G}. \quad (2.41)$$

Definition 2.2.19 ([SS25]). Let $\pi_{\mathcal{P}} : \mathcal{P} \rightarrow B$ be a principaloid bundle, and let $\mathcal{O} := \{O_i\}_{i \in I}$ be a trivializing cover of B . The *local connection data* associated with $\mathcal{O}$ is a collection of sections $A_i \in \Gamma(\mathrm{pr}_1^* \mathrm{T}^* O_i \otimes \mathrm{pr}_2^* E_R)$ over $O_i \times \mathrm{Ob} \mathcal{G}$, subject to the following gluing conditions over $O_{ij} \times \mathrm{Ob} \mathcal{G} := (O_i \cap O_j) \times \mathrm{Ob} \mathcal{G}$:

$$A_i(b, (t \circ \beta_{ij}(b))(x)) = \mathrm{T}_{\mathrm{Id}(x)} C_{\beta_{ij}(b)} \circ A_j(b, x) - \theta_R \circ \mathrm{T}_b(\mathrm{ev}_x \circ \beta_{ij}), \quad (2.42)$$

where (β_{ij}) is the transition 1-cocycle of $\mathcal{P}$, and

$$\mathrm{ev} : \mathbb{B}(\mathcal{G}) \times \mathrm{Ob} \mathcal{G} \rightarrow \mathrm{Mor} \mathcal{G} : (\beta, x) \mapsto \beta(x). \quad (2.43)$$

Theorem 2.2.20 ([SS25, 3.15.]). *On $\mathcal{P}$, connection 1-forms are in a one-to-one correspondence with local connection data. The relation is given by*

$$((\mathcal{P}\tau_i^{-1})^* \mathcal{A})(b, g) = \mathrm{id}_{\mathrm{T}\mathcal{G}}|_g + \mathrm{T}_{\mathrm{Id}(t(g))} r_g \circ A_i(b, t(g)), \quad (b, g) \in O_i \times \mathcal{G}. \quad (2.44)$$

Proposition 2.2.21 ([SS25, 4.17.]). *Let $\mathcal{A}$ be a connection 1-form on a principaloid bundle $\mathcal{P}$. Given a trivializing cover $\{O_i\}_{i \in I}$ of the base B , let $\{A_i\}_{i \in I}$ be the local connection data for $\mathcal{A}$ associated to $\mathcal{O}$. Further, for a gauge transformation $\Phi \in \mathrm{Gauge}(\mathcal{P})$, let $\{\gamma_i\}_{i \in I}$ be its local presentation as in Proposition 2.2.12, and denote by $\{A_i^\Phi\}_{i \in I}$ the local connection data for $\mathcal{A}^\Phi$. Then, for each $i \in I$ and every $(b, x) \in O_i \times \mathrm{Ob} \mathcal{G}$ the following equality holds:*

$$A_i^\Phi(b, (t \circ \gamma_i(b))(x)) = \mathrm{T}_{\mathrm{Id}(x)} C_{\gamma_i(b)} \circ A_i(b, x) - \theta_R \circ \mathrm{T}_b(\mathrm{ev}_x \circ \beta_{ij}). \quad (2.45)$$

Chapter 3

Gauging Tensorial Sigma Models

Definition 3.0.1. An n -dimensional (closed) sigma model is a triple $(\Sigma, (M, g, H), \mathcal{M}_{\text{DF}})$, where:

- Σ is a compact, closed and oriented manifold with $\dim \Sigma = n$, called the *worldvolume*,
- (M, g, H) is a Riemannian manifold (M, g) together with a closed differential $(n+1)$ -form H with integer periods¹; we call M the *target space*,
- $\mathcal{M}_{\text{DF}}$ is a function, called the *Dirac-Feynman amplitude* (later the DF-amplitude), from $\Gamma(\Sigma \times M) := \Gamma(\Sigma \times M \rightarrow \Sigma)$, whose elements are called *fields*, into $U(1) \simeq \mathbb{R}/\mathbb{Z}$. The DF-amplitude is defined below.

For any $\varphi \in \Gamma(\Sigma \times M)$ set $\phi := \text{pr}_2 \circ \varphi$. The general form of the DF-amplitude is

$$\mathcal{M}_{\text{DF}}(\varphi) = \exp\left(i \int_{\Sigma} \sqrt{\det(\phi^*g)}\right) \cdot \chi_H(\varphi), \quad (3.1)$$

where χ_H is determined by the properties:

- independence of the metric g on M ,
- additivity on worldvolumes²,
- the variational contribution³:

$$\frac{1}{i} \delta \log \chi_H(\varphi) = \phi^*(\iota_{\delta\phi} H). \quad (3.2)$$

Thus, the sigma model is a theory of minimal embeddings $\Sigma \rightarrow M$ perturbed by Lorentz-type forces sourced by H from the topological term. Note that we adopt the Dirac's quantum-mechanical interpretation of the classical action functional, which leads us to replace S by $\exp(iS)$ (for $\hbar = 1$) and proceed to generalize the amplitude through imposition of the above properties – after Alvarez and Gawędzki [Alv85][Gaw88]. The definition of the sigma model covers the natural prototypes: the theory of geodesic motions perturbed by Lorentz forces ($n = 1$), and the theory of minimal surfaces perturbed by a background field of a 3-form ($n = 2$). In what follows, we will discuss a concrete definition of χ_H .

¹Definition 5.1.1

²Equivalent to the postulate of non-interaction of separate trajectories in the target space.

³Here, $\delta \log$ is the logarithmic variation of χ_H induced by a variation $\delta\varphi$ of the embedding field. We model the latter by a smooth vector field in the neighborhood of $\varphi(\Sigma)$.

In our approach to the gauging of a rigid symmetry of a field theory, assume the target space M to be the object manifold $\text{Ob } \mathcal{G}$ of a Lie groupoid $\mathcal{G}$. The target space admits the action

$$t \circ - : \mathbb{B}(\mathcal{G}) \rightarrow \text{Diff}(M), \quad (3.3)$$

which induces the following action on fields:

$$\forall \varphi \in \Gamma(\Sigma \times M) \forall \beta \in \mathbb{B}(\mathcal{G}) : \beta \triangleright \varphi := (\text{id}_\Sigma \times (t \circ \beta)) \circ \varphi. \quad (3.4)$$

We assume the DF-amplitude to be $\mathbb{B}(\mathcal{G})$ -invariant:

$$\forall \varphi \in \Gamma(\Sigma \times M) \forall \beta \in \mathbb{B}(\mathcal{G}) : \mathcal{M}_{\text{DF}}(\beta \triangleright \varphi) = \mathcal{M}_{\text{DF}}(\varphi). \quad (3.5)$$

We then call the action $\phi \mapsto \beta \triangleright \phi$ a *rigid symmetry* of the sigma model.

An important class of examples of sigma models is given by those with $H = dB$, for B and g both $\mathbb{B}(\mathcal{G})$ -invariant. In this case, we take

$$\chi_H(\varphi) = \exp\left(i \int_\Sigma \phi^* B\right). \quad (3.6)$$

In order to gauge the rigid symmetry of such a sigma model, we choose a principaloid $\mathcal{G}$ -bundle $\mathcal{P}$ over Σ with a trivializing open cover $\{O_i\}_{i \in I}$. We also pick a *smooth tessellation* of Σ , that is, a decomposition

$$\Sigma = \bigcup_{i \in I_0} \Sigma_i \quad (3.7)$$

into manifolds with boundary such that $D_{ij} := \Sigma_i \cap \Sigma_j$ are codimension-1 manifolds with boundary (whenever non-empty). Moreover, we want the tessellation to be *subordinate* to the open cover $\{O_i\}_{i \in I}$:

$$\forall i \in I_0 \exists j \in I : \Sigma_i \subseteq O_j. \quad (3.8)$$

By compactness of Σ , one can assume finiteness of I_0 . Motivated by Example 2.1.15 and the subsequent remark, we wish to construct a gauge-invariant DF-amplitude $\tilde{\mathcal{M}}_{\text{DF}} : \Gamma(\mathcal{F}) \rightarrow U(1)$. Let $\{\mathcal{F}\tau_i\}$ be the local trivializations of $\mathcal{F}$ over the same cover $\{O_i\}$. We set

$$\tilde{S}_g(\varphi) := \sum_{i \in I_0} \int_{\Sigma_i} \sqrt{\det(g \circ \Psi_i^2)}, \quad (3.9)$$

and

$$\tilde{S}_{\chi_H}(\varphi) := \sum_{i \in I_0} \int_{\Sigma_i} B \circ \Psi_i^n, \quad (3.10)$$

where

$$\Psi_i^k := \left(T(\text{pr}_2 \circ \mathcal{F}\tau_i) \nabla^{\mathcal{A}_i} \varphi_i|_{O_i} \right)^{\otimes k} : \Gamma(T\Sigma)^{\otimes k} \rightarrow TM^{\otimes k}, \quad k \in \{2, n\}. \quad (3.11)$$

Then, we define

$$\tilde{\mathcal{M}}_{\text{DF}}(\varphi) := \tilde{S}_g + \tilde{S}_{\chi_H} \mod \mathbb{Z}. \quad (3.12)$$

Proposition 3.0.2. *The theory governed by the DF-amplitude $\tilde{\mathcal{M}}_{\text{DF}}$ does not depend on the choice of the tessellation subordinate to the open cover $\{O_i\}_{i \in I}$. Moreover, it is invariant with respect to gauge transformations $(\varphi, \mathcal{A}) \mapsto (\varphi^\Phi, \mathcal{A}^\Phi)$.*

Proof. Let $\mathcal{T} \in \{g, B\}$. The gauge invariance of the DF-amplitude follows from Proposition 2.2.11, Proposition 2.2.12, the verticality of $\mathcal{F}(\Phi)$, and the $\mathbb{B}(\mathcal{G})$ -invariance of $\mathcal{T}$. Indeed, we observe that

$$\begin{aligned} T(\text{pr}_2 \circ \mathcal{F}\tau_i) \nabla^{\mathcal{A}^\Phi} \varphi^\Phi|_{O_i} &= T(\text{pr}_2 \circ \mathcal{F}\tau_i) \circ T(\mathcal{F}_*(\Phi)) \circ \nabla^{\mathcal{A}} \varphi|_{O_i} \\ &= T(t \circ \gamma_i) \circ T(\text{pr}_2 \circ \mathcal{F}\tau_i) \circ \nabla^{\mathcal{A}} \varphi|_{O_i}. \end{aligned} \quad (3.13)$$

To see the invariance under a change of tessellation, note that every such tessellation can be realized as a fundamental cycle of Σ [Hat02, 3.3]. Since the transition maps can be expressed in terms of a transition 1-cocycle, as

$$\mathcal{F}\tau_i \circ (\mathcal{F}\tau_j)^{-1}(b, x) = (b, (t \circ \beta_{ij})(b)(x)), \quad (3.14)$$

and again, by the $\mathbb{B}(\mathcal{G})$ -invariance of $\mathcal{T}$, the pullbacks $(\mathcal{F}\tau_i)^* \text{pr}_2^* \mathcal{T}$ glue to a global section over $\mathcal{F}$. Moreover, by Proposition 2.2.11, and by verticality of $\mathcal{F}_*(\Phi)$, we know that on intersections $\pi_{\mathcal{F}}^{-1}(O_i) \cap \pi_{\mathcal{F}}^{-1}(O_j)$ the respective restrictions of $T(\text{pr}_2 \circ \mathcal{F}\tau_i) \nabla^{\mathcal{A}} \varphi|_{O_i}$ and $T(\text{pr}_2 \circ \mathcal{F}\tau_j) \nabla^{\mathcal{A}} \varphi|_{O_j}$ differ by a left action by an element of the transition 1-cocycle. Consequently, the expressions 3.9 and 3.10 are integrals over a fundamental cycle of Σ . Since $\dim \Sigma = n$, the n -forms are closed, so 3.9, 3.10 depend only on the fundamental class of Σ . $\square$

As can be seen from the proof of Proposition 3.0.2, the language of smooth singular chains is not only more general and less rigid, but also arguably simpler than the one coming from smooth tessellations. For this reason, we continue to adapt the smooth singular approach to the theory of the gauging of sigma-model symmetries.

Chapter 4

Notes on The Metric Term

Let us consider the metric term of a sigma model. In particular, let $g \in \Gamma((TM \otimes TM)^*)$ be a metric on M —the object manifold of $\mathcal{G}$. The following should convince us of the necessity of reducing the structure group as $\mathbb{B} := \mathbb{B}(\mathcal{G}) \searrow \mathbb{B}_g$, where $\mathbb{B}_g$ denotes isometric bisections:

$$\mathbb{B}_g = (t \circ -)^{-1}(\text{Isom}(M, g)). \quad (4.1)$$

Definition 4.0.1. We say that a Lie groupoid $\mathcal{G}$ is $\mathbb{B}$ -complete whenever there exists a global bisection through every arrow of $\mathcal{G}$. Precisely,

$$\forall g \in \text{Mor } \mathcal{G} \exists \beta \in \mathbb{B} : \beta(s(g)) = g. \quad (4.2)$$

Proposition 4.0.2. Let $\mathcal{G}$ be $\mathbb{B}$ -complete. Suppose that $t \circ - : \mathbb{B} \rightarrow \text{Isom}(M, g) \subseteq \text{Diff}(M)$ or, in other words,

$$\forall \beta \in \mathbb{B} : (t \circ \beta)^* g = g. \quad (4.3)$$

Then, all $\mathcal{G}$ -orbits in M are 0-dimensional.

Before we prove Proposition 4.0.2, we prove the following

Lemma 4.0.3. Let $\mathcal{G}$ be a transitive Lie groupoid (i.e., $\forall x, y \in M : t^{-1}(y) \cap s^{-1}(x) \neq \emptyset$). Then, the map $(t, s) : \text{Mor } \mathcal{G} \rightarrow M \times M$ is a surjective submersion.

Proof. For $\dim(M) = 0$ the hypothesis clearly holds. Thus, we can suppose $\dim(M) > 0$. Pick a point $b \in M$. Since $t_b := t|_{s^{-1}(b)} : s^{-1}(b) \rightarrow M$ is a surjection, by Sard's Theorem, there exists a regular point of t_b . Combining this with the constancy of the rank of t_b [Mac87, Ch. III. Cor. 1.7.], we infer that it is a submersion, and, being such, it admits smooth local sections. Pick arbitrary $x, y \in M$. Let $x \in U \xrightarrow{\sigma_U} s^{-1}(b)$ and $y \in V \xrightarrow{\sigma_V} s^{-1}(b)$ be smooth local sections of t_b . Take any $g \in t^{-1}(V) \cap s^{-1}(U)$ and define

$$V \times U \ni (b, a) \mapsto \Sigma(b, a) = \sigma_V(b) \cdot \sigma_V(t(g))^{-1} \cdot g \cdot \sigma_U(s(g)) \cdot \sigma_U(a)^{-1} \in \text{Mor } \mathcal{G}. \quad (4.4)$$

Clearly, the map Σ is smooth, and, moreover, it satisfies

$$((t, s) \circ \Sigma)(b, a) = (t(\sigma_V(b)), s(\sigma_U(a)^{-1})) = (t_b(\sigma_V(b)), t_b(\sigma_U(a))) = (b, a), \quad (4.5)$$

for each $(b, a) \in V \times U$. Therefore, we have constructed a smooth local section of (t, s) . A standard argument

$$T((t, s) \circ \Sigma) = T(t, s) \circ T\Sigma = \text{id}_{T(V \times U)} = \text{id}_{TV \oplus TU} \quad (4.6)$$

shows that (t, s) is a submersion at (b, a) , and since the choice of x, y was arbitrary, it is a submersion everywhere. $\square$

Remark 4.0.4. We can generalize this theorem to transitivity components (orbits) of a Lie groupoid which is not necessarily globally transitive. This follows from the fact that, for a transitivity component $M_T \subseteq M$, there exists a manifold structure on $t^{-1}(M_T) \cap s^{-1}(M_T)$ with respect to which it is a submanifold of $\text{Mor } \mathcal{G}$ and a Lie—now transitive—groupoid over M_T . This assertion is proven in [Mac87, III 1.8].

Let $\mathcal{O} \subseteq M$ be an orbit with $\dim \mathcal{O} \geq 1$. Take $x, y \in \mathcal{O}$ together with $0 \neq u \in T_x M$ and $0 \neq v \in T_y M$. Then, by Lemma 4.0.3, there exists $g \in \text{Mor } \mathcal{G}$, and $0 \neq w \in T_g \text{Mor } \mathcal{G}$ such that $T_g s(w) = u$ and $T_g t(w) = v$.

Proof of Proposition 4.0.2. We will show that there exists a global bisection $\beta \in \mathbb{B}$ satisfying

$$T_x(t \circ \beta)(u) = v. \quad (4.7)$$

The arbitrariness of u, v then contradicts $t \circ \beta \in \text{Isom}(M, g)$. In order to construct such β , we start by invoking the assumption of $\mathbb{B}$ -completeness to assert the existence of a bisection $\beta' \in \mathbb{B}$ satisfying $\beta'(x) = g$. We have

$$T_x \beta'(u) = w' \in T_g \text{Mor } \mathcal{G}. \quad (4.8)$$

Importantly, $w' \neq 0$ as $u \neq 0$. Note that if $w' = w$ we obtain

$$T_x(t \circ \beta')(u) = T_g t \circ T_x \beta'(u) = T_g t(w) = v, \quad (4.9)$$

and the hypothesis holds. Otherwise, we need to nontrivially deform the bisection β' so that the resulting β satisfies $T_x \beta(u) = w$. We will achieve it by finding $\psi \in \text{Diff}(\text{Mor } \mathcal{G})$ with the property $\beta = \psi \circ \beta'$. The source map s is a submersion, so, by the Constant Rank Theorem, and for an open neighborhood $g \in O_g$, there exists a local trivialization $\tau_g : O_g \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$, where $n = \dim \text{Mor } \mathcal{G}$, the first k coordinates parametrize s -fibers, and the remaining ones parametrize the directions transverse to them. Note that under this trivialization we obtain

$$T_g \tau_g(w) = (w_s, u_\tau), \quad T_g \tau_g(w') = (w'_s, u_\tau), \quad (4.10)$$

so the images of w, w' differ only in the first k coordinates. Denote by $\text{pr}_s : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the projection onto the coordinates parametrizing s -fibers, and by $\text{pr}_t : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ the projection onto the transverse part. We let $g \in O'_g \subseteq O''_g \subseteq O_g$ be other open neighborhoods, and take $h : \mathbb{R}^{n-k} \rightarrow [0, 1]$ to be a smooth bump function with the property

$$h|_{\text{pr}_t \circ \tau_g(O'_g)} \equiv 1, \quad h|_{\text{pr}_t \circ \tau_g(O_g \setminus O''_g)} \equiv 0. \quad (4.11)$$

It is always possible to choose such neighborhoods and a corresponding bump function on a smooth manifold [Nes20, pp. 13-16]. Now, let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be equal to $\tilde{f} \times \text{id}_{\mathbb{R}^{n-k}}$, where $\tilde{f} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is any invertible orientation-preserving linear function mapping w'_s to w_s . It certainly exists due to $w, w' \neq 0$. We use the fact that

$$\text{GL}^+(k, \mathbb{R}) := \{A \in \text{GL}(k, \mathbb{R}) : \det A > 0\} \quad (4.12)$$

is path-connected [HN12, Cor. 2.1.8.] to pick a smooth path

$$\gamma : [0, 1] \rightarrow \text{GL}^+(k, \mathbb{R}) : \gamma(0) = \text{id}_{\mathbb{R}^k}, \gamma(1) = \tilde{f}. \quad (4.13)$$

It obviously corresponds to the map (its transpose)

$$\gamma^\tau : \mathbb{R}^k \times [0, 1] \rightarrow \mathbb{R}^k : \gamma(t)(x) = \gamma^\tau(x, t). \quad (4.14)$$

Observe that f preserves the τ_g images of s -fibers. We finally define ψ by

$$\psi = \begin{cases} \tau_g^{-1} \circ (\gamma^\tau \times \text{id}_{\mathbb{R}^{n-k}}) \circ (\text{id}_{\mathbb{R}^k} \times (h, \text{id}_{\mathbb{R}^{n-k}})) \circ \tau_g & \text{on } O_g, \\ \text{id}_{\text{Mor } \mathcal{G} \setminus O_g} & \text{on } \text{Mor } \mathcal{G} \setminus O_g. \end{cases} \quad (4.15)$$

Mind that by $(h, \text{id}_{\mathbb{R}^{n-k}})$ we mean $(h \times \text{id}_{\mathbb{R}^{n-k}}) \circ \Delta$, where Δ is the diagonal map. By construction, the map ψ is smooth. Moreover, since h depends only on the transverse component, the smooth inverse to ψ is obtained by inverting γ pointwise and inserting $(\gamma^{-1})^\tau$ into 4.15. Consequently, the map ψ is an s -fiber-preserving diffeomorphism of $\text{Mor } \mathcal{G}$, so the composition $\psi \circ \beta'$ is, indeed, a bisection. Crucially,

$$\text{T}_g \psi(w') = \text{T}_g(\tau_g^{-1} \circ f \circ \tau_g)(w') = \text{T}_{\tau_g(g)} \tau_g^{-1} \circ \text{T}_{\tau_g(g)} f(w'_s, u_\tau) = \text{T}_{\tau_g(g)} \tau_g^{-1}(w_s, u_\tau) = w, \quad (4.16)$$

so we managed to obtain the desired $\beta = \psi \circ \beta'$. $\square$

It follows that in order for the $\mathcal{G}$ -orbits to be of non-zero dimension, we need the group $\text{Isom}(M, \mathfrak{g}) \cap \text{Diff}(M)$ to be contained in $t \circ \mathbb{B}(\mathcal{G})$. However, the transition maps must be isometric, and hence, the structure group should be reduced.

Chapter 5

The Gauging for Non-Tensorial Sigma Models

Now, we consider the gauging of rigid Lie-groupoidal symmetries for more general sigma model, whose action functionals are determined by differential characters. Before we define the DF-amplitude, we give an axiomatic and a cohomological description of differential characters.

5.1. Differential Characters

Definition 5.1.1. Let M be a manifold. The graded subalgebra $\Omega_{\mathbb{Z}}^{\bullet}(M) \subseteq \Omega^{\bullet}(M)$ of *differential forms with integer periods* is composed of those closed differential forms whose integrals over every cycle lie in $\mathbb{Z}$. That is,

$$\Omega_{\mathbb{Z}}^k(M) := \left\{ \omega \in \Omega^k(M) \mid d\omega = 0 \wedge \forall c \in Z_k(M) : \int_c \omega \in \mathbb{Z} \right\}. \quad (5.1)$$

Remark 5.1.2. By de Rham's Theorem, the classes $[\omega] \in \Omega_{\mathbb{Z}}^k(M)/d\Omega^{k-1}(M)$ are in isomorphism with the image of $i_* : H^k(M, \mathbb{Z}) \rightarrow H^k(M, \mathbb{R})$ in singular cohomology, induced by the coefficient morphism $i : \mathbb{Z} \hookrightarrow \mathbb{R}$.

Lemma 5.1.3. The map $\iota : \Omega^{\bullet}(M) \rightarrow C^{\bullet}(M, \mathbb{R}/\mathbb{Z})$ given by

$$\iota(\omega)(c) := \int_c \omega \bmod \mathbb{Z} \quad (5.2)$$

is an injection.

Proof. Suppose $0 \neq \omega \in \ker \iota$. Then $\omega(p) \neq 0$ for p in some coordinate chart (U, ϕ) . In local coordinates $\phi = (x^1, \dots, x^n)$ on U , the form ω is expressed as

$$\omega(x) = f(x) dx^{i_1} \wedge \dots \wedge dx^{i_k} + \dots \quad (5.3)$$

Without loss of generality, we may assume that $f(p) \neq 0$. By the continuity of f , there exist $\delta > 0$ and $\varepsilon > 0$ such that we can fit an n -cube

$$K = \{x : |x^j - p^j| \leq \varepsilon, 1 \leq j \leq \dim M\} \subseteq U \cap \{x : |f(x)| \geq \delta\}. \quad (5.4)$$

Consider a smooth k -simplex $\sigma : \Delta^k \rightarrow M$ defined in local coordinates as

$$\sigma(t^1, \dots, t^k) = \phi^{-1}(p^1, \dots, p^{i_1-1}, p^{i_1} + \varepsilon t^1, p^{i_1+1}, \dots, p^{i_k-1}, p^{i_k} + \varepsilon t^k, \dots, p_n). \quad (5.5)$$

Clearly, $\sigma(\Delta^k) \subseteq K$. The pullback of ω calculates as

$$\sigma^*\omega = (f \circ \sigma(t))\varepsilon^k dt^1 \wedge \dots \wedge dt^k. \quad (5.6)$$

We obtain

$$\int_{\sigma} \omega = \int_{\Delta^k} (f \circ \sigma(t))\varepsilon^k dt^1 \wedge \dots \wedge dt^k. \quad (5.7)$$

Since $|f \circ \sigma(t)| \geq \delta$ on Δ^k , we get an estimate

$$\left| \int_{\sigma} \omega \right| \geq \delta \frac{\varepsilon^k}{k!}. \quad (5.8)$$

Combining this with another estimate

$$\left| \int_{\sigma} \omega \right| \leq \sup_{x \in K} |f(x)| \frac{\varepsilon^k}{k!}, \quad (5.9)$$

which is finite by the compactness of K , we conclude that for sufficiently small $\varepsilon > 0$:

$$0 < \left| \int_{\sigma} \omega \right| < 1, \quad (5.10)$$

which contradicts the assumption that the value of any such integral lies in $\mathbb{Z}$. $\square$

Corollary 5.1.4. *In Definition 5.1.1 one does not have to assume closedness on ω .*

Proof. Let $\omega \in \Omega^k(M)$ be such that

$$\forall c \in Z_k(M) : \int_c \omega \in \mathbb{Z}. \quad (5.11)$$

Then, by Stokes Theorem, for any $c' \in C_{k+1}(M)$ we have

$$\int_{\partial c'} \omega = \int_{c'} d\omega \in \mathbb{Z}, \quad (5.12)$$

and thus, $d\omega = 0$. $\square$

Definition 5.1.5 ([CS85]). Let M be a manifold. The k -th group of differential characters is defined as

$$\hat{H}^k(M, \mathbb{R}/\mathbb{Z}) := \{f \in \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \mid \exists \omega_f \in \Omega^k(M) : f \circ \partial = \iota(\omega_f)\}. \quad (5.13)$$

That is, for any $c \in C_k(M)$ we have

$$f(\partial c) = \int_c \omega_f \bmod \mathbb{Z}. \quad (5.14)$$

It is clear from the construction that $\omega_f \in \Omega_{\mathbb{Z}}^k(M)$. Lemma 5.1.3 implies that ω_f is uniquely determined by f . Moreover, the assignment $M \mapsto \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$ is functorial. Indeed, for a map $h : M \rightarrow N$ we have

$$\forall f \in \hat{H}^k(N, \mathbb{R}/\mathbb{Z}) : f \circ h_* \circ \partial = f \circ \partial \circ h_* = \iota(\omega_f) \circ h_* = \iota(h^* \omega_f). \quad (5.15)$$

We will call the form ω_f the *curvature* of f .

Definition 5.1.6 ([SiSu07]). A *character functor* is a 5-tuple $(\hat{G}^\bullet, i_1, i_2, \delta_1, \delta_2)$, where $\hat{G}^\bullet$ is a contravariant functor from the category of smooth manifolds to graded abelian groups, and $i_1, i_2, \delta_1, \delta_2$ are natural transformations rendering the following *character diagram* commutative, and its diagonal sequences exact for each k :

$$\begin{array}{ccccc}
0 & & & & 0 \\
& \searrow & & & \nearrow \\
& & H^{k-1}(\cdot, \mathbb{R}/\mathbb{Z}) & \xrightarrow{-B} & H^k(\cdot, \mathbb{Z}) \\
& \nearrow \alpha & & \nearrow \delta_2 & \searrow r \\
H^{k-1}(\cdot, \mathbb{R}) & & & \hat{G}^k & & H^k(\cdot, \mathbb{R}) \\
& \searrow \beta & & \nearrow i_2 & \searrow \delta_1 & \nearrow s \\
& & \Omega^{k-1}/\Omega_{\mathbb{Z}}^{k-1} & \xrightarrow{d} & \Omega_{\mathbb{Z}}^k & \\
& \nearrow & & & \searrow & \\
0 & & & & 0.
\end{array} \tag{5.16}$$

The maps α, B, r are obtained from the long exact sequence

$$\dots \rightarrow H^k(\cdot, \mathbb{Z}) \xrightarrow{r} H^k(\cdot, \mathbb{R}) \xrightarrow{\alpha} H^k(\cdot, \mathbb{R}/\mathbb{Z}) \xrightarrow{B} H^{k+1}(\cdot, \mathbb{Z}) \rightarrow \dots \tag{5.17}$$

associated to the coefficient short exact sequence of abelian groups $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$, and β, d, s are defined as follows. The map β -by

$$H^{k-1}(\cdot, \mathbb{R}) \xrightarrow{\simeq} H_{\text{dR}}^{k-1}(\cdot) \hookrightarrow \Omega^{k-1}/d\Omega^{k-2} \twoheadrightarrow \Omega^{k-1}/\Omega_{\mathbb{Z}}^{k-1}, \tag{5.18}$$

using de Rham Theorem, and the fact that $d\Omega^{k-2} \subseteq \Omega_{\mathbb{Z}}^{k-1}$. Since $d\Omega_{\mathbb{Z}}^{k-1} = 0$, the de Rham differential is well-defined on classes in $\Omega^{k-1}/\Omega_{\mathbb{Z}}^{k-1}$. Finally,

$$s : \Omega_{\mathbb{Z}}^k \hookrightarrow \ker d^k \twoheadrightarrow \ker d^k / d\Omega_{\mathbb{Z}}^{k-1} = H_{\text{dR}}^k(\cdot) \xrightarrow{\simeq} H^k(\cdot, \mathbb{R}). \tag{5.19}$$

Proposition 5.1.7 ([SiSu07]). *The differential characters $\hat{H}^\bullet$ substituted for $\hat{G}^\bullet$ fit into the character diagram, and, as such, form a character functor together with appropriate natural transformations $i_1, i_2, \delta_1, \delta_2$.*

Proof. Fix a manifold M and let $f \in \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$. We begin by defining $\delta_1(f) := \omega_f$. Naturality of this assignment follows from 5.15. For surjectivity, let $\omega \in \Omega_{\mathbb{Z}}^k(M)$ be arbitrary. Define

$$f(c) = \begin{cases} \iota(\omega)(b) & c = \partial b \in B_{k-1}(M) \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Then, $\delta_1(f) = \omega$. Since $\mathbb{R}/\mathbb{Z}$ is a divisible group, and thus an injective abelian group, the Universal Coefficient Theorem asserts

$$H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R}/\mathbb{Z}). \tag{5.21}$$

Therefore, by the left exactness of the left hom-functor (Definition A.0.3), the canonical projection in the short exact sequence

$$0 \rightarrow B_{k-1}(M) \hookrightarrow Z_{k-1}(M) \twoheadrightarrow Z_{k-1}(M)/B_{k-1}(M) = H_{k-1}(M) \rightarrow 0 \tag{5.22}$$

induces the inclusion

$$H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R}/\mathbb{Z}) \hookrightarrow \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}). \quad (5.23)$$

This gives the map $i_1 : H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$. Indeed, by construction for any $[r] \in H^{k-1}(M, \mathbb{R}/\mathbb{Z})$ and any $\partial b \in B_{k-1}(M)$ we have

$$i_1([r])(\partial b) = \delta r(b) = 0. \quad (5.24)$$

Therefore, by Universal Coefficient Theorem and the left exactness of the hom-functor, the image of i_1 corresponds to the differential characters with zero curvature. In other words, $\text{im}(i_1) = \ker(\delta_1)$.

The restriction

$$\text{res} : C^{k-1}(M, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(C_{k-1}(M), \mathbb{R}/\mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \quad (5.25)$$

composed with ι gives a map

$$\text{res} \circ \iota : \Omega^{k-1}(M) \rightarrow \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}), \quad (5.26)$$

whose kernel is composed of those $(k-1)$ -forms, which integrate to integers on cycles. This is precisely $\Omega_{\mathbb{Z}}^{k-1}(M)$ (Corollary 5.1.4). Note that, by Stokes Theorem, for any $\partial b \in B_{k-1}(M)$, we have

$$(\text{res} \circ \iota)(\omega)(\partial b) = \iota(d\omega)(b). \quad (5.27)$$

Thus, $\text{res} \circ \iota$ defines a map to $\hat{H}^k(M, \mathbb{R}/\mathbb{Z})$. We define i_2 as its factorization through $\Omega^{k-1}(M)/\Omega_{\mathbb{Z}}^{k-1}(M)$, which is manifestly injective. Moreover, 5.27 proves commutativity of the bottom triangle in 5.16.

In order to define δ_2 consider the following diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\quad} & \mathbb{R}/\mathbb{Z} \\ & \nwarrow T & \uparrow \\ & C_{k-1}(M) & \\ & \uparrow & \\ & Z_{k-1}(M) & \end{array} \quad \begin{array}{c} \nearrow \tilde{f} \\ \searrow f \end{array} \quad (5.28)$$

Given $f \in \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$, we use the fact that $Z_{k-1}(M)$ is a free abelian group to lift f to $\tilde{f}$ (Proposition A.0.14). Then, we use the injectivity of $\mathbb{R}$ as an abelian group (Corollary A.0.17) to factor $\tilde{f}$ through $C_{k-1}(M)$, and we call this factorization T . Observe that for any $c \in C_k(M)$ we get

$$\delta T(c) \bmod \mathbb{Z} = T(\partial c) \bmod \mathbb{Z} = \tilde{f}(\partial c) \bmod \mathbb{Z} = f(\partial c) = \iota(\omega_f)(c) = \omega_f(c) \bmod \mathbb{Z}, \quad (5.29)$$

where after the last equality we treat ω_f as a cochain by integration, i.e.,

$$\omega_f \mapsto \left(c \mapsto \int_c \omega_f \right). \quad (5.30)$$

Therefore, under this identification, $\omega_f - \delta T \in C^k(M, \mathbb{Z})$ and it is closed, since ω_f is closed. Indeed, by de Rham Theorem, the cochain associated to ω_f is closed if and only if ω_f is d-closed. Moreover, if T' is another map making 5.28 commutative, we compute

$$(T - T')|_{Z_{k-1}(M)} = \tilde{f} - \tilde{f} = 0 \quad \Rightarrow \quad T - T' = \delta d, \quad d \in C^{k-2}(M, \mathbb{R}). \quad (5.31)$$

Here, we used Universal Coefficient Theorem to infer from $T - T' = 0$ in $\text{Hom}_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R})$ the equality $[T - T'] = 0$ in $H^{k-1}(M, \mathbb{R})$. If we pick a different lift $\tilde{f}'$, we get $\tilde{f}' - \tilde{f} \bmod \mathbb{Z} = f - f = 0$, so $\tilde{f} - \tilde{f}' = c|_{Z_{k-1}(M)}$ for some $c \in C^{k-1}(M, \mathbb{Z})$. We conclude that in general

$$T - T' = \delta d + c \quad \Rightarrow \quad \delta(T - T') = \delta c. \quad (5.32)$$

This means that the cohomology class $[\omega_f - \delta T] \in H^k(M, \mathbb{Z})$ depends only on f . We define $\delta_2(f) := [\omega_f - \delta T]$.

To see that it is surjective, let $[u] \in H^k(M, \mathbb{Z})$ be arbitrary. By Remark 5.1.2, there exists $\omega \in \Omega_{\mathbb{Z}}^k(M)$ with $[\omega] = i_*[u]$ under indentification by de Rham's isomorphism. Then, since $[\omega - i_*u] = 0$, for any representative u , the cochain $\omega - i_*u \in C^k(M, \mathbb{R})$ is exact, so we can find $T \in C^{k-1}(M, \mathbb{R})$ with $\delta T = \omega - i_*u$. By postcomposing the restricted cochain $T|_{Z_{k-1}(M)}$ with the natural projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$, we obtain

$$f \in \text{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}), \quad f \circ \partial = \iota(\omega), \quad (5.33)$$

a differential character, which satisfies $\delta_2(f) = [u]$. Indeed, we have $\omega - \delta T = i_*u$ by construction.

Now, suppose $\delta_2(f) = 0$, that is $[\omega_f - \delta T] = 0$. Since δT is exact, using de Rham Theorem, we infer that

$$\exists \theta \in \Omega^{k-1}(M) : d\theta = \omega_f, \quad (5.34)$$

and

$$\exists e \in C^{k-1}(M, \mathbb{Z}) : \omega_f - \delta T = \delta e. \quad (5.35)$$

We calculate $\delta(\theta - T - e) = 0$, so there exists $\zeta \in Z^{k-1}(M, \mathbb{R})$ such that

$$\theta - T - e = \zeta. \quad (5.36)$$

By de Rham Theorem we can find

$$\phi \in \Omega^{k-1}(M) : (\theta - T - e)|_{Z_{k-1}(M)} = \phi|_{Z_{k-1}(M)}. \quad (5.37)$$

We have $T|_{Z_{k-1}(M)} = (\theta - \phi - e)|_{Z_{k-1}(M)}$. By postcomposing both sides with the natural projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$, we obtain $f = \iota(\theta - \phi)|_{Z_{k-1}(M)} = \iota(\theta - \phi)|_{Z_{k-1}(M)} = i_2(\theta - \phi)$, as e is $\mathbb{Z}$ -valued. Hence, $f \in \text{im}(i_2)$. We conclude that both diagonal sequences are exact. We proceed by checking commutativity of the left side of the character diagram.

Since i_2 was defined as a factorization of $\text{res} \circ \iota$ through the quotient by its kernel, let $[r] \in H^{k-1}(M, \mathbb{R})$ and let us pick a representative $\omega \in \Omega^{k-1}(M)$ of $\beta([r])$ within the class in $\Omega^{k-1}(M)/\Omega_{\mathbb{Z}}^{k-1}(M)$. Then, $(i_2 \circ \beta)([r]) = (\text{res} \circ \iota)(\omega)$. Using Universal Coefficient Theorem we identify α with

$$\begin{array}{ccc} H^{k-1}(M, \mathbb{R}) & \xrightarrow{\alpha} & H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Hom}_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R}) & \xrightarrow{\pi_*} & \text{Hom}_{\mathbb{Z}}(H_{k-1}(M), \mathbb{R}/\mathbb{Z}) \end{array} \quad (5.38)$$

Now, it follows from de Rham Theorem that

$$(\text{res} \circ \iota)(\omega) = \text{pr}^* \circ \pi_*([r]) = (i_1 \circ \alpha)([r]), \quad (5.39)$$

where $\text{pr} : Z_{k-1}(M) \twoheadrightarrow H_{k-1}(M)$ of 5.22 was used to define i_1 .

We already showed under 5.27 that the bottom triangle commutes. The top triangle commutes by the construction of δ_2 above, where taking $f \in \text{im}(i_1)$ implies $\omega_f = 0$ by exactness of the diagonal sequence in 5.16, and we recover the usual connecting homomorphism $-B$. Finally, the right side of the diagram commutes by Remark 5.1.2, since by construction, $r(\delta_2(f)) = r([\omega_f - \delta T]) = [\omega_f] \in H^k(M, \mathbb{R})$. Naturality of δ_1 follows from 5.15. The map i_2 is a natural transformation by the change of variables formula for integrals. Pullbacks and pushforwards induce natural transformation for groups of cochains, hence i_1 is a natural transformation. Finally, δ_2 was built using universal constructions, so it is also natural. $\square$

Definition 5.1.8. For any $f \in \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$ we call $\delta_2(f)$ *the characteristic class of f* .

In fact, the axiomatic description of differential characters we just laid down is exhaustive, and therefore we may interchangeably refer to the functor of differential characters and the associated character functor. This result is expressed in

Theorem 5.1.9 ([SiSu07]). *For every character functor $(\hat{G}^\bullet, i_1, i_2, \delta_1, \delta_2)$, there exists a unique natural isomorphism $\Phi : \hat{G}^\bullet \rightarrow \hat{H}^\bullet(\cdot, \mathbb{R}/\mathbb{Z})$, which commutes with identity transformations on all other functors in the diagram. In other words, the diagram formed by two character diagrams—corresponding to $\hat{G}^\bullet$ and $\hat{H}^\bullet$ —connected with identity transformations and Φ is commutative.*

Besides the axiomatic functorial description of differential characters given in [SiSu07], a cohomological classification was developed by Hopkins and Singer in [HS05]. Despite the fact that $\hat{H}^\bullet(\cdot, \mathbb{R}/\mathbb{Z})$ does not form a genuine sheaf over **Man**, it has been shown [LM08a] that the cocycle category associated to $\hat{H}^2(\cdot, \mathbb{R}/\mathbb{Z})$ forms a stack over **Man**. What this means is that given a collection of local characters on elements of an open cover such that their restrictions to double intersections match, there might not be a global character which restricts to the local ones. Stackiness, however, guarantees that given particular extra structure on double and triple intersections, rendering the local characters "coherently aligned", there does indeed exist a unique global gluing. In what follows, we will delineate what we mean by "coherently aligned". In this thesis, we only give proofs of assertions for the case of degree-2 characters. However, we believe that a generalization to arbitrary $k \in \mathbb{N}$ should be straightforward, albeit technically involved.

Definition 5.1.10 ([HS05]). Let M be a manifold and let $s \in \mathbb{N}$. The cochain complex $\text{DC}_s^\bullet(M)$ is defined by

$$\text{DC}_s^n(M) = \{(c, h, \omega) : \omega = 0 \text{ for } n < s\} \subseteq C^n(M, \mathbb{Z}) \times C^{n-1}(M, \mathbb{R}) \times \Omega^n(M), \quad (5.40)$$

and

$$d(c, h, \omega) = (\delta c, \omega - c - \delta h, d\omega), \quad (5.41)$$

where we identify ω with its real cochain given by integration. We note that $\text{DC}_s^\bullet$ forms a presheaf of complexes on **Man** (Definition A.0.22), as a product of such presheaves.

Proposition 5.1.11. *For each $n \in \mathbb{N}$ there is a natural isomorphism $H^n(\text{DC}_n^\bullet) \simeq \hat{H}^n(\cdot, \mathbb{R}/\mathbb{Z})$*

Proof. Let $M \in \mathbf{Man}$. We set

$$\psi_M : H^n(\text{DC}_n^\bullet(M)) \ni [(c, h, \omega)] \mapsto h|_{Z_{n-1}(M)} \bmod \mathbb{Z} \in \hat{H}^n(M, \mathbb{R}/\mathbb{Z}). \quad (5.42)$$

It is a linear map, so its well-definedness follows from

$$\psi_M(d(c, h, 0)) = \psi_M(\delta c, -c - \delta h, 0) = (-c - \delta h)|_{Z_{n-1}(M)} \bmod \mathbb{Z} = 0, \quad (5.43)$$

and

$$\psi_M([(c, h, \omega)])(\partial b) = \delta h(b) \bmod \mathbb{Z} = \omega(b) - c(b) \bmod \mathbb{Z} = \omega(b) \bmod \mathbb{Z} = \iota(\omega)(b). \quad (5.44)$$

To construct the inverse, given $f \in \hat{H}^n(M, \mathbb{R}/\mathbb{Z})$, we take $T(f) \in C^{n-1}(M, \mathbb{R})$ as in 5.28. For ω simply take $\omega_f = \delta_1(f)$ (5.16). Just as it was shown in the proof of Proposition 5.1.7, the cochain $T(f)$ satisfies

$$\delta T(f) = \omega_f - c, \quad (5.45)$$

where $c \in Z^n(M, \mathbb{Z})$ is a representative of $\delta_2(f)$. Moreover, $T(f)$ is determined up to

$$c' + \delta d, \quad c' \in C^{n-1}(M, \mathbb{Z}), \quad d \in C^{n-2}(M, \mathbb{R}). \quad (5.46)$$

This makes the class $[(c, T(f), \omega_f)]$ uniquely determined. Indeed, let $(\tilde{c}, \tilde{T}(f), \omega_f)$ be a different choice. Then,

$$\begin{aligned} (\tilde{c}, \tilde{T}(f), \omega_f) - (c, T(f), \omega_f) &= (\delta(T(f) - \tilde{T}(f)), \tilde{T}(f) - T(f), 0) = (-\delta c', c' + \delta d, 0) \\ &= d(-c', -d, 0). \end{aligned} \quad (5.47)$$

We should check that $\psi_M^{-1} \circ \psi_M = \text{id}_{H^n(\text{DC}_n^\bullet(M))}$. But this follows from the fact that the class $[(c, h, \omega)]$ is uniquely determined by $f = h|_{Z_{n-1}(M)} \bmod \mathbb{Z}$. The equality $\psi_M \circ \psi_M^{-1} = \text{id}_{\hat{H}^n(M)}$ follows directly from the construction of $T(f)$. Naturality of ψ is straightforward. $\square$

For an open cover $\mathcal{O} = \bigsqcup_{i \in I} O_i$ of a manifold M we construct the Čech nerve (Remark A.0.24), which is a simplicial manifold $\mathcal{O}_\bullet : \Delta \rightarrow \mathbf{Man}$ whose object component is

$$\mathcal{O}_n = \underbrace{\mathcal{O} \times_M \mathcal{O} \times_M \dots \times_M \mathcal{O}}_{n+1}, \quad (5.48)$$

and the morphism component is fixed by the choice of the face and degeneracy maps (Definition A.0.23). The face maps are the canonical projections

$$d_i^{(n)} : \mathcal{O}_n \rightarrow \mathcal{O}_{n-1} : (x_0, x_1, \dots, x_n) \mapsto (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \quad 0 \leq i \leq n. \quad (5.49)$$

The degeneracy maps are

$$s_i^{(n)} : \mathcal{O}_n \rightarrow \mathcal{O}_{n+1} : (x_0, x_1, \dots, x_n) \mapsto (x_0, \dots, x_i, x_i, \dots, x_n) \quad 0 \leq i \leq n. \quad (5.50)$$

Note that there are diffeomorphisms

$$\mathcal{O}_n \simeq \bigsqcup_{\bar{i} \in I^{n+1}} O_{i_0 i_1 \dots i_n} \quad \bar{i} = (i_0, \dots, i_n), \quad (5.51)$$

where $O_{i_0 i_1 \dots i_n} := O_{i_1} \cap O_{i_2} \cap \dots \cap O_{i_n}$. We tacitly assume that $X \sqcup \emptyset = X$, and the same for an infinite disjoint union of empty sets. Any smooth map $\psi : N \rightarrow M$ induces a pullback of the Čech nerve:

$$(\psi^{-1} \mathcal{O})_\bullet \rightarrow \mathcal{O}_\bullet. \quad (5.52)$$

The simplicial manifold $(\psi^{-1}\mathcal{O})_\bullet$ is just the Čech nerve of the pullback cover

$$\psi^{-1}\mathcal{O} = \bigsqcup_{i \in I} \psi^{-1}(O_i). \quad (5.53)$$

Clearly,

$$(\psi^{-1}\mathcal{O})_n \simeq \bigsqcup_{\bar{i} \in I^{n+1}} \psi^{-1}(O_{i_0 i_1 \dots i_n}) = \bigsqcup_{\bar{i} \in I^{n+1}} \psi^{-1}(O_{i_0}) \cap \dots \cap \psi^{-1}(O_{i_n}). \quad (5.54)$$

The canonical morphism $(\psi^{-1}\mathcal{O})_\bullet \rightarrow \mathcal{O}_\bullet$ is given by

$$\psi_n = \psi^{\times(n+1)} : (\psi^{-1}\mathcal{O})_n \rightarrow \mathcal{O}_n. \quad (5.55)$$

Denote by j the covering map $\mathcal{O} \rightarrow M$, which is defined as the respective inclusion on every connected component of $\mathcal{O}$. Using the Čech nerve, we construct the double complex:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow & & \uparrow & & \uparrow & \\ \text{DC}_2^0(\mathcal{O}_2) & \xrightarrow{\text{d}} & \text{DC}_2^1(\mathcal{O}_2) & \xrightarrow{\text{d}} & \text{DC}_2^2(\mathcal{O}_2) & \longrightarrow & \dots \\ \delta^1 \uparrow & & \delta^1 \uparrow & & \delta^1 \uparrow & & \\ \text{DC}_2^0(\mathcal{O}_1) & \xrightarrow{-\text{d}} & \text{DC}_2^1(\mathcal{O}_1) & \xrightarrow{-\text{d}} & \text{DC}_2^2(\mathcal{O}_1) & \longrightarrow & \dots \\ \delta^0 \uparrow & & \delta^0 \uparrow & & \delta^0 \uparrow & & \\ \text{DC}_2^0(\mathcal{O}_0) & \xrightarrow{\text{d}} & \text{DC}_2^1(\mathcal{O}_0) & \xrightarrow{\text{d}} & \text{DC}_2^2(\mathcal{O}_0) & \longrightarrow & \dots \\ j^* \uparrow & & j^* \uparrow & & j^* \uparrow & & \\ \text{DC}_2^0(M) & \xrightarrow{-\text{d}} & \text{DC}_2^1(M) & \xrightarrow{-\text{d}} & \text{DC}_2^2(M) & \longrightarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & & 0 & & 0 & & \end{array} \quad (5.56)$$

where

$$\delta^k := \sum_{i=0}^{k+1} (-1)^i \left(d_i^{(k+1)} \right)^*. \quad (5.57)$$

We call δ^k the $(k\text{-th})$ *Dupont operator*. The fact that columns of 5.56 form cochain complexes follows from simplicial identities of the Čech nerve. The fact that d and δ^k commute is a consequence of the fact that d commutes with pullbacks. Note that for any representative (c, h, ω) of a class in $H^2(\text{DC}_2^\bullet(M))$, the chain map j^* induces a cocycle in the total complex of the truncation

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \uparrow & & \uparrow & & \uparrow & \\ & 0 & & 0 & & 0 & \\ \text{DC}_2^0(\mathcal{O}_2) & \xrightarrow{\text{d}} & \text{DC}_2^1(\mathcal{O}_2) & \xrightarrow{\text{d}} & \text{DC}_2^2(\mathcal{O}_2) & \longrightarrow & \dots \\ \delta^1 \uparrow & & \delta^1 \uparrow & & \delta^1 \uparrow & & \\ \text{DC}_2^0(\mathcal{O}_1) & \xrightarrow{-\text{d}} & \text{DC}_2^1(\mathcal{O}_1) & \xrightarrow{-\text{d}} & \text{DC}_2^2(\mathcal{O}_1) & \longrightarrow & \dots \\ \delta^0 \uparrow & & \delta^0 \uparrow & & \delta^0 \uparrow & & \\ \text{DC}_2^0(\mathcal{O}_0) & \xrightarrow{\text{d}} & \text{DC}_2^1(\mathcal{O}_0) & \xrightarrow{\text{d}} & \text{DC}_2^2(\mathcal{O}_0) & \longrightarrow & \dots \end{array} \quad (5.58)$$

Simply take $j^*(c, h, \omega) \in \text{DC}_2^2(\mathcal{O}_0)$, and $(0, 0, 0)$ in $\text{DC}_2^1(\mathcal{O}_1)$ and $\text{DC}_1^0(\mathcal{O}_2)$:

$$(0, 0, 0)$$

$$(0, 0, 0) \tag{5.59}$$

$$j^*(c, h, \omega).$$

This defines a map

$$p : H^2(\text{DC}_2^\bullet(M)) \rightarrow H_{\text{tot}}^2(\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T), \tag{5.60}$$

where $\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T$ is the truncated double complex 5.58.

Theorem 5.1.12 ([LM08a]). *For any $M \in \mathbf{Man}$ and any open cover $\mathcal{O}$ of M , the map $p : H^2(\text{DC}_2^\bullet(M)) \rightarrow H_{\text{tot}}^2(\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T)$ is an isomorphism.*

Theorem 5.1.12 gives us the necessary and sufficient conditions for a collection of characters of degree-2 on the elements of a cover that agree on double overlaps to glue to a global character. Later in this thesis, we will use the gluing of characters to construct a sigma model with a groupoidal symmetry gauged. However, the proof which appears in [LM08a] is not constructive, and an explicit construction of the inverse to p will prove important in subsequent sections. This is why we give a

Proof of Theorem 5.1.12. Let

$$\begin{array}{ccc} (c_{ijk}, 0, 0) & & 0 \\ & & \\ (c_{ij}, h_{ij}, 0) & & 0 \\ & & \\ (c_i, h_i, \omega_i) & & 0 \end{array} \tag{5.61}$$

be a general cocycle in the total complex of $\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T$. The second component in $(c_{ijk}, 0, 0)$ is zero for dimensional reasons – by definition it belongs to $C^{-1}(\mathcal{O}_2, \mathbb{R})$. Note that we use 5.51 and interpret $a_{i_0, \dots, i_n}$, $a \in \{c, h, \omega\}$ as collection of objects on connected components of $\mathcal{O}_n$. We can give the inverse to p as an assignment to the class of 5.61 of a homomorphism

$$f \in \text{Hom}_{\mathbb{Z}}(Z_1(M), \mathbb{R}/\mathbb{Z}), \quad f \circ \partial = \iota(\omega_f), \tag{5.62}$$

where ω_f comes from the class $[(c_i, h_i, \omega_i)]$. We will rely on the Subdivision Theorem (Theorem A.0.12), and write every 1-cycle z on M as a sum $z_{\mathcal{O}} + \partial b$ for $z_{\mathcal{O}}$ subordinate to $\mathcal{O}$. We will denote by $Z_1^{\mathcal{O}}(M)$ the group of subordinate cycles on M . We can define f by introducing $f' \in \text{Hom}_{\mathbb{Z}}(Z_1^{\mathcal{O}}(M), \mathbb{R}/\mathbb{Z})$, and set

$$f(z) = f'(z_{\mathcal{O}}) + \iota(\omega_f)(b). \tag{5.63}$$

As long as $f' \circ \partial = \iota(\omega_f)|_{\partial^{-1}Z_1^{\mathcal{O}}(M)}$, the expression 5.63 does not depend on the choice of a decomposition of z . Indeed, let $z = z'_{\mathcal{O}} + \partial b'$ be another decomposition. We have

$$z_{\mathcal{O}} - z'_{\mathcal{O}} = \partial(b' - b). \tag{5.64}$$

Since both sides are subordinate to $\mathcal{O}$, we can use Lemma A.0.13 to find $b_{\mathcal{O}} \in C_2^{\mathcal{O}}(M)$ satisfying $\partial b_{\mathcal{O}} = \partial(b' - b)$, and get $z_{\mathcal{O}} - z'_{\mathcal{O}} = \partial b_{\mathcal{O}}$, and thus $\exists d \in Z_2(M) : b' - b - b_{\mathcal{O}} = z$. Applying 5.63 yields

$$\begin{aligned} f'(z_{\mathcal{O}}) + \iota(\omega_f)(b) &= f'(z'_{\mathcal{O}} + \partial b_{\mathcal{O}}) + \iota(\omega_f)(b) = f'(z'_{\mathcal{O}}) + \iota(\omega_f)(b_{\mathcal{O}} + b) \\ &= f'(z'_{\mathcal{O}}) + \iota(\omega_f)(b' - b + b - d) = f'(z'_{\mathcal{O}}) + \iota(\omega_f)(b'). \end{aligned} \quad (5.65)$$

We read off the equations affirming cocyclicity of 5.61:

$$(\delta c_i, \omega_i - c_i - \delta h_i, d\omega_i) = 0, \quad (5.66)$$

$$(c_j - c_i - \delta c_{ij}, h_j - h_i + c_{ij} + \delta h_{ij}, \omega_j - \omega_i) = 0, \quad (5.67)$$

$$(c_{jk} - c_{ik} + c_{ij} + \delta c_{ijk}, h_{jk} - h_{ik} + h_{ij} - c_{ijk}, 0) = 0. \quad (5.68)$$

It follows that the local forms ω_i agree on double intersections. Since $\Omega^{\bullet}$ forms a sheaf, this is enough to define ω_f as the unique gluing of the ω_i . In order to construct f' , let $z_{\mathcal{O}} \in Z_1^{\mathcal{O}}(M)$ decompose into individual simplices as

$$z_{\mathcal{O}} = \sum_{j=1}^{m_z} z_j \quad \text{im}(z_j) \subseteq O_{i_j}. \quad (5.69)$$

This comes with a choice of a function

$$\{1, \dots, m_z\} \ni j \mapsto i_j \in I. \quad (5.70)$$

We will have to prove that $f'(z_{\mathcal{O}})$ does not depend on this choice. Let d_0, d_1 denote the face maps defining the boundary maps

$$\partial z_j = d_0 z_j - d_1 z_j. \quad (5.71)$$

Because $z_{\mathcal{O}}$ is a cycle, for any z_j there exists a $z_{j'}$ such that

$$d_0 z_j = d_1 z_{j'}. \quad (5.72)$$

This also involves a choice, which—as we will show—does not change $f'(z_{\mathcal{O}})$. We set

$$f'(z_{\mathcal{O}}) = \sum_{j=1}^{m_z} h_{i_j}(z_j) + h_{i_{j'}, i_j}(d_0 z_j) \mod \mathbb{Z}. \quad (5.73)$$

Note that the expression is well-defined since $\text{im}(d_0 z_j) \subseteq O_{i_{j'}, i_j}$. Moreover, 5.73 is linear in the cocycle 5.61. Therefore, in order to prove that f' only depends on the class in total cohomology of 5.58, it is enough to show that 5.73 vanishes on any coboundary (the upper diagonal):

$$\begin{aligned} (d_{jk} - d_{ik} + d_{ij}, 0, 0) \\ (d_{ij}, 0, 0) \quad (a_j - a_i - \delta d_{ij}, b_j - b_i + d_{ij}, 0) \end{aligned} \quad (5.74)$$

$$(a_i, b_i, 0) \quad (\delta a_i, -a_i - \delta b_i, 0).$$

The expression for f' becomes

$$\begin{aligned}
f'(z_{\mathcal{O}}) &= \sum_{j=1}^{m_z} -a_{i_j}(z_j) - \delta b_{i_j}(z_j) + b_{i_j}(d_0 z_j) - b_{i_{j'}}(d_0 z_j) + d_{i_j, i_{j'}}(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} -b_{i_j}(d_0 z_j) + b_{i_j}(d_1 z_j) + b_{i_j}(d_0 z_j) - b_{i_{j'}}(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} b_{i_j}(d_1 z_j) - b_{i_{j'}}(d_1 z_{j'}) \mod \mathbb{Z} = 0,
\end{aligned} \tag{5.75}$$

where we use $a_i, d_{ij} \in C^\bullet(M, \mathbb{Z})$, and the fact that $z_{\mathcal{O}}$ is a cycle (a sum over j' is just a shifted sum over j). Now, let us consider a different choice function

$$\{1, \dots, m_z\} \ni j \mapsto i'_j \in I. \tag{5.76}$$

Using 5.67 and 5.68 we compute

$$\begin{aligned}
&\sum_{j=1}^{m_z} h_{i_j}(z_j) - h_{i'_j}(z_j) + h_{i_j, i_j}(d_0 z_j) - h_{i'_j, i'_j}(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} \delta h_{i_j, i'_j}(z_j) + c_{i_j, i'_j}(z_j) + h_{i_j, i_j}(d_0 z_j) - h_{i'_j, i'_j}(d_0 z_j) \mod \mathbb{Z} \\
&\sum_{j=1}^{m_z} h_{i_j, i'_j}(d_0 z_j) - h_{i_j, i'_j}(d_1 z_j) + h_{i_j, i_j}(d_0 z_j) - h_{i'_j, i'_j}(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} h_{i_j, i'_j}(d_0 z_j) - h_{i_{j'}, i'_{j'}}(d_0 z_j) + h_{i_j, i_j}(d_0 z_j) - h_{i'_{j'}, i'_{j'}}(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} (h_{i_j, i'_j} - h_{i'_{j'}, i'_{j'}})(d_0 z_j) + (h_{i_j, i_j} - h_{i'_{j'}, i'_{j'}})(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} (h_{i_j, i'_{j'}} - c_{i_j, i'_{j'}, i'_j})(d_0 z_j) + (h_{i'_{j'}, i_j} - c_{i_j, i'_{j'}, i'_j})(d_0 z_j) \mod \mathbb{Z} \\
&= \sum_{j=1}^{m_z} (h_{i_j, i'_{j'}} + h_{i'_{j'}, i_j})(d_0 z_j) \mod \mathbb{Z} = \sum_{j=1}^{m_z} (c_{i_j, i'_{j'}, i_j} + c_{i_j, i_j, i'_{j'}})(d_0 z_j) \mod \mathbb{Z} = 0.
\end{aligned} \tag{5.77}$$

That 5.73 is invariant under the change of assignment $j \mapsto j'$ to $j \mapsto j''$ follows by replacing $j \mapsto i_j$ so that $i'_{j''} = i_{j'}$, the invariance under which we just showed. Now suppose $z_{\mathcal{O}} = \partial(b_{\mathcal{O}})$ for a subordinate 2-chain $b_{\mathcal{O}}$. Note that

$$\partial b_{\mathcal{O}} = \partial \sum_{j=1}^{m_b} b_j = \sum_{j=1}^{m_b} \partial b_j, \tag{5.78}$$

so one can decompose $\partial b_{\mathcal{O}}$ into simplices as follows:

$$\partial(b_{\mathcal{O}}) = \sum_{l=1}^{m_b} d_0 b_l - d_1 b_l + d_2 b_l = \sum_{j=1}^{m_{\partial b}} z_j. \tag{5.79}$$

Importantly, as $\text{im}(\partial b_j) \subseteq O_{i_j}$ and it is a boundary, we can choose $i_{j'} = i_j$ in the latter decomposition of 5.79. By the linearity of h_i and h_{ij} , one can write

$$\begin{aligned} f'(\partial(b_{\mathcal{O}})) &= \sum_{j=1}^{m_b} h_{i_j}(\partial b_j) + h_{i_j i_j}(\text{d}_0 \partial b_j) \mod \mathbb{Z} = \sum_{j=1}^{m_b} \delta h_{i_j}(b_j) + c_{i_j i_j i_j}(\text{d}_0 \partial b_j) \mod \mathbb{Z} \\ &= \sum_{j=1}^{m_b} (\omega_{i_j} - c_{i_j})(b_j) \mod \mathbb{Z} = \omega_f(b_{\mathcal{O}}) \mod \mathbb{Z} = \iota(\omega_f)(b_{\mathcal{O}}), \end{aligned} \quad (5.80)$$

where we use $c_{iii} = h_{ii} - h_{ii} + h_{ii} = h_{ii}$. This completes the proof of well-definedness of 5.63. It remains to show that the assignment of f is an isomorphism. First, let $f_M \in \hat{H}^2(M, \mathbb{R}/\mathbb{Z})$ be a character represented by a cocycle $(c, h, \omega) \in \text{DC}_2^2(M)$, i.e., $f_M = h|_{Z_1(M)} \mod \mathbb{Z}$. On the other hand, the character assigned to $p([(c, h, \omega)])$ can be written as (cf. 5.59)

$$\begin{aligned} f(z_{\mathcal{O}} + \partial b) &= \sum_{j=1}^{m_z} h_{i_j}(z_j) \mod \mathbb{Z} + \iota(\omega)(b) \\ &= \sum_{j=1}^{m_z} h|_{O_{i_j}}(z_j) \mod \mathbb{Z} + \iota(\omega)(b) \\ &= \sum_{j=1}^{m_z} h(z_j) \mod \mathbb{Z} + \iota(\omega)(b) \\ &= f_M(z_{\mathcal{O}} + \partial b). \end{aligned} \quad (5.81)$$

It follows that our map $H_{\text{tot}}^2(\text{DC}_2^{\bullet}(\mathcal{O}_{\bullet})_{\text{T}}) \rightarrow H^2(\text{DC}_2^{\bullet}(M))$ is surjective. It is now enough to show injectivity. Suppose that for a cocycle 5.61 the corresponding character f is zero. That is, in particular

$$f'(z_{\mathcal{O}}) = \sum_{j=1}^{m_z} h_{i_j}(z_j) + h_{i_j i_j}(\text{d}_0 z_j) \mod \mathbb{Z} = 0 \quad (5.82)$$

for each $z_{\mathcal{O}} \in Z_1^{\mathcal{O}}(M)$. Note that for $z = z_{\mathcal{O}}$, we may take $b = 0$. By considering a cycle z_{O_i} supported in O_i , picking $i_{j'} = i_j$, and using the fact that $h_{ii} = c_{iii} \in C^0(\mathcal{O}_1, \mathbb{Z})$ we find that $h_i(z_{O_i}) \equiv 0 \pmod{\mathbb{Z}}$ for any such z_{O_i} . Therefore, by Universal Coefficient Theorem, there exist $b_i \in C^0(\mathcal{O}_0, \mathbb{R})$ and $c'_i \in C^1(\mathcal{O}_0, \mathbb{Z})$ such that $h_i = -\delta b_i + c'_i$. The expression for f' becomes

$$\begin{aligned} f'(z_{\mathcal{O}}) &= \sum_{j=1}^{m_z} b_{i_j}(\text{d}_1 z_j) - b_{i_j}(\text{d}_0 z_j) + h_{i_j i_j}(\text{d}_0 z_j) \mod \mathbb{Z} \\ &= \sum_{j=1}^{m_z} b_{i_{j'}}(\text{d}_1 z_{j'}) - b_{i_j}(\text{d}_0 z_j) + h_{i_{j'} i_j}(\text{d}_0 z_j) \mod \mathbb{Z} \\ &= \sum_{j=1}^{m_z} b_{i_{j'}}(\text{d}_0 z_j) - b_{i_j}(\text{d}_0 z_j) + h_{i_{j'} i_j}(\text{d}_0 z_j) \mod \mathbb{Z} \\ &= \sum_{j=1}^{m_z} (-\delta^0 b)_{i_{j'} i_j} + h_{i_{j'} i_j}(\text{d}_0 z_j) \mod \mathbb{Z}. \end{aligned} \quad (5.83)$$

Since the z_j are arbitrary, it follows that $h_{ij} = (\delta^0 b)_{ij} + d_{ij}$ for $d_{ij} \in C^0(M, \mathbb{Z})$. Moreover, by cocyclicity $\text{d}(c_i, h_i, \omega_i) = 0$, we obtain

$$\text{d}(c_i, -\delta b_i + c'_i, \omega_i) = (\delta c_i, \omega_i - c_i - \delta c'_i, \text{d}\omega_i) = 0, \quad (5.84)$$

5.2. Equivariant Structures on $\mathcal{G}$

The authors of [GSW13] establish a fundamental link between equivariant structures on the action groupoid (formulated in terms of geometric realizations of differential characters – bundle gerbes), including a weaker condition, in which a gerbe is equivariant up to a trivial gerbe corresponding to a global differential 2-form, which is known to satisfy an extra condition of multiplicativity to be defined (for more general Lie-groupoid) below. Here, we translate their constructions to the language of differential characters, and we replace the action groupoid by an arbitrary Lie groupoid.

We observe that any Lie groupoid comes with a structure of a simplicial manifold. Indeed, since s, t are surjective submersions, the fibered product of composable morphisms $\text{Mor } \mathcal{G} \times_s \text{Mor } \mathcal{G} \times_s \dots \times_s \text{Mor } \mathcal{G}$ is in fact a smooth manifold. This enables us to state the following

Definition 5.2.1. Given a Lie groupoid $\mathcal{G}$ we construct a simplicial manifold ([GM02, Ch. I]) $\mathcal{G}_\bullet : \Delta \rightarrow \mathbf{Man}$ with the object component

$$\mathcal{G}_n \equiv \begin{cases} \text{Ob } \mathcal{G} & n = 0, \\ \underbrace{\text{Mor } \mathcal{G} \times_s \text{Mor } \mathcal{G} \times_s \dots \times_s \text{Mor } \mathcal{G}}_n & n \neq 0, \end{cases} \quad (5.91)$$

and the morphism component fixed by a choice of the face and degeneracy maps. For $n = 1$ the face maps are $d_0^{(1)} = s$ and $d_1^{(1)} = t$. For $n > 1$ we define them as

$$d_i^{(n)} : \mathcal{G}_n \rightarrow \mathcal{G}_{n-1} : (g_1, g_2, \dots, g_n) \mapsto \begin{cases} (g_2, g_3, \dots, g_n) & i = 0, \\ (g_1, \dots, g_i \circ g_{i+1}, \dots, g_n) & 0 < i < n, \\ (g_1, g_2, \dots, g_{n-1}) & i = n. \end{cases} \quad (5.92)$$

The degeneracy maps are

$$s_i^{(n)} : \mathcal{G}_n \rightarrow \mathcal{G}_{n+1} : (g_1, g_2, \dots, g_n) \mapsto (g_1, g_2, \dots, g_i, \text{Id}(t(g_i)), g_{i+1}, \dots, g_n) \quad (5.93)$$

for $n > 0$ and $0 \leq i \leq n$, and $s_0^{(0)} = \text{Id}$. From now on, we will use the shorthand notation $\mathcal{G}$ for $\mathcal{G}_1 = \text{Mor } \mathcal{G}$.

Definition 5.2.2 ([BSS76]). The *Bott-Shulman-Stasheff complex* of a Lie groupoid $\mathcal{G}$ is the double complex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow \delta^2 & & \uparrow \delta^2 & & \uparrow \delta^2 & \\ \Omega^0(\mathcal{G}_2) & \xrightarrow{d^0} & \Omega^1(\mathcal{G}_2) & \xrightarrow{d^1} & \Omega^2(\mathcal{G}_2) & \xrightarrow{d^2} & \dots \\ & \uparrow \delta^1 & & \uparrow \delta^1 & & \uparrow \delta^1 & \\ \Omega^0(\mathcal{G}_1) & \xrightarrow{-d^0} & \Omega^1(\mathcal{G}_1) & \xrightarrow{-d^1} & \Omega^2(\mathcal{G}_1) & \xrightarrow{-d^2} & \dots \\ & \uparrow \delta^0 & & \uparrow \delta^0 & & \uparrow \delta^0 & \\ \Omega^0(\mathcal{G}_0) & \xrightarrow{d^0} & \Omega^1(\mathcal{G}_0) & \xrightarrow{d^1} & \Omega^2(\mathcal{G}_0) & \xrightarrow{d^2} & \dots \end{array} \quad (5.94)$$

where the horizontal arrows are just signed components of the de Rham differential, while

$$\delta^k := \sum_{i=0}^{k+1} (-1)^i \left(d_i^{(k+1)} \right)^* \quad (5.95)$$

is the Dupont operator. We say that a form $\rho \in \Omega^k(\mathcal{G})$ is *multiplicative* if $\delta^1(\rho) = 0$. Comparing with Definition 5.2.1, we see that this condition is equivalent to

$$m^*\rho = \text{pr}_1^*\rho + \text{pr}_2^*\rho, \quad (5.96)$$

where $\text{pr}_i : \mathcal{G}_2 = \text{Mor } \mathcal{G} \times_s \text{Mor } \mathcal{G} \rightarrow \text{Mor } \mathcal{G}$ are the canonical projections. If $\phi \in \Omega^{k+1}(\text{Ob } \mathcal{G})$ is closed, we say that $\rho \in \Omega^k(\mathcal{G})$ is ϕ -relatively closed if

$$d\rho = \delta^0(\phi), \quad (5.97)$$

or, in other words, $d\rho = s^*\phi - t^*\phi$. Note that a multiplicative ϕ -relatively closed form ρ amounts to the condition that $\rho + \phi$ be a $(k+1)$ -cocycle with respect to the differential of the total Bott-Shulman-Stasheff complex:

$$D^k = \sum_{i=0}^k (\delta^i + (-1)^i d^{k-i}). \quad (5.98)$$

Definition 5.2.3. Given a multiplicative form $\rho \in \Omega^{k-1}(\mathcal{G})$, we call a differential character $f \in \hat{H}^k(\text{Ob } \mathcal{G}, \mathbb{R}/\mathbb{Z})$ ρ -pre-equivariant if

$$s^*f - t^*f = i_2([\rho]). \quad (5.99)$$

By naturality of the map δ_1 , the following diagram is commutative:

$$\begin{array}{ccc} \hat{H}^k(\mathcal{G}, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta_1} & \Omega_{\mathbb{Z}}^k(\mathcal{G}) \\ \delta_{\hat{H}^k}^0 \uparrow & & \uparrow \delta^0 \\ \hat{H}^k(\text{Ob } \mathcal{G}, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\delta_1} & \Omega_{\mathbb{Z}}^k(\text{Ob } \mathcal{G}), \end{array} \quad (5.100)$$

where

$$\delta_{\hat{H}^k}^0 := \hat{H}^k(s, \mathbb{R}/\mathbb{Z}) - \hat{H}^k(t, \mathbb{R}/\mathbb{Z}). \quad (5.101)$$

Now, since $\delta_1 \circ i_2 = d$, the commutativity of the above diagram shows that whenever f is ρ -pre-equivariant, the form ρ is $\delta_1(f)$ -relatively closed.

Remark 5.2.4. By the exactness of the sequence

$$0 \rightarrow \Omega^{k-1}/\Omega_{\mathbb{Z}}^{k-1} \xrightarrow{i_2} \hat{H}^k(\cdot, \mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_2} H^k(\cdot, \mathbb{Z}) \rightarrow 0 \quad (5.102)$$

we observe that for any $f \in \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$, a form $\rho \in \Omega^k(\mathcal{G})$ satisfying 5.99 exists if and only if

$$\delta_2\left(\delta_{\hat{H}^k}^0(f)\right) = 0. \quad (5.103)$$

In other words, the characteristic class of $\delta_{\hat{H}^k}^0(f)$ measures the obstruction to the pre-equivariance of f . We will say that a character f satisfying 5.103 is *relatively pre-equivariant*.

In order to define equivariant differential characters, we construct a double complex $DC_k^\bullet(\mathcal{G}_\bullet)_T$:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
DC_k^0(\mathcal{G}_k) & \xrightarrow{(-1)^k d} & DC_k^1(\mathcal{G}_k) & \xrightarrow{(-1)^k d} & DC_k^2(\mathcal{G}_k) & \longrightarrow & \dots \\
& \uparrow \delta^{k-1} & & \uparrow \delta^{k-1} & & \uparrow \delta^{k-1} & \\
& \vdots & & \vdots & & \vdots & \\
& \uparrow \delta^2 & & \uparrow \delta^2 & & \uparrow \delta^2 & \\
DC_k^0(\mathcal{G}_2) & \xrightarrow{d} & DC_k^1(\mathcal{G}_2) & \xrightarrow{d} & DC_k^2(\mathcal{G}_2) & \longrightarrow & \dots \\
& \uparrow \delta^1 & & \uparrow \delta^1 & & \uparrow \delta^1 & \\
DC_k^0(\mathcal{G}_1) & \xrightarrow{-d} & DC_k^1(\mathcal{G}_1) & \xrightarrow{-d} & DC_k^2(\mathcal{G}_1) & \longrightarrow & \dots \\
& \uparrow \delta^0 & & \uparrow \delta^0 & & \uparrow \delta^0 & \\
DC_k^0(\mathcal{G}_0) & \xrightarrow{d} & DC_k^1(\mathcal{G}_0) & \xrightarrow{d} & DC_k^2(\mathcal{G}_0) & \longrightarrow & \dots
\end{array} \tag{5.104}$$

Definition 5.2.5. An *equivariant differential character* on $\mathcal{G}$ is a character $f \in \hat{H}^k(\mathcal{G}_0, \mathbb{R}/\mathbb{Z})$, whose class extends to a class in $H_{\text{tot}}^k(DC_k^\bullet(\mathcal{G}_\bullet)_T)$. Given a multiplicative $\rho \in \Omega^{k-1}(\mathcal{G})$, we say that f is ρ -equivariant if one can extend any cocycle representing f to a cochain $\mathfrak{C}$ in $DC_k^\bullet(\mathcal{G}_\bullet)_T$ whose total derivative is concentrated in $DC_k^k(\mathcal{G}_1)$ and equals $(0, \rho, d\rho)$. This means that $D\mathfrak{C}$ equals

$$\begin{array}{c}
\vdots \\
(0, 0, 0)
\end{array} \tag{5.105}$$

$$(0, \rho, d\rho).$$

Remark 5.2.6. Any ρ -equivariant character f is also ρ -pre-equivariant. Indeed, let (c, h, ω) be a cocycle representing f . Then, by Definition 5.2.5,

$$s^*(c, h, \omega) - t^*(c, h, \omega) - d(a, b, 0) = (0, \rho, d\rho), \tag{5.106}$$

where $(a, b, 0)$ is the first step of the extension of (c, h, ω) . Taking classes in cohomology of both sides in 5.106 we see that $s^*f - t^*f = i_2([\rho])$, as $d(a, b, 0)$ is exact. Clearly, a 0-equivariant character is just an equivariant character. In this sense, we generalize the notion of equivariance of characters studied in [LM08b].

We will consider a special class of bisections, to which we will later reduce the structure group of a principaloid bundle. We expect the rigid symmetries from $\mathbb{B}(\mathcal{G})$, modeled by the γ_i , to preserve ω_f , which has a concrete interpretation of the (Maxwell) field strength. We calculate

$$0 \stackrel{!}{=} (t \circ \beta)^* \omega_f - \omega_f = (t \circ \beta)^* \omega_f - (s \circ \beta)^* \omega_f = \beta^*(t^* \omega_f - s^* \omega_f). \tag{5.107}$$

But for a ρ -pre-equivariant character this translates to (5.106):

$$0 = \beta^*(\delta_2(0, \rho, d\rho)) = \beta^*d\rho. \quad (5.108)$$

In [GSW13], the distinguished bisections from Example 2.1.15 were used to model gauged transformations. These can be shown to satisfy the stronger condition $\beta^*\rho = 0$. This motivates us to restrict our attention to this class of bisections in what follows.

Lemma 5.2.7. *Given a multiplicative form $\rho \in \Omega^k(\mathcal{G})$, the set*

$$\mathbb{B}_\rho(\mathcal{G}) := \{\beta \in \mathbb{B}(\mathcal{G}) : \beta^*\rho = 0\} \quad (5.109)$$

is a subgroup of $\mathbb{B}(\mathcal{G})$.

Proof. Since $m \circ (\text{Id}, \text{Id}) = \text{Id}$, we get

$$\text{Id}^*\rho = \text{Id}^*\rho + \text{Id}^*\rho, \quad (5.110)$$

so $\text{Id}^*\rho = 0$ and $\text{Id} \in \mathbb{B}_\rho(\mathcal{G})$. Now, let $\beta_1, \beta_2 \in \mathbb{B}_\rho(\mathcal{G})$. We compute

$$(\beta_1 \cdot \beta_2)^*\rho = (m \circ (\beta_1 \circ t \circ \beta_2, \beta_2))^*\rho = (t \circ \beta_2)^*\beta_1^*\rho + \beta_2^*\rho = 0. \quad (5.111)$$

Finally, for $\beta \in \mathbb{B}_\rho(\mathcal{G})$ we check

$$\begin{aligned} (\beta^{-1})^*\rho &= (i \circ \beta \circ (t \circ \beta)^{-1})^*\rho = ((t \circ \beta)^{-1})^*\beta^*i^*\rho \\ &= ((t \circ \beta)^{-1})^*\beta^*((m \circ (i, \text{id}_\mathcal{G}))^*\rho - \rho) = ((t \circ \beta)^{-1})^*\beta^*(\text{Id}^*\rho - \rho) = 0. \end{aligned} \quad (5.112)$$

This completes the proof. $\square$

We will refer to elements of $\mathbb{B}_\rho(\mathcal{G})$ as ρ -holonomic bisections. It is worth noting that holonomic bisections play a role in the study of the structural relation [CSS12] between multiplicative forms and the *Spencer operators* on Lie groupoids. The relation will be implicit in our considerations below.

Proposition 5.2.8. *Let $\mathcal{G}$ be a Lie groupoid equipped with a multiplicative form $\rho \in \Omega^k(\mathcal{G})$. Then for any $\beta \in \mathbb{B}_\rho(\mathcal{G})$ we have $C_\beta^*\rho = \rho$, where C_β denotes the adjoint action of $\mathbb{B}(\mathcal{G})$ on $\mathcal{G}$:*

$$\forall g \in \mathcal{G} : C_\beta(g) \equiv \beta(t(g)) \circ g \circ \beta(s(g))^{-1}. \quad (5.113)$$

Proof. Consider a map $F : \mathcal{G} \rightarrow \mathcal{G}_{s \times t} \mathcal{G}_{s \times t} \mathcal{G}$ given by

$$F(g) \equiv (\beta(t(g)), g, \beta(s(g))^{-1}). \quad (5.114)$$

The action $C_\beta(g)$ can be viewed as the composition $(m^{(3)} \circ F)(g)$, where $m^{(3)} = m \circ (m \times \text{id}_\mathcal{G})$. Thus, we have

$$C_\beta^*\rho = (m^{(3)} \circ F)^*\rho = F^*(m^{(3)})^*\rho = F^*(\text{pr}_1^*\rho + \text{pr}_2^*\rho + \text{pr}_3^*\rho), \quad (5.115)$$

where the last equality follows from the multiplicativity of ρ . Indeed,

$$(m^{(3)})^*\rho = (m \times \text{id}_\mathcal{G})^*m^*\rho = (m \times \text{id}_\mathcal{G})^*(\text{pr}_1^*\rho + \text{pr}_2^*\rho) = \text{pr}_1^*\rho + \text{pr}_2^*\rho + \text{pr}_3^*\rho. \quad (5.116)$$

We compute

$$\text{pr}_1 \circ F = \beta \circ t, \quad (5.117)$$

$$\mathrm{pr}_2 \circ F = \mathrm{id}_{\mathcal{G}}, \quad (5.118)$$

$$\mathrm{pr}_3 \circ F = i \circ \beta \circ s. \quad (5.119)$$

This applies to ρ as follows:

$$(\mathrm{pr}_1 \circ F)^* \rho = (\beta \circ t)^* \rho = t^* \beta^* \rho = 0, \quad (5.120)$$

$$(\mathrm{pr}_2 \circ F)^* \rho = \mathrm{id}_{\mathcal{G}}^* \rho = \rho, \quad (5.121)$$

$$(\mathrm{pr}_3 \circ F)^* \rho = (i \circ \beta \circ s)^* \rho = s^* \beta^* i^* \rho = s^* \beta^* (-\rho) = 0. \quad (5.122)$$

These results yield

$$C_{\beta}^* \rho = \rho. \quad (5.123)$$

□

Whenever there exists a ρ -equivariant structure on $\mathcal{G}$, we will consider another subgroup of $\mathbb{B}(\mathcal{G})$.

Lemma 5.2.9 ([CSS12]). *Let $\rho \in \Omega^k(\mathcal{G})$ be a multiplicative form and let $v_1 \in (\ker \mathrm{Ts})_g$ for some $g \in \mathcal{G}$. Then for any $v_2, \dots, v_k \in \mathrm{T}_g \mathcal{G}$ the following equality holds:*

$$\begin{aligned} \rho(g)(v_1, v_2, \dots, v_k) \\ = \rho(\mathrm{Id}(t(g))) (\mathrm{T}_g r_{g^{-1}}(v_1), (\mathrm{T}_{t(g)} \mathrm{Id} \circ \mathrm{T}_g t)(v_2), \dots, (\mathrm{T}_{t(g)} \mathrm{Id} \circ \mathrm{T}_g t)(v_k)). \end{aligned} \quad (5.124)$$

Proof. We invoke the identity

$$\mathrm{T}_{(g,h)} m(v_g, v_h) = \mathrm{T}_h l_g(v_h) + \mathrm{T}_g r_h(v_g). \quad (5.125)$$

Note that, unlike for Lie groups, we must choose g and h to be composable, and we must demand that $(v_g, v_h) \in (\ker \mathrm{Ts})_g \oplus (\ker \mathrm{T}t)_h \subseteq \mathrm{T}_{(g,h)}(\mathcal{G}_{s \times t} \mathcal{G})$. Otherwise, 5.125 is not well-defined. Using 5.125, we express

$$v_1 = (\mathrm{T}_{\mathrm{Id}(t(g))} r_g \circ \mathrm{T}_g r_{g^{-1}})(v_1) = \mathrm{T}_{(\mathrm{Id}(t(g)),g)} m(\mathrm{T}_g r_{g^{-1}}(v_1), 0_g). \quad (5.126)$$

Then, we differentiate

$$\mathrm{id}_{\mathcal{G}} = m \circ ((\mathrm{Id} \circ t) \times \mathrm{id}_{\mathcal{G}}) \quad (5.127)$$

to obtain

$$v_j = \mathrm{T}_{(\mathrm{Id}(t(g)),g)} m((\mathrm{T}_{t(g)} \mathrm{Id} \circ \mathrm{T}_g t)(v_j), v_j), \quad 2 \leq j \leq k. \quad (5.128)$$

Equation 5.124 is obtained by acting with ρ on 5.126 and 5.128, and using the multiplicativity of the ρ . □

Lemma 5.2.10. *Let $\rho \in \Omega^1(\mathcal{G})$ be multiplicative, and let θ_{R} be the right-invariant Maurer-Cartan form (Definition 2.2.18). Then, for every $g \in \mathcal{G}$ and $v \in (\ker \mathrm{Ts})_g$, the following equality holds:*

$$\mathrm{Id}^*(\iota_{(\iota_{\mathrm{R}}(\theta_{\mathrm{R}}(v)))} \rho)(t(g)) = \rho(g)(v), \quad (5.129)$$

where we make use of the isomorphism ι_{R} introduced in 2.34 of Definition 2.2.16.

Proof. This follows by direct calculation and using Lemma 5.2.9:

$$\mathrm{Id}^*(\iota_{(\iota_{\mathrm{R}}(\theta_{\mathrm{R}}(v)))} \rho)(t(g)) = \rho(\mathrm{Id}(t(g))(\mathrm{T}_g r_{g^{-1}}(v))) = \rho(g)(v). \quad (5.130)$$

□

5.3. Constructing the DF-amplitude

If Σ is a compact, oriented and (cohomologically) closed¹ n -dimensional smooth manifold, then there exists a smooth triangulation of Σ , and the corresponding fundamental chain c_Σ is a smooth cycle [Hat02, 3.3]. Consequently, any smooth map $\phi \in C^\infty(\Sigma, M)$ induces a smooth cycle $\phi_*c_\Sigma \in Z_n(M)$. Consider $f \in \hat{H}^{n+1}(M, \mathbb{R}/\mathbb{Z})$, and $(\Sigma, (M, g, H), \mathcal{M}_{\text{DF}})$ be an n -dimensional sigma model. Define the topological term of the DF-amplitude as

$$\chi_{\omega_f}(\phi) = f(\phi_*c_\Sigma). \quad (5.131)$$

A physical justification for adopting the above form of the topological term of the DF-amplitude follows from the condition imposed on χ_{ω_f} by Definition 3.0.1. The postulate of non-interaction of trajectories separated in the target space implies additivity (in $U(1)$) in ϕ_*c_Σ . The other condition is that the variational contribution of the topological term should be

$$\frac{1}{i} \delta \log \chi_{\omega_f}(\phi) = \phi^*(\iota_{\delta\phi} \omega_f). \quad (5.132)$$

We model the variation on a smooth vector field $\delta\phi \equiv \mathcal{V} \in \Gamma(TM)$ and the corresponding flow $\Phi_{\mathcal{V}}^t$. Let $K^t(\phi, c_\Sigma) := \{(\Phi_{\mathcal{V}}^{t'} \circ \phi)_*c_\Sigma : 0 \leq t' \leq t\}$ and calculate:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \log f((\Phi_{\mathcal{V}}^t \circ \phi)_*c_\Sigma) &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\log f((\Phi_{\mathcal{V}}^t \circ \phi)_*c_\Sigma) - \log f(\phi_*c_\Sigma) \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \log f\left(\partial\{(\Phi_{\mathcal{V}}^{t'} \circ \phi)_*c_\Sigma : 0 \leq t' \leq t\}\right) = \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_{K^t(\phi, c_\Sigma)} \omega_f \right) \\ &= \phi^* \iota_{\mathcal{V}}(\omega_f). \end{aligned} \quad (5.133)$$

The purpose of the gauging is to construct a gauge-invariant DF-amplitude on sections of the shadow bundle $\mathcal{F}$ of a principaloid $\mathcal{G}$ -bundle for $\text{Ob } \mathcal{G} = M$. This is conceptually motivated by Example 2.1.15 and the subsequent remark. The case when $\mathcal{G}$ is an action groupoid and $n = 2$ was studied exhaustively in [GSW10] and [GSW13], and it is known that, in that case, the bundle $\mathcal{F}$ coincides with a smooth realization $P \times_\lambda M$ of the homotopy quotient of [Car50] (see also [Tu20]).

¹The assumption of closedness for the spacetime Σ of the sigma model does not correspond to ordinary trajectories of the probe charge, but instead to compositions of cobordant trajectories in a model of Aharonov-Bohm type experiment. We make the assumption for the sake of simplicity. The variant with spacetimes with boundary calls for relative version of characters. See [RS09], [GSW13] for the 2-dimensional analogue.

Conjecture 5.3.1. For any $n \geq 1$, let $\rho \in \Omega^n(\mathcal{G})$ be a multiplicative form on a Lie groupoid $\mathcal{G}$ and let $f \in \hat{H}^{n+1}(M, \mathbb{R}/\mathbb{Z})$ be a ρ -pre-equivariant differential character on $M = \text{Ob } \mathcal{G}$. For any principaloid $\mathcal{G}$ -bundle $\mathcal{P} \rightarrow \Sigma$ with compatible connection 1-form $\mathcal{A}$ and structure group reduced to $\mathbb{B}_\rho(\mathcal{G})$, there exists a gauge-invariant differential character $f_i^{\mathcal{A}}$ on the pullback of a trivializing cover $\pi_{\mathcal{F}}^{-1}\mathcal{O} \rightarrow \mathcal{F}$, satisfying $\delta^0(f_i^{\mathcal{A}}) = 0$. Its characteristic class is a pullback of that of f .

Theorem 5.3.2. Conjecture 5.3.1 holds for $n = 1$.

Proof. Note that the form ρ induces the following morphism:

$$\kappa : \Gamma(E_{\mathbb{R}}) \rightarrow C^\infty(M) : \alpha \mapsto \text{Id}^*(\iota_{\iota_{\mathbb{R}}(\alpha)}(\rho)), \quad (5.134)$$

Let $\mathcal{O} = \bigsqcup_{i \in I} O_i$ be a common trivializing cover of the base Σ for the bundles $\mathcal{P}$ and $\mathcal{F}$, and let $\pi_{\mathcal{F}}^{-1}\mathcal{O}$ be its pullback by $\pi_{\mathcal{F}}$. Fix a trivialization $\{\mathcal{F}\tau_i\}$ of $\mathcal{F}$. Let $\{A_i \in \Gamma(\text{pr}_1^*\text{T}^*O_i \otimes \text{pr}_2^*E_{\mathbb{R}})\}$ be the local connection data of $\mathcal{A}$ for $\mathcal{O}$. We define forms $\rho^{A_i} \in \Omega^1(O_i \times M)$ by

$$\rho^{A_i}(\sigma, m) = \kappa(m) \circ A_i(\sigma, m), \quad (5.135)$$

which we will write, in a shorthand notation, as $\text{pr}_2^*\kappa \circ A_i$.

We construct a collection of differential characters $f_i^{\mathcal{A}}$ (in what follows we write $i_2(\omega) := i_2([\omega])$ for any differential form ω). Over $\pi_{\mathcal{F}}^{-1}(O_i)$, we set

$$f_i^{\mathcal{A}} := (\mathcal{F}\tau_i)^*(\text{pr}_2^*f - i_2(\rho^{A_i})) \in \hat{H}^2(\pi_{\mathcal{F}}^{-1}(O_i), \mathbb{R}/\mathbb{Z}). \quad (5.136)$$

Our next goal is to show that these characters agree on intersections, so that $\delta^0(f_i^{\mathcal{A}}) = 0$. Note that the characteristic class of $f_i^{\mathcal{A}}$ is

$$\delta_2(f_i^{\mathcal{A}}) = (\text{pr}_2 \circ \mathcal{F}\tau_i)^*\delta_2(f), \quad (5.137)$$

because $\text{im}(i_2) \subseteq \ker(\delta_2)$. Consider an intersection $\pi_{\mathcal{F}}^{-1}(O_{ij}) = \pi_{\mathcal{F}}^{-1}(O_i) \cap \pi_{\mathcal{F}}^{-1}(O_j)$. We observe that $\mathcal{F}\tau_i \circ \mathcal{F}\tau_j^{-1}|_{O_{ij} \times M} = \Lambda_{\beta_{ij}}$, where

$$\Lambda_{\beta_{ij}} = (\text{pr}_1, t \circ \text{ev} \circ (\beta_{ij} \times \text{id}_M)) : O_{ij} \times M \rightarrow O_{ij} \times M : (\sigma, m) \mapsto (\sigma, (t \circ \beta_{ij}(\sigma))(m)), \quad (5.138)$$

with $\beta_{ij} : O_{ij} \rightarrow \mathbb{B}_\rho(\mathcal{G})$ being elements of the transition 1-cocycle of $\mathcal{P}$. In order for

$$f_i^{\mathcal{A}} = f_j^{\mathcal{A}} \quad (5.139)$$

over $\pi_{\mathcal{F}}^{-1}(O_{ij})$, it suffices that

$$\begin{aligned} (\mathcal{F}\tau_i)^*(\text{pr}_2^*f - i_2(\rho^{A_i}))|_{\pi_{\mathcal{F}}^{-1}(O_{ij})} &= (\mathcal{F}\tau_j)^*\Lambda_{\beta_{ij}}^*(\text{pr}_2^*f - i_2(\rho^{A_i}))|_{\pi_{\mathcal{F}}^{-1}(O_{ij})} \\ &\stackrel{!}{=} (\mathcal{F}\tau_j)^*(\text{pr}_2^*f - i_2(\rho^{A_j}))|_{\pi_{\mathcal{F}}^{-1}(O_{ij})}, \end{aligned} \quad (5.140)$$

which reduces to the condition that

$$\Lambda_{\beta_{ij}}^*(\text{pr}_2^*f - i_2(\rho^{A_i}))|_{O_{ij} \times M} = (\text{pr}_2^*f - i_2(\rho^{A_j}))|_{O_{ij} \times M}. \quad (5.141)$$

We proceed with the calculation of the left-hand side:

$$\begin{aligned} \Lambda_{\beta_{ij}}^*(\text{pr}_2^*f - i_2(\rho^{A_i})) &= (t \circ \text{ev} \circ (\beta_{ij} \times \text{id}_M))^*f - i_2(\Lambda_{\beta_{ij}}^*\rho^{A_i}) \\ &= (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*t^*f - i_2(\Lambda_{\beta_{ij}}^*\rho^{A_i}) \\ &= (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*s^*f - (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*i_2(\rho) - i_2(\Lambda_{\beta_{ij}}^*\rho^{A_i}) \\ &= (s \circ \text{ev} \circ (\beta_{ij} \times \text{id}_M))^*f - i_2((\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*\rho) - i_2(\Lambda_{\beta_{ij}}^*\rho^{A_i}) \\ &= \text{pr}_2^*f - i_2((\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*\rho + \Lambda_{\beta_{ij}}^*\rho^{A_i}), \end{aligned} \quad (5.142)$$

where we use the ρ -pre-equivariance of f and the naturality² of i_2 (5.16). Clearly, it is enough to show that

$$(\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* \rho + \Lambda_{\beta_{ij}}^* \rho^{A_i} = \rho^{A_j}. \quad (5.143)$$

Before continuing, note that while $\text{im}(\text{T}(\text{ev} \circ (\beta_{ij} \times \text{id}_M))) \not\subseteq \ker \text{T}s$, when we investigate the decomposition into $\ker \text{T}s$ and its transverse:

$$\text{T}_{(\sigma, m)}(\text{ev} \circ (\beta_{ij} \times \text{id}_M))(v, w) = \text{T}_\sigma(\text{ev}_m \circ \beta_{ij})(v) + \text{T}_m \beta_{ij}(\sigma)(w), \quad (5.144)$$

where $(v, w) \in \text{T}_\sigma O_{ij} \oplus \text{T}_m M \simeq \text{T}_{(\sigma, m)}(O_{ij} \times M)$, the ρ -holonomicity of $\beta_{ij}(\sigma)$ for any $\sigma \in O_{ij}$ implies³ that only the first term in 5.144 gives a nonzero contribution to the pullback of the 1-form ρ by $\text{ev} \circ (\beta_{ij} \times \text{id}_M)$. Now, we are ready to apply 2.42 to $\Lambda_{\beta_{ij}}^* \rho^{A_i}$. Let $(\sigma, m) \in O_{ij} \times M$, and compute

$$\begin{aligned} \Lambda_{\beta_{ij}}^* \rho^{A_i}(\sigma, m) &= \text{pr}_2^*(\kappa \circ \text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)}) \circ A_j(\sigma, m) - \text{pr}_2^*(\kappa \circ \theta_R \circ \text{T}_\sigma(\text{ev}_m \circ \beta_{ij})) \\ &= \text{pr}_2^*(\kappa \circ \text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)}) \circ A_j(\sigma, m) - \text{pr}_2^*(\rho \circ \text{T}_\sigma(\text{ev}_m \circ \beta_{ij})) \\ &= \text{pr}_2^*(\kappa \circ \text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)}) \circ A_j(\sigma, m) - (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* \rho, \end{aligned} \quad (5.145)$$

where we used Lemma 5.2.10 and the remark about holonomicity.

Comparing the result in 5.145 with 5.143, we make yet another reduction

$$\text{pr}_2^*(\kappa \circ \text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)}) \circ A_j(\sigma, m) = \rho^{A_j}, \quad (5.146)$$

Note that, by Proposition 5.2.8,

$$\forall g \in \mathcal{G} \forall v \in \text{T}_g \mathcal{G} \forall \sigma \in O_{ij} : \rho(g)(v) = \rho(C_{\beta_{ij}(\sigma)}(g))(\text{T}_g C_{\beta_{ij}(\sigma)}(v)), \quad (5.147)$$

so, in particular, for any $m \in M$ and $w \in \text{T}_m M$, we have

$$\rho(\text{Id}(m))(\text{T}_m \text{Id}(w)) = \rho((C_{\beta_{ij}(\sigma)} \circ \text{Id})(m))((\text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)} \circ \text{T}_m \text{Id})(w)). \quad (5.148)$$

Combining this with 2.37 we obtain

$$\rho(\text{Id}(m))(\text{T}_m \text{Id}(w)) = \rho(\text{Id}(t(\beta_{ij}(\sigma)(m))))((\text{T}_{\text{Id}(m)} C_{\beta_{ij}(\sigma)} \circ \text{T}_m \text{Id})(w)), \quad (5.149)$$

which proves 5.146. Thus, we managed to show that the local characters $f_i^{\mathcal{A}}$ agree on all intersections. Next, we prove that $f_i^{\mathcal{A}}$ are gauge-invariant:

$$\forall \Phi \in \text{Gauge}(\mathcal{P}) : (\mathcal{F}_*(\Phi)|_{\pi_{\mathcal{F}}^{-1}(O_i)})^* f_i^{\mathcal{A}^\Phi} = f_i^{\mathcal{A}}. \quad (5.150)$$

This follows simply from Proposition 2.2.21 and the fact that in 2.42 and 2.45 the maps β_{ij} play essentially the same role as the γ_i , the equality on intersections being now replaced by the equality on each element of the cover. Importantly, for each $i \in I$, we have $\text{im}(\gamma_i) \subseteq \mathbb{B}_\rho(\mathcal{G})$. $\square$

Example 5.3.3. The naturality of i_2 (5.16) implies that, whenever there exists $B \in \Omega^n(M)$ such that $f = i_2(B)$, the character f is ρ -pre-equivariant, with $\rho = \delta^0(B)$. Moreover, so defined ρ is manifestly multiplicative, as $\delta^1 \circ \delta^0 = 0$. This means that, for tensorial couplings, the ρ -pre-equivariance is automatic.

²This follows from the formula $\int_c f^* \omega = \int_{f \circ c} \omega$.

³The second term is of the kind $\beta^* \rho$, which is zero by ρ -holonomicity.

Theorem 5.3.4. *A differential character $f_i^A \in \hat{H}^2(\pi_{\mathcal{F}}^{-1}\mathcal{O}, \mathbb{R}/\mathbb{Z})$ that descends to $\mathcal{F}$ induces the topological term of a DF-amplitude $\mathcal{M}_{\text{DF}}^A$ on $\Gamma(\mathcal{F})$, which does not depend on the choice of a fundamental cycle of Σ . Moreover, for any $\Phi \in \text{Gauge}(\mathcal{P})$, the character $f_i^{A^\Phi}$ also descends to $\mathcal{F}$, and the topological term $\mathcal{M}_{\text{DF}, \text{top}}^A$ is gauge-invariant:*

$$\forall \varphi \in \Gamma(\mathcal{F}) : \mathcal{M}_{\text{DF}, \text{top}}^{A^\Phi}(\varphi^\Phi) = \mathcal{M}_{\text{DF}, \text{top}}^A(\varphi). \quad (5.151)$$

Proof. Let $f^A \in \hat{H}^2(\mathcal{F}, \mathbb{R}/\mathbb{Z})$ be the descended character, and let c_Σ be a fundamental cycle of Σ . We define $\mathcal{M}_{\text{DF}, \text{top}}^A$ by

$$\mathcal{M}_{\text{DF}, \text{top}}^A(\varphi) := f^A(\varphi_* c_\Sigma) = \varphi^* f^A(c_\Sigma). \quad (5.152)$$

By the exactness of

$$0 \rightarrow H^1(\Sigma, \mathbb{R}/\mathbb{Z}) \rightarrow \hat{H}^2(\Sigma, \mathbb{R}/\mathbb{Z}) \rightarrow \Omega_{\mathbb{Z}}^2(\Sigma) \rightarrow 0, \quad (5.153)$$

and since Σ is 1-dimensional, the character $\varphi^* f^A$ is given by a class in cohomology of Σ with coefficients in $\mathbb{R}/\mathbb{Z}$. It follows, that $\varphi^* f^A(c_\Sigma)$ depends only on the fundamental class $[c_\Sigma]$, that is—only on the topology of Σ . Now, let

$$\begin{aligned} (c_{ijk}^A, 0, 0) & \quad 0 \\ (c_{ij}^A, h_{ij}^A, 0) & \quad 0 \end{aligned} \quad (5.154)$$

$$(c_i^A, h_i^A, \omega_i^A) \quad 0$$

be a chosen cocycle representing f^A . By 5.150, there exists $(c_i^\Phi, h_i^\Phi, 0) \in \text{DC}_2^1((\pi_{\mathcal{F}}^{-1}\mathcal{O})_0)$ such that

$$\begin{aligned} (\mathcal{F}_*(\Phi))^*(c_i^{A^\Phi}, h_i^{A^\Phi}, \omega_i^{A^\Phi}) &:= \left(\mathcal{F}_*(\Phi) \Big|_{\pi_{\mathcal{F}}^{-1}(\mathcal{O}_i)} \right)^* (c_i^A, h_i^A, \omega_i^A) \\ &= (c_i^A, h_i^A, \omega_i^A) + d(c_i^\Phi, h_i^\Phi, 0), \end{aligned} \quad (5.155)$$

where $(c_i^{A^\Phi}, h_i^{A^\Phi}, \omega_i^{A^\Phi})$ represents $f_i^{A^\Phi}$. Thus, we may pick the cocycle

$$\begin{aligned} (c_{ijk}^A, 0, 0) \\ (c_{ij}^A, h_{ij}^A, 0) + \delta^0(c_i^\Phi, h_i^\Phi, 0) \end{aligned} \quad (5.156)$$

$$(\mathcal{F}_*(\Phi))^*(c_i^{A^\Phi}, h_i^{A^\Phi}, \omega_i^{A^\Phi}),$$

which is cohomologous to 5.154. Using the fact that $\mathcal{F}_*(\Phi)$ is a vertical automorphism of $\mathcal{F}$, we may pull back the above cocycle by $\mathcal{F}_*(\Phi^{-1}) = \mathcal{F}_*(\Phi)^{-1}$ and obtain a cocycle

$$\begin{aligned} (\mathcal{F}_*(\Phi^{-1}))^*(c_{ijk}^A, 0, 0) \\ (\mathcal{F}_*(\Phi^{-1}))^*((c_{ij}^A, h_{ij}^A, 0) + \delta^0(c_i^\Phi, h_i^\Phi, 0)) \\ (c_i^{A^\Phi}, h_i^{A^\Phi}, \omega_i^{A^\Phi}). \end{aligned} \quad (5.157)$$

We choose its global character as the descent of $f_i^{\mathcal{A}^\Phi}$. Again, by the verticality of $\mathcal{F}_*(\Phi)$, the local pullbacks commute through j with the global ones. We have constructed $f^{\mathcal{A}^\Phi}$ such that

$$(\mathcal{F}_*(\Phi))^* f^{\mathcal{A}^\Phi} = f^{\mathcal{A}}. \quad (5.158)$$

Therefore, we compute

$$\begin{aligned} \mathcal{M}_{\text{DF,top}}^{\mathcal{A}^\Phi}(\varphi^\Phi) &= f^{\mathcal{A}^\Phi}((\mathcal{F}_*(\Phi) \circ \varphi)_* c_\Sigma) = (\mathcal{F}_*(\Phi))^* f^{\mathcal{A}^\Phi}(\varphi_* c_\Sigma) = f^{\mathcal{A}}(\varphi_* c_\Sigma) \\ &= \mathcal{M}_{\text{DF,top}}^{\mathcal{A}}(\varphi). \end{aligned} \quad (5.159)$$

□

Conjecture 5.3.5. *For any $n \geq 1$, let $\rho \in \Omega^n(\mathcal{G})$ be a multiplicative form on a Lie groupoid $\mathcal{G}$, and let $f \in \dot{H}^{n+1}(M, \mathbb{R}/\mathbb{Z})$ be a ρ -equivariant differential character on $M = \text{Ob } \mathcal{G}$. For any principaloid $\mathcal{G}$ -bundle $\mathcal{P} \rightarrow \Sigma$ with compatible connection 1-form $\mathcal{A}$ and structure group reduced to $\mathbb{B}_\rho(\mathcal{G})$, the differential character $f_i^{\mathcal{A}}$ from Theorem 5.3.2 descends to $\mathcal{F}$ in a canonical way.*

Theorem 5.3.6. *Conjecture 5.3.5 holds for $n = 1$.*

Proof. We are going to utilize a simplicial manifold $(\mathcal{F}\tau)_\bullet : \Delta \rightarrow \mathbf{Man}$ induced by a trivialization $\mathcal{F}\tau : \mathcal{F} \rightarrow \mathcal{O} \times M$, and the fact that it is isomorphic to $(\pi_{\mathcal{F}}^{-1}\mathcal{O})_\bullet$. The object component of $(\mathcal{F}\tau)_\bullet$ is defined as

$$(\mathcal{F}\tau)_n = \mathcal{O}_n \times M. \quad (5.160)$$

The degeneracy maps are simply $s_i^{(n)} \times \text{id}_M$, where $s_i^{(n)}$ are the degeneracy maps of $\mathcal{O}_\bullet$. To define face maps, recall the diffeomorphisms

$$\mathcal{O}_n \simeq \bigsqcup_{\bar{i} \in I^{n+1}} \mathcal{O}_{i_0 i_1 \dots i_n} \quad \bar{i} = (i_0, \dots, i_n). \quad (5.161)$$

We can write elements of $\mathcal{O}_n$ as $(i_0, \dots, i_n, x)$ with $x \in \mathcal{O}_{i_0 i_1 \dots i_n}$. In this notation, we set

$$\begin{aligned} d_k^{(n)} : (\mathcal{F}\tau)_n &\rightarrow (\mathcal{F}\tau)_{n-1} : \\ (i_0, \dots, i_n, x, m) &\mapsto \begin{cases} (i_0, \dots, i_{k-1}, i_{k+1}, \dots, i_n, x, m) & 0 \leq k \leq n-1, \\ (i_0, \dots, i_{n-1}, \Lambda_{\beta_{i_{n-1} i_n}}(x, m)) & k = n, \end{cases} \end{aligned} \quad (5.162)$$

where $\Lambda_{\beta_{ij}} = \mathcal{F}\tau_i \circ \mathcal{F}\tau_j^{-1}$ are the transition maps 5.138. It is an easy check to verify that the above maps satisfy simplicial identities (Definition A.0.23), making $(\mathcal{F}\tau)_\bullet$ into a simplicial manifold. One uses the fact that $\mathcal{O}_\bullet$ is a simplicial manifold, leaving only the identities involving $d_n^{(n)}$ to be checked. But those follow from the cocyclicity of $\Lambda_{\beta_{ij}}$:

$$\Lambda_{\beta_{ik}} = \Lambda_{\beta_{ij}} \circ \Lambda_{\beta_{jk}} \quad \text{over } \mathcal{O}_{ijk}. \quad (5.163)$$

The isomorphism $\alpha : (\mathcal{F}\tau)_\bullet \rightarrow (\pi_{\mathcal{F}}^{-1}\mathcal{O})_\bullet$ has components

$$\alpha_n : (\mathcal{F}\tau)_n \rightarrow (\pi_{\mathcal{F}}^{-1}\mathcal{O})_n : (i_0, \dots, i_n, x, m) \mapsto (i_0, \dots, i_n, \mathcal{F}\tau_{i_n}^{-1}(x, m)). \quad (5.164)$$

Each component is, clearly, an isomorphism, since the $\mathcal{F}\tau_i$ are isomorphisms onto their images. To see that α is a map of simplicial manifolds, the only nontrivial check to be made is that

$$\begin{array}{ccc} (\mathcal{F}\tau)_{n-1} & \xleftarrow{d_n^{(n)}} & (\mathcal{F}\tau)_n \\ \downarrow \alpha_{n-1} & & \downarrow \alpha_n \\ (\pi_{\mathcal{F}}^{-1}\mathcal{O})_{n-1} & \xleftarrow{d_n^{(n)}} & (\pi_{\mathcal{F}}^{-1}\mathcal{O})_n \end{array} \quad (5.165)$$

commutes. We compute

$$\begin{aligned}
\alpha_{n-1} \circ d_n^{(n)}(i_0, \dots, i_n, x, m) &= \alpha_{n-1}(i_0, \dots, i_{n-1}, \Lambda_{\beta_{i_{n-1}i_n}}(x, m)) \\
&= (i_0, \dots, i_{n-1}, \mathcal{F}\tau_{i_{n-1}}^{-1} \circ \Lambda_{\beta_{i_{n-1}i_n}}(x, m)) = (i_0, \dots, i_{n-1}, \mathcal{F}\tau_{i_n}^{-1}(x, m)) \\
&= d_n^{(n)} \circ \alpha_n(i_0, \dots, i_n, x, m).
\end{aligned} \tag{5.166}$$

Now, let us consider a ρ -equivariant character $f \in \hat{H}^2(M, \mathbb{R}/\mathbb{Z})$. By Definition 5.2.5, it is represented by a cochain

$$\begin{aligned}
(d, 0, 0) & \quad 0 \\
(a, b, 0) & \quad (0, \rho, d\rho) \\
(c, h, \omega) & \quad 0.
\end{aligned} \tag{5.167}$$

Replacing characters by their respective cocycles in 5.142, we obtain

$$\begin{aligned}
& \Lambda_{\beta_{ij}}^*(\text{pr}_2^*(c, h, \omega) - (0, \rho^{A_i}, d\rho^{A_i})) \\
&= (t \circ \text{ev} \circ (\beta_{ij} \times \text{id}_M))^*(c, h, \omega) - \Lambda_{\beta_{ij}}^*(0, \rho^{A_i}, d\rho^{A_i}) \\
&= (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* t^*(c, h, \omega) - \Lambda_{\beta_{ij}}^*(0, \rho^{A_i}, d\rho^{A_i}) \\
&= (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* s^*(c, h, \omega) - (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*(0, \rho, d\rho) - \Lambda_{\beta_{ij}}^*(0, \rho^{A_i}, d\rho^{A_i}) \\
& \quad + (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* d(a, b, 0) \\
&= \text{pr}_2^*(c, h, \omega) - (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*(0, \rho, d\rho) - \Lambda_{\beta_{ij}}^*(0, \rho^{A_i}, d\rho^{A_i}) \\
& \quad + (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* d(a, b, 0).
\end{aligned} \tag{5.168}$$

Following the proof of Theorem 5.3.2, we know that

$$(\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* \rho + \Lambda_{\beta_{ij}}^* \rho^{A_i} = \rho^{A_j}, \tag{5.169}$$

and so we conclude that

$$\begin{aligned}
& \Lambda_{\beta_{ij}}^*(\text{pr}_2^*(c, h, \omega) - (0, \rho^{A_i}, d\rho^{A_i})) \\
&= \text{pr}_2^*(c, h, \omega) - (0, \rho^{A_j}, d\rho^{A_j}) + (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^* d(a, b, 0).
\end{aligned} \tag{5.170}$$

If we consider $\text{pr}_2^*(c, h, \omega) - (0, \rho^{A_i}, d\rho^{A_i})$ as components of an element in $\text{DC}_2^2((\mathcal{F}\tau)_0)$, we see that 5.170 translates to

$$\delta^0(\text{pr}_2^*(c, h, \omega) - (0, \rho^{A_i}, d\rho^{A_i})) = -d((\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*(a, b, 0)), \tag{5.171}$$

where δ^0 is the Dupont operator for $(\mathcal{F}\tau)_\bullet$. We view

$$(a_{ij}, b_{ij}, 0) := (\text{ev} \circ (\beta_{ij} \times \text{id}_M))^*(a, b, 0) \tag{5.172}$$

as components of an element in $\text{DC}_2^1((\mathcal{F}\tau)_1)$. From the isomorphism $(\mathcal{F}\tau)_\bullet \simeq (\pi_{\mathcal{F}}^{-1}\mathcal{O})_\bullet$, it is clear that the next step is to calculate $\delta^1(a_{ij}, b_{ij}, 0)$ and see whether it is exact in the relevant Hopkins-Singer complex. We adopt the notation

$$g_{ij} := (\text{ev} \circ (\beta_{ij} \times \text{id}_M)), \tag{5.173}$$

and

$$\tilde{g}_{ijk} := (g_{ij} \circ \Lambda_{\beta_{jk}}|_{O_{ijk}}, g_{jk}|_{O_{ijk}}) : O_{ijk} \times M \rightarrow \mathcal{G}_2. \quad (5.174)$$

Moreover, in virtue of the multiplication law in $\mathbb{B}(\mathcal{G})$, we have $m \circ \tilde{g}_{ijk} = g_{ik}$. Now, we are ready to compute:

$$\begin{aligned} (\delta^1(a_{ij}, b_{ij}, 0))_{ijk} &= (a_{jk}, b_{jk}, 0) - (a_{ik}, b_{ik}, 0) + \Lambda_{\beta_{jk}}^*(a_{ij}, b_{ij}, 0) \\ &= (\text{pr}_2 \circ \tilde{g}_{ijk})^*(a, b, 0) + (\text{pr}_1 \circ \tilde{g}_{ijk})^*(a, b, 0) - (m \circ \tilde{g}_{ijk})^*(a, b, 0) \\ &= \tilde{g}_{ijk}^*(\text{pr}_2^* + \text{pr}_1^* - m^*)(a, b, 0) = \tilde{g}_{ijk}^*d(d, 0, 0) = d(\tilde{g}_{ijk}^*d, 0, 0). \end{aligned} \quad (5.175)$$

We conclude that $(\delta^1(a_{ij}, b_{ij}, 0)) \in d(\text{DC}_2^0((\mathcal{F}\tau)_2))$. Therefore, the isomorphic pullback of the above cocycle by $\alpha^{-1} : (\pi_{\mathcal{F}}^{-1}\mathcal{O})_{\bullet} \rightarrow (\mathcal{F}\tau)_{\bullet}$ defines a unique class in $H_{\text{tot}}^2(\text{DC}_2^{\bullet}((\pi_{\mathcal{F}}^{-1}\mathcal{O})_{\bullet})_{\text{T}})$, which descends to a character $f^{\mathcal{A}}$ on $\mathcal{F}$. The proof is complete. $\square$

Remark 5.3.7. Since the symmetry of the metric can be gauged through minimal coupling, described in Chapter 5, under the assumptions of Conjecture 5.3.5, and upon further restriction of the structure group to $\mathbb{B}_{\rho}(\mathcal{G}) \cap \mathbb{B}_{\text{g}}(\mathcal{G})$, one can canonically gauge the full sigma model.

For a ρ -equivariant character $f \in \hat{H}^2(M, \mathbb{R}/\mathbb{Z})$ and its cochain 5.167, the descended character $f^{\mathcal{A}}$ takes the form

$$\begin{aligned} f^{\mathcal{A}}(z_{\pi_{\mathcal{F}}^{-1}\mathcal{O}} + \partial v) \\ = \sum_{j=1}^{m_z} (\mathcal{F}\tau)_{i_j}^*(\text{pr}_2^*h - \rho^{A_i})(z_j) + (\mathcal{F}\tau)_{i_j}^*(\text{ev} \circ (\beta_{i_j, i_j} \times \text{id}_M))^*b \mod \mathbb{Z} + \iota(\omega_{f^{\mathcal{A}}})(v) \end{aligned} \quad (5.176)$$

where $\omega_{f^{\mathcal{A}}}$ is the gluing of the $(\mathcal{F}\tau_i)^*(\text{pr}_2^*\omega_f + d\rho^{A_i})$.

From the point of view of finding an explicit expression for $\mathcal{M}_{\text{DF}, \text{top}}^{\mathcal{A}}$, it might be useful to obtain the local data of $\varphi^*f^{\mathcal{A}}$ on a convenient open cover of Σ . Note that since $\varphi \in \Gamma(\mathcal{F})$, we have

$$\varphi^{-1}\pi_{\mathcal{F}}^{-1}\mathcal{O} = (\pi_{\mathcal{F}} \circ \varphi)^{-1}\mathcal{O} = \mathcal{O}. \quad (5.177)$$

We may use the canonical pullback map 5.52

$$\varphi : \mathcal{O}_{\bullet} \rightarrow \pi_{\mathcal{F}}^{-1}\mathcal{O}_{\bullet} \quad (5.178)$$

to pull back the cocycle in the total complex of $\text{DC}_2^{\bullet}(\pi_{\mathcal{F}}^{-1}(\mathcal{O})_{\bullet})_{\text{T}}$. Indeed, the components 5.55 of φ are smooth, and thus induce chain maps

$$\varphi_k^* := (\varphi^{\times(k+1)})^* : \text{DC}_2^{\bullet}((\pi_{\mathcal{F}}^{-1}\mathcal{O})_k) \rightarrow \text{DC}_2^{\bullet}(\mathcal{O}_k). \quad (5.179)$$

Moreover, since φ is a map of simplicial manifolds, the above pullbacks commute with δ^k of the double complexes. Finally, the diagram

$$\begin{array}{ccc} \text{DC}_2^{\bullet}((\pi_{\mathcal{F}}^{-1}\mathcal{O})_0) & \xrightarrow{\varphi_0^*} & \text{DC}_2^{\bullet}(\mathcal{O}_0) \\ j^* \uparrow & & \uparrow (\varphi^{-1}j)^* \\ \text{DC}_2^{\bullet}(\mathcal{F}) & \xrightarrow{\varphi^*} & \text{DC}_2^{\bullet}(\Sigma), \end{array} \quad (5.180)$$

where $\varphi^{-1}j$ is the covering map of the pullback cover, is commutative. We just showed that we can pull back a class $j^*[f] \in H_{\text{tot}}^2(\text{DC}_2^{\bullet}((\pi_{\mathcal{F}}^{-1}\mathcal{O})_{\bullet})_{\text{T}})$ by the simplicial map φ , and that the

resultant class in $H_{\text{tot}}^2(\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T)$ descends to $\varphi^*[f]$. We will discuss the case where $\mathcal{O}$ is a good cover. You can always find a finite good trivializing cover of Σ [BT82, Theorem 5.1.]. We consider the pullback

$$\begin{array}{ccc} \varphi_2^*(c_{ijk}^A, 0, 0) & & 0 \\ & \varphi_1^*(c_{ij}^A, h_{ij}^A, 0) & 0 \end{array} \quad (5.181)$$

$$\varphi_0^*(c_i^A, h_i^A, \omega_i^A) \quad 0$$

in $\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T$. For dimensional reasons, $\varphi_0^*c_i^A = 0 = \varphi_0^*\omega_i^A$. The cocyclicity of 5.181 then implies $\delta(\varphi_0^*h_i) = 0$, and so, by the goodness of $\mathcal{O}$, we get $\varphi_0^*h_i = \delta b_i$, for $b_i \in C^1(\mathcal{O}_0, \mathbb{R})$. Let us add the coboundary of

$$(0, 0, 0) \quad (5.182)$$

$$(0, b_i, 0)$$

to 5.181. We arrive at the following cocycle

$$\begin{array}{ccc} (\varphi_2^*c_{ijk}^A, 0, 0) & & 0 \\ & (\varphi_1^*c_{ij}^A, \varphi_1^*h_{ij}^A + b_i - b_j, 0) & 0 \end{array} \quad (5.183)$$

$$(0, 0, 0) \quad 0.$$

Again, by the cocyclicity and the goodness, we obtain $\varphi_1^*c_{ij}^A = \delta d_{ij}$, $d \in C^0(\mathcal{O}_1, \mathbb{Z})$. Adding the coboundary of

$$(d_{ij}, 0, 0) \quad (5.184)$$

$$(0, 0, 0)$$

to 5.183 leads us to

$$\begin{array}{ccc} (\varphi_2^*c_{ijk}^A + d_{jk} - d_{ik} + d_{ij}, 0, 0) & & \\ & (0, \varphi_1^*h_{ij}^A + b_j - b_i + d_{ij}, 0) & \end{array} \quad (5.185)$$

$$(0, 0, 0).$$

The cocyclicity of 5.185 implies

$$\varphi_1^*h_{ij}^A + b_j - b_i + d_{ij} \in H^0(\mathcal{O}_1, \mathbb{R}) \quad (5.186)$$

and

$$\varphi_2^*(h_{jk}^A - h_{ik}^A + h_{ij}^A) = \varphi_2^*c_{ijk}^A. \quad (5.187)$$

In particular, the amplitude $\mathcal{M}_{\text{DF,top}}^A$ can be expressed as

$$\mathcal{M}_{\text{DF,top}}^A(\varphi) = \varphi^*f^A(c_\Sigma) = \sum_{j=1}^{m_z} (\varphi_1^*h_{i_j,i_j}^A + b_{i_j} - b_{i_{j'}})(d_0z_j) \mod \mathbb{Z}, \quad (5.188)$$

where

$$c_\Sigma = \sum_{j=1}^{m_z} z_j \quad (5.189)$$

is a decomposition of a subordinate fundamental cycle into simplices. The data fixing the amplitude is a Čech [BT82, Ch. II. §10] cocycle

$$(\varphi_1^*h_{ij}^A + b_j - b_i) \in \check{Z}^1(\mathcal{O}, \mathbb{R}/\mathbb{Z}). \quad (5.190)$$

Clearly, every Čech coboundary $(\delta^0 b')_{ij}$ for $b'_i \in \check{C}^0(\mathcal{O}, \mathbb{R}/\mathbb{Z})$ can be extended by zero to a coboundary with respect to the total derivative of $\text{DC}_2^\bullet(\mathcal{O}_\bullet)_T$, and, thus, it does not affect the character φ^*f^A . In fact, we just provided an explicit construction of a map

$$\hat{H}^2(\Sigma, \mathbb{R}/\mathbb{Z}) \rightarrow \check{H}^1(\mathcal{O}, \mathbb{R}/\mathbb{Z}) \simeq H^1(\Sigma, \mathbb{R}/\mathbb{Z}), \quad (5.191)$$

where the last isomorphism follows from Leray's Acyclicity Theorem and the goodness of $\mathcal{O}$ ([Bry07, 1.3.6.]). This map is an isomorphism, inverse to i_1 in the short exact sequence (5.153):

$$0 \rightarrow H^1(\Sigma, \mathbb{R}/\mathbb{Z}) \rightarrow \hat{H}^2(\Sigma, \mathbb{R}/\mathbb{Z}) \rightarrow \Omega_{\mathbb{Z}}^2(\Sigma) \rightarrow 0. \quad (5.192)$$

In summary, the map $f^A \mapsto \varphi^*f^A$ factors through $\check{H}^1(\Sigma, \mathbb{R}/\mathbb{Z})$ and 5.188 is the explicit form of this factorization.

The differential-topological approach adopted in the present thesis proves itself useful and arguably simpler than the one utilizing geometric realizations whose generalized holonomies replace differential characters in defining the DF-amplitude. Although the category of iso-classes of bundle gerbes is equivalent to the cocycle category of differential characters, there are a few additional difficulties in applying the theory of the former to the gauging of groupoidal symmetries. For once, in order to describe equivariant structures using Beilinson-Deligne hypercohomology, one needs to choose particular refinements of the cover, the existence of which is far from straightforward to prove. Moreover, an important tool, easily applicable when working with differential characters, and which has not been used in the theory of gauging so far, is the Subdivision Theorem. This theorem lies at the core of the proof of the effective descent of characters given in this thesis. Finally, the language of defects can be successfully translated to the language of (higher) stacks, providing an accessible bridge in communication between theoretical physicists and mathematicians.

Appendix A

Prerequisites in Homological and Categorical Algebra, and Topology

Definition A.0.1. A *chain complex* of abelian groups is a graded abelian group $C_\bullet = \bigoplus_{k \in \mathbb{Z}} C_k$ together with a degree-(-1) endomorphism ∂ of $C_\bullet$ satisfying $\partial \circ \partial = 0$. We call it the *boundary operator* for $C_\bullet$. A morphism of chain complexes is a degree-0 morphism $f : C_\bullet \rightarrow D_\bullet$ of graded abelian groups which commutes with the boundary operators: $f \circ \partial_C = \partial_D \circ f$. A cochain complex is defined analogously, but with the degree-(-1) boundary replaced by a degree-1 *coboundary* δ . We also use an upper index $C^\bullet$. A morphism of cochain complexes must commute with the coboundary operators. Chain complexes and cochain complexes of abelian groups form categories denoted by $\mathbf{Ch}(\mathbf{Ab})$ and $\mathbf{CCh}(\mathbf{Ab})$, respectively.

Definition A.0.2. We say that a chain (cochain) complex is *exact* if $\ker \partial = \operatorname{im} \partial$ ($\ker \delta = \operatorname{im} \delta$). We call the elements of $\ker \partial$ ($\ker \delta$) *cycles* (*cocycles*) and the elements of $\operatorname{im} \partial$ ($\operatorname{im} \delta$) *boundaries* (*coboundaries*).

Definition A.0.3. A functor $F : \mathbf{Ab} \rightarrow \mathbf{Ab}$ is called *left exact* if it sends any short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (\text{A.1})$$

in $\mathbf{Ab}$ into an exact sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C). \quad (\text{A.2})$$

For a contravariant functor $F : \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{Ab}$, left exactness means that, for the short exact sequence A.1, the following sequence is exact:

$$0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A). \quad (\text{A.3})$$

Proposition A.0.4. *For every abelian group A , the functor*

$$\operatorname{Hom}_{\mathbb{Z}}(\cdot, A) : C \mapsto \operatorname{Hom}_{\mathbb{Z}}(C, A) \quad (\text{A.4})$$

is left exact.

Definition A.0.5. A *double cochain complex* is a diagram $C_{\bullet,\bullet}$:

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{n,m+1} & \xrightarrow{\delta_h^n} & C^{n+1,m+1} & \longrightarrow & \cdots \\
& & \delta_v^m \uparrow & & \delta_v^m \uparrow & & \\
\cdots & \longrightarrow & C^{n,m} & \xrightarrow{\delta_h^n} & C^{n+1,m} & \longrightarrow & \cdots, \\
& & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & &
\end{array} \tag{A.5}$$

in which each row and each column is a complex and $\delta_v \circ \delta_h = -\delta_h \circ \delta_v$. Any double complex defines the *total complex*

$$\text{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{n=p+q} C^{p,q}, \tag{A.6}$$

whose coboundary is

$$D^n = \sum_{n=p+q} (\delta_v^p + (-1)^p \delta_h^q). \tag{A.7}$$

It is easy to check that $D \circ D = 0$. We denote the cohomology of the total complex by $H_{\text{tot}}^n(C^{\bullet,\bullet})$. By a cocycle resp. a coboundary in a double complex we mean a cocycle resp. a coboundary relative to D .

Definition A.0.6. We say that chain complexes $(C_{\bullet}, \partial_C)$ and $(D_{\bullet}, \partial_D)$ are *chain-homotopy equivalent* if there exist chain maps $f : (C_{\bullet}, \partial_C) \rightarrow (D_{\bullet}, \partial_D)$ and $g : (D_{\bullet}, \partial_D) \rightarrow (C_{\bullet}, \partial_C)$, together with degree-1 morphisms of graded abelian groups $h_C : C_{\bullet} \rightarrow C_{\bullet}$ and $h_D : D_{\bullet} \rightarrow D_{\bullet}$ satisfying

$$g \circ f - \text{id}_{C_{\bullet}} = \partial_C \circ h_C + h_C \circ \partial_C, \tag{A.8}$$

and

$$f \circ g - \text{id}_{D_{\bullet}} = \partial_D \circ h_D + h_D \circ \partial_D. \tag{A.9}$$

We then say that $f \circ g$ and $g \circ f$ are *homotopic* to identities.

Definition A.0.7. *Homology* is a functor $H : \mathbf{Ch}(\mathbf{Ab}) \rightarrow \mathbf{Ab}^{\mathbb{Z}}$ into the category of graded abelian groups. It is defined on objects as

$$H(C_{\bullet}) = \ker \partial / \text{im } \partial. \tag{A.10}$$

This is well-defined since $\partial \circ \partial = 0$. Similarly, *cohomology* is a functor $H : \mathbf{CCh}(\mathbf{Ab}) \rightarrow \mathbf{Ab}^{\mathbb{Z}}$ defined on objects as

$$H(C^{\bullet}) = \ker \delta / \text{im } \delta. \tag{A.11}$$

We often consider a single degree. Then we write

$$(H(C_{\bullet}))_n =: H_n(C_{\bullet}), \tag{A.12}$$

resp.

$$(H(C^{\bullet}))_n =: H^n(C^{\bullet}). \tag{A.13}$$

For a chain map $f : (C_{\bullet}, \partial_C) \rightarrow (D_{\bullet}, \partial_D)$ and the canonical projection $\pi_D : \ker \partial_D \twoheadrightarrow \ker \partial_D / \text{im } \partial_D$, the map $H(f)$ is defined as a factorization of $\pi_D \circ f|_{\ker \partial_C}$ through $\ker \partial_C / \text{im } \partial_C$. This factorization relies on the condition that chain maps commute with boundary and coboundary maps.

Proposition A.0.8. *Chain-homotopy equivalences induce isomorphisms in homology/cohomology.*

Definition A.0.9. A (smooth) *singular n -simplex* in a manifold M is a continuous (smooth) map from the standard n -simplex

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}_{\geq 0}^{n+1} : x_0 + \dots + x_n = 1\} \quad (\text{A.14})$$

to M . For $n > 0$ and $0 \leq i \leq n$, we define *faces* of the standard n -simplex:

$$\Delta_i^n := \{(x_0, \dots, x_n) \in \Delta^n : x_i = 0\} \simeq \Delta^{n-1}. \quad (\text{A.15})$$

The faces of a singular simplex $\sigma : \Delta^n \rightarrow M$ are given as

$$d_i \sigma = \sigma|_{\Delta_i^n} : \Delta_i^n \rightarrow M. \quad (\text{A.16})$$

We treat them as singular $(n-1)$ -simplices. Denote by $C_n(M)$ the free abelian group generated by singular simplices in M . We construct a chain complex, called the *singular chain complex*, by introducing a boundary operator:

$$\forall \sigma \in C_n(M) : \partial \sigma = \sum_{i=0}^n (-1)^i d_i \sigma \in C_{n-1}(M). \quad (\text{A.17})$$

It is an easy check that $\partial \circ \partial = 0$. We denote

$$Z_k(M) := \ker(C_k(M) \xrightarrow{\partial} C_{k-1}(M)) \quad B_k(M) := \text{im}(C_{k+1}(M) \xrightarrow{\partial} C_k(M)), \quad (\text{A.18})$$

and the homology of $C_\bullet(M)$ by $H_n(M)$. For an abelian group Λ , we define a *singular cochain* (of degree n) with values in Λ as a map $c \in \text{Hom}_{\mathbb{Z}}(C_n(M), \Lambda)$. By linearity, they form a graded abelian group of singular cochains, and we make it into a cochain complex by setting

$$\delta c(\sigma) = c(\partial \sigma). \quad (\text{A.19})$$

It is clear that $\delta \circ \delta = 0$. We denote

$$Z^k(M, \Lambda) := \ker(C^k(M, \Lambda) \xrightarrow{\delta} C^{k+1}(M, \Lambda)), \quad (\text{A.20})$$

$$B^k(M, \Lambda) := \text{im}(C^{k-1}(M, \Lambda) \xrightarrow{\delta} C^k(M, \Lambda)), \quad (\text{A.21})$$

and the cohomology of $C^\bullet(M, \Lambda)$ by $H^n(M, \Lambda)$. For $n < 0$ we set all the groups to zero.

Proposition A.0.10. *Let M be a manifold and let*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0 \quad (\text{A.22})$$

be a short exact sequence of abelian groups. Then, there exists a degree-1 morphism of graded abelian groups

$$B : H^\bullet(M, C) \rightarrow H^{\bullet+1}(M, A), \quad (\text{A.23})$$

called the connecting homomorphism or the Bockstein homomorphism, rendering the following sequence exact

$$\dots \rightarrow H^k(M, A) \xrightarrow{H(i)} H^k(M, B) \xrightarrow{H(p)} H^k(M, C) \xrightarrow{B} H^{k+1}(M, A) \rightarrow \dots \quad (\text{A.24})$$

Definition A.0.11. Let $\mathcal{O} = \{O_i\}_{i \in I}$ be an open cover of a manifold M . We say that a (smooth) singular chain c is *subordinate to $\mathcal{O}$* if every simplex of c is supported in some element of $\mathcal{O}$. Denote by $C_\bullet^\mathcal{O}(M)$ the group of subordinate chains in M . It is clear that, upon restricting ∂ , the graded group $C_\bullet^\mathcal{O}(M)$ forms a chain complex.

Theorem A.0.12 ([Hat02, Proposition 2.21.]). *The inclusion*

$$\iota : C_\bullet^\mathcal{O}(M) \hookrightarrow C_\bullet(M) \quad (\text{A.25})$$

is a chain-homotopy equivalence. In particular, there exists a map

$$\rho : C_\bullet(M) \rightarrow C_\bullet^\mathcal{O}(M) \quad (\text{A.26})$$

such that $\rho \circ \iota$ and $\iota \circ \rho$ are homotopic to identities. As a consequence, if $z \in Z_k(M)$, we can write

$$(\iota \circ \rho)(z) - z = (\partial \circ h)(z) + h(\partial z) = \partial(h(z)), \quad (\text{A.27})$$

for some degree-1 morphism of abelian groups $h : C_\bullet(M) \rightarrow C_\bullet(M)$. Thus, the cycle z differs from a subordinate cycle $z_\mathcal{O} := (\iota \circ \rho)(z)$ by a boundary.

Lemma A.0.13. *Let $b \in C_k(M)$ be such that $\partial b \in C_{k-1}^\mathcal{O}(M)$. Then, there exists $b' \in C_k^\mathcal{O}(M)$ satisfying $\partial b' = \partial b$.*

Proof. We write down the homotopy

$$\rho \circ \iota - \text{id}_{C_\bullet^\mathcal{O}} = \partial \circ h + h \circ \partial. \quad (\text{A.28})$$

Now, define $b' := \rho(b) - h(\partial b)$. This lies in $C_k^\mathcal{O}(M)$, since $\partial b \in C_{k-1}^\mathcal{O}(M)$. We compute

$$\begin{aligned} \partial b' &= \partial \rho(b) - \partial(h(\partial b)) = \rho(\partial b) - ((\rho \circ \iota - \text{id}_{C_\bullet^\mathcal{O}})(\partial b) - h(\partial(\partial b))) \\ &= \rho(\partial b) - \rho(\iota(\partial b)) + \partial b = \rho(\partial b) - \rho(\partial b) + \partial b = \partial b, \end{aligned} \quad (\text{A.29})$$

where we used the fact that ρ is a chain map, and that $\iota(\partial b) = \partial b$. $\square$

Proposition A.0.14. *Every free abelian group C satisfies the following property. For every epimorphism of abelian groups $p : A \twoheadrightarrow B$, and every morphism $f : C \rightarrow B$, there exists a morphism $g : C \rightarrow A$ satisfying $f = p \circ g$. We say that every free abelian group is projective.*

Definition A.0.15. An abelian group C is called *injective* if, for every monomorphism of abelian groups $j : A \hookrightarrow B$, and any morphism $f : A \rightarrow C$, there exists a morphism $g : B \rightarrow C$ satisfying $f = g \circ j$.

Proposition A.0.16. *An abelian group C is called divisible if the following condition is satisfied:*

$$\forall n \in \mathbb{N} \forall c \in C \exists c' \in C : nc' = c, \quad (\text{A.30})$$

where by nc' we understand the n -fold sum of c' . Any abelian group is divisible if and only if it is injective.

Corollary A.0.17. *The abelian groups $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z} \simeq \text{U}(1)$ are injective.*

Proposition A.0.18 (A special case of Universal Coefficient Theorem). *Let M be a manifold and let Λ be an injective abelian group. Then, the natural map*

$$H^n(M, \Lambda) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(M), \Lambda) : [c] \mapsto ([\sigma] \mapsto c(\sigma)) \quad (\text{A.31})$$

is an isomorphism. To see that the map is well-defined, note that, for $\sigma \in Z_n(M)$ and $c' \in C^n(M, \Lambda)$, we have $\delta c'(\sigma) = c'(\partial\sigma) = 0$, and that, for $c \in Z^n(M, \Lambda)$ and $\sigma' \in C_n(M)$, we have $c(\partial\sigma') = \delta c(\sigma') = 0$. In fact, the map A.31 gives rise to a natural isomorphism of functors $H^\bullet(\cdot, \Lambda) \simeq \text{Hom}_{\mathbb{Z}}(H_\bullet(\cdot), \Lambda)$.

Theorem A.0.19 (de Rham Theorem). *Let $H_{\text{dR}}^n(M)$ be the cohomology of the cochain complex $(\Omega^\bullet(M), d)$ of differential forms with the exterior derivative. There is a natural isomorphism*

$$H^n(M, \mathbb{R}) \simeq H_{\text{dR}}^n(M) \quad (\text{A.32})$$

for all $n \in \mathbb{N}$.

Definition A.0.20. When M is a connected orientable closed manifold of dimension n , the top homology group $H_n(M) \simeq \mathbb{Z}$, and an orientation is a choice of the generator. The chosen generator is called the *fundamental class* of M . Any of its representatives $c \in Z_n(M)$ is called a *fundamental chain/cycle*. For every n -form ω on M and every fundamental cycle c we have

$$\int_c \omega = \int_M \omega. \quad (\text{A.33})$$

In this manner, fundamental cycles correspond to integration over the whole manifold.

Proposition A.0.21. *The constructions $C_\bullet, Z_\bullet, B_\bullet, H_\bullet$ are covariant functors on $\mathbf{Man}$. The constructions $C^\bullet(\cdot, \Lambda), Z^\bullet(\cdot, \Lambda), B^\bullet(\cdot, \Lambda), H^\bullet(\cdot, \Lambda), \Omega^\bullet(\cdot, \Lambda), H_{\text{dR}}^\bullet(\cdot, \Lambda)$ are contravariant functors on $\mathbf{Man}$, and covariant functors in the coefficient group.*

Definition A.0.22. A *presheaf* on a category $\mathcal{C}$ valued in a category $\mathcal{D}$ is a functor $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

Definition A.0.23. The *simplex category* Δ is a category whose objects are sets

$$[n] = \{0, 1, \dots, n\}, \quad n \in \mathbb{N}, \quad (\text{A.34})$$

and whose morphisms are order-preserving functions $f : [n] \rightarrow [m]$. Every such map is a composition of coface and codegeneracy maps. The coface maps are

$$\delta_i^{(n)} : [n] \rightarrow [n+1] : \delta_i^{(n)}(k) = \begin{cases} k & k < i \\ k+1 & k \geq i, \end{cases} \quad 0 \leq i \leq n. \quad (\text{A.35})$$

The codegeneracy maps are

$$\sigma_i^{(n)} : [n] \rightarrow [n-1] : \sigma_i^{(n)}(k) = \begin{cases} k & k \leq i \\ k-1 & k > i, \end{cases} \quad 0 \leq i \leq n-1. \quad (\text{A.36})$$

A *simplicial object* is a presheaf on Δ . Note that every simplicial object is defined by specifying the object component and the images of coface and codegeneracy maps, which we call *face* and *degeneracy* maps, respectively. For example, to specify a simplicial manifold $M_\bullet$, we fix $M_n = M_\bullet([n])$, the maps $d_i^{(n)} = M_\bullet(\delta_i^{(n)})$, and the maps $s_i^{(n)} = M_\bullet(\sigma_i^{(n+1)})$. In fact, one

can show that different morphism components of a simplicial manifold with a fixed object component are in a one-to-one correspondence with a collection of maps

$$d_i^{(n)} : M_n \rightarrow M_{n-1} \quad s_i^{(n)} : M_n \rightarrow M_{n+1} \quad (\text{A.37})$$

satisfying the following *simplicial identities*, true, whenever well-defined:

1. $d_i^{(n-1)} \circ d_j^{(n)} = d_j^{(n-1)} \circ d_i^{(n)}$ for $i < j$,
2. $d_i^{(n+1)} \circ s_j^{(n)} = s_{j-1}^{(n-1)} \circ d_i^{(n)}$ for $i < j$,
3. $d_i^{(n+1)} \circ s_j^{(n)} = \text{id}$ if $i = j$ or $i = j + 1$,
4. $d_i^{(n+1)} \circ s_j^{(n)} = s_j^{(n-1)} \circ d_{i-1}^{(n)}$ for $i > j + 1$,
5. $s_i^{(n-1)} \circ s_j^{(n)} = s_{j+1}^{(n-1)} \circ s_i^{(n)}$ for $i \leq j$.

A morphism of simplicial manifolds is a natural transformation $\varphi : M_\bullet \rightarrow N_\bullet$.

Remark A.0.24. To every category $\mathcal{C}$, one can associate a simplicial object $N(\mathcal{C})$, called the *nerve* of $\mathcal{C}$, in the following manner. The object component is

$$N(\mathcal{C})_n = \{X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n : X_i \in \text{Ob } \mathcal{C} \text{ for } 0 \leq i \leq n\}, \quad (\text{A.38})$$

the set of composable arrows in $\mathcal{C}$. The degeneracy maps of the nerve are

$$d_i^{(n)} : (X_0 \rightarrow \dots \rightarrow X_n) \mapsto (X_0 \rightarrow \dots \rightarrow X_{i-1} \rightarrow X_{i+1} \rightarrow \dots \rightarrow X_n), \quad (\text{A.39})$$

where we obtain $X_{i-1} \rightarrow X_{i+1}$ by composing $X_{i-1} \rightarrow X_i \rightarrow X_{i+1}$. The degeneracy maps are

$$s_i^{(n)} : (X_0 \rightarrow \dots \rightarrow X_n) \mapsto (X_0 \rightarrow \dots \rightarrow X_i \rightarrow X_i \rightarrow X_{i+1} \rightarrow \dots \rightarrow X_n), \quad (\text{A.40})$$

where we insert the identity map $X_i \rightarrow X_i$.

An example of a nerve of a category is the *Čech nerve* [GM02, Ch. I] of an open cover $\mathcal{O}$ of M . The underlying category for that nerve is the Čech groupoid $\text{Pair}_M(\mathcal{O})$, which is the pair groupoid but with morphisms restricted to only those pairs (u_1, u_2) which map to the same point in M under the covering map.

Theorem A.0.25 ([Ser92, LG 3.27]). *Let M be a manifold and let $\sim$ be an equivalence relation on M , with graph*

$$\begin{array}{ccc} \mathcal{R}_\sim & \hookrightarrow & M \times M \\ & \searrow \text{pr}_1 & \swarrow \text{pr}_2 \\ & M & M. \end{array} \quad (\text{A.41})$$

There exists a smooth structure on the quotient

$$M//\sim := \{[m]_\sim : m \in M\}, \quad (\text{A.42})$$

compatible with the quotient topology, and such that $\pi : M \rightarrow M//\sim$ is a submersion, if and only if the graph $\mathcal{R}_\sim$ is a proper submanifold of $M \times M$ and the restriction of the projection $\text{pr}_1 : M \times M \rightarrow M$ to $\mathcal{R}_\sim$ is a submersion.

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