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Twisted Abelian Differential Cohomology and Weakly Abelian Lie Groups

Master's thesis
in MATHEMATICS

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Abstract

We develop the theory of differential cohomology with local coefficients. We formulate it within the axiomatic approach introduced by J. Simons and D. Sullivan. We prove de Rham theorem for invariant forms valued in local systems. We construct twisted differential characters, generalizing the original construction by J. Cheeger and J. Simons, and serving as a model for the twisted cohomology. We prove essential uniqueness of the twisted character functor. Finally, we conjecture that twisted differential characters of arbitrary degree form a stack over the category of smooth connected based manifolds. We prove the assertion for degree-2 twisted differential characters.

Keywords

differential cohomology, local coefficients, twisted de Rham theorem, differential characters, character functor

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Contents

Introduction

Differential cohomology theory [HS05] provides a natural differential enhancement of the singular cohomology of manifolds. It grew out of the classical subject of differential characters, which already appears in the seminal work of Cheeger and Simons [CS85]. It has a well-known axiomatic characterization due to Simons and Sullivan [SS07] as a simultaneous extension of singular cohomology by "nonintegral" differential forms and of integral differential forms by circle-valued cohomology which fits into a hexagonal *character diagram*. Together with the four maps defined by that diagram, it forms a so-called *character functor* – a notion which enhances that of contravariant functor from the category of smooth manifolds to that of graded abelian groups. The axiomatic characterization given by Simons and Sullivan fixes this functor up to unique natural isomorphism, thus giving an abstract description of Cheeger-Simons characters.

Differential cohomology theory has numerous applications to manifold topology and mathematical physics. In the second realm, it allows for a convenient description of higher rank gauge fields which arise in various physical models pertinent to supergravity, string theory and topological field theory and turns out to be useful in problems pertaining to their quantization (see, for example, [Sza12]).

In this thesis, we generalize differential cohomology theory to produce a natural differential enhancement of the singular cohomology of manifolds *with local coefficients*. This generalized theory, which we call *differential cohomology with local coefficients*, arises in mathematics and physics problems which depend on systems of local coefficients – such as the study of various versions of higher gauge theory which involve certain types of nonabelian gerbe twist. Such situations abound in supergravity, string theory and topological field theory and have deep connections with topological problems of "non-principal type" – i.e. those where the fundamental group of a manifold plays a crucial role and hence forces the use of non-principal obstruction theory [Bau06]. One of the simplest classes of examples of this kind is provided by *weakly-abelian gauge theories* – defined as those ordinary nonabelian gauge theories whose structure group is *weakly abelian* in the sense that its Lie algebra is abelian (such a Lie group is generally disconnected and it is necessarily an extension of a discrete group by a connected abelian Lie group). As shown in [LS22], weakly-abelian gauge theories play a crucial role in the self-dual formulation of N=1 supergravity theory in four dimensions and hence are of major interest for the problem of classifying configurations and solutions of such theories (including solutions of so-called "U-fold type"). In the application to $N = 1$, supergravity, the relevant structure group is an extension of an even-dimensional torus group $U(1)^{2n}$ by a so-called *modified Siegel modular group* $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$, a certain arithmetic group which contains the ordinary modular group $\mathrm{Sp}(2n, \mathbb{Z})$ as a subgroup. Irrespective of any physics applications, this generalization of differential cohomology is mathematically natural and bound to have numerous applications to various problems in manifold topology.

Below, we provide an axiomatic characterization of differential cohomology with local coefficients which extends the Simons-Sullivan characterization of ordinary differential cohomology. We also give an explicit construction of this theory using an appropriately "twisted" version of Cheeger-Simons characters, which depend on a choice of a local coefficient system on the underlying manifold. After giving the proofs of functoriality, essential unicity and existence of this theory (the last of which is by construction of the twisted Cheeger-Simons characters), we show that twisted differential characters of degree two form an appropriate stack over the category of manifolds; this generalizes a result proved by Lerman and Malkin [LM07] for ordinary differential cohomology. We then discuss some applications, focusing on the motivating example of Dirac quantization in weakly abelian gauge theories and briefly point out some connections to problems in physics and number theory.

The thesis is organized as follows. Chapter 1 briefly recalls the definition of Cheeger-Simons characters [CS85] and the axiomatic characterization of ordinary differential cohomology due to Simons and Sullivan [SS07]. Chapter 2 discusses differential cohomology with local coefficients. The chapter starts with a brief review of cohomology with local coefficients. Section 2.1 explains the precise sense in which such a cohomology theory is functorial while Section 2.2 discusses twisted de Rham cohomology for forms valued in a local coefficient system and proves a corresponding version of the de Rham theorem.

Chapter 3 introduces and studies the twisted character functor, which generalizes the ordinary character functor of [SS07] to differential cohomology with local coefficients. After showing that twisted differential characters provide such a functor, we prove in Section 3.1 the relevant essential uniqueness theorem for differential cohomology with local coefficients. In Section 3.2 give a cohomological description of twisted differential characters in terms of an appropriate presheaf of cochain complexes, which generalizes a similar description given by Hopkins and Singer for ordinary differential cohomology [HS05]. Using the presheaf of cochain complexes, in Section 3.3 we constuct an appropriate double complex on the Čech nerve of an open cover, and we prove that its total cohomology classifies twisted differential characters of degree-2 on the covered manifold. This amounts to the effective descent of the respective cocycle category, or in other words – "stackiness" of twisted differential characters of degree-2. The generalization to arbitrary natural degree is conjectured. Some of our results require restrictions on the coefficient group of the local coefficients, which we state explicitly. Finally, Chapter 4 discusses briefly a few applications that illustrate the general theory. In particular, in Section 4.2 we show how the theory of twisted differential characters applies to weakly-abelian gauge theories.

Chapter 1

Ordinary Differential Characters

We assume every manifold and every map between manifolds to be smooth. We will denote by $C_\bullet$ the graded abelian group of *smooth* singular chains, and by $Z_\bullet$, $B_\bullet$ its respective subgroups of smooth singular cycles and smooth singular boundaries respectively. We make use of the fact, that there is a chain homotopy equivalence between the complex of smooth singular chains and the complex of continuous singular chains [Lee00, Ch. 18]. We will denote by $\mathbf{Ab}^{\mathbb{Z}}$ the category of graded abelian groups.

Definition 1.0.1. Let M be a manifold. The graded subalgebra $\Omega^\bullet_{\mathbb{Z}}(M) \subseteq \Omega^\bullet(M)$ of *differential forms with integer periods* is composed of those closed differential forms, whose integrals over every cycle lie in $\mathbb{Z}$. That is,

$$\Omega^k_{\mathbb{Z}}(M) \coloneqq \left\{\omega \in \Omega^k(M) \,\middle|\, \mathrm{d}\omega = 0 \wedge \forall c \in Z_k(M) : \int_c \omega \in \mathbb{Z}\right\}. \tag{1.1}$$

Remark 1.0.2. By de Rham's Theorem, the classes $[\omega] \in \Omega^k_{\mathbb{Z}}(M)/\mathrm{d}\Omega^{k-1}(M)$ are in isomorphism with the image of $i_* : H^k(M,\mathbb{Z}) \to H^k(M,\mathbb{R})$ in singular cohomology, induced by the coefficient morphism $i : \mathbb{Z} \hookrightarrow \mathbb{R}$.

Lemma 1.0.3. *The map $\iota : \Omega^\bullet(M) \to C^\bullet(M, \mathbb{R}/\mathbb{Z})$ given by*

$$\iota(\omega)(c) \coloneqq \int_c \omega \bmod \mathbb{Z} \tag{1.2}$$

is an injection.

Proof. Suppose $0 \neq \omega \in \ker \iota$. Then $\omega(p) \neq 0$ for p in some coordinate chart (U, ϕ). In local coordinates $\phi = (x^1, \dots, x^n)$ on U, the form ω is expressed as

$$\omega(x) = f(x)\,\mathrm{d}x^{i_1} \wedge \dots \mathrm{d}x^{i_k} + \dots \tag{1.3}$$

Without loss of generality, we may assume that $f(p) \neq 0$. By the continuity of f, there exist $\delta > 0$ and $\varepsilon > 0$ such that we can fit an n-cube

$$K = \{x : |x^j - p^j| \leqslant \varepsilon,\ 1 \leqslant j \leqslant \dim M\} \subseteq U \cap \{x : |f(x)| \geqslant \delta\}. \tag{1.4}$$

Consider a smooth k-simplex $\sigma : \Delta^k \to M$ defined in local coordinates as

$$\sigma(t^1, \dots, t^k) = \phi^{-1}(p^1, \dots, p^{i_1-1}, p^{i_1} + \varepsilon t^1, p^{i_1+1}, \dots, p^{i_k-1}, p^{i_k} + \varepsilon t^k, \dots, p_n). \tag{1.5}$$

Clearly, $\sigma(\Delta^k) \subseteq K$. The pullback of ω calculates as

$$\sigma^*\omega = \big(f \circ \sigma(t)\big)\varepsilon^k \, \mathrm{d}t^1 \wedge \ldots \wedge \mathrm{d}t^k. \tag{1.6}$$

We obtain

$$\int_\sigma \omega = \int_{\Delta^k} \big(f \circ \sigma(t)\big)\varepsilon^k \, \mathrm{d}t^1 \wedge \ldots \wedge \mathrm{d}t^k. \tag{1.7}$$

Since $\big|f \circ \sigma(t)\big| \geqslant \delta$ on Δ^k, we get an estimate

$$\Big|\int_\sigma \omega\Big| \geqslant \delta \frac{\varepsilon^k}{k!}. \tag{1.8}$$

Combining this with another estimate

$$\Big|\int_\sigma \omega\Big| \leqslant \sup_{x \in K} |f(x)| \frac{\varepsilon^k}{k!}, \tag{1.9}$$

which is finite by the compactness of K, we conclude that for sufficiently small $\varepsilon > 0$:

$$0 < \Big|\int_\sigma \omega\Big| < 1, \tag{1.10}$$

which contradicts the assumption that the value of any such integral lies in $\mathbb{Z}$. $\square$

Corollary 1.0.4. *In Definition 1.0.1 one does not have to assume the closedness of* ω*.*

Proof. Let $\omega \in \Omega^k(M)$ be such that

$$\forall c \in Z_k(M) : \int_c \omega \in \mathbb{Z}. \tag{1.11}$$

Then, by Stokes Theorem, for any $c' \in C_{k+1}(M)$ we have

$$\int_{\partial c'} \omega = \int_{c'} \mathrm{d}\omega \in \mathbb{Z}, \tag{1.12}$$

and thus, $\mathrm{d}\omega = 0$, by an argument similar to the one leading to 1.10. $\square$

Definition 1.0.5 ([CS85])**.** Let M be a manifold. The abelian group *differential characters of degree* $k \in \mathbb{Z}_{>0}$ is defined as

$$\hat{H}^k(M, \mathbb{R}/\mathbb{Z}) \coloneqq \big\{ f \in \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}\big) \,\big|\, \exists\, \omega_f \in \Omega^k(M) : f \circ \partial = \iota(\omega_f) \big\}. \tag{1.13}$$

That is, for any $c \in C_k(M)$ we have

$$f(\partial c) = \int_c \omega_f \bmod \mathbb{Z}. \tag{1.14}$$

It is clear from the construction that $\omega_f \in \Omega^k_{\mathbb{Z}}(M)$. Lemma 1.0.3 implies that ω_f is uniquely determined by f. Moreover, the assignment $M \mapsto \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$ is functorial. Indeed, for a map $h : M \to N$ we have

$$\forall f \in \hat{H}^k(N, \mathbb{R}/\mathbb{Z}) : f \circ h_* \circ \partial = f \circ \partial \circ h_* = \iota(\omega_f) \circ h_* = \iota(h^*\omega_f). \tag{1.15}$$

Definition 1.0.6 ([SS07])**.** A *character functor* is a 5-tuple $(\hat{G}^\bullet, i_1, i_2, \delta_1, \delta_2)$, where

$$\hat{G}^\bullet : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}, \tag{1.16}$$

and $i_1, i_2, \delta_1, \delta_2$ are natural transformations rendering the following *character diagram* commutative, and its diagonal sequences exact for each $k \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(\,\cdot\,,\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "i_1"] & & H^k(\,\cdot\,,\mathbb{Z}) \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1}(\,\cdot\,,\mathbb{R}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & \hat{G}^k \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^k(\,\cdot\,,\mathbb{R}) \\
& \Omega^{k-1}/\Omega^{k-1}_{\mathbb{Z}} \arrow[rr, "\mathrm{d}"] \arrow[ur, "i_2"] & & \Omega^k_{\mathbb{Z}} \arrow[ur, "s"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0.
\end{tikzcd} \tag{1.17}$$

The maps α, B, r are obtained from the long exact sequence

$$\ldots \to H^k(\,\cdot\,,\mathbb{Z}) \xrightarrow{r} H^k(\,\cdot\,,\mathbb{R}) \xrightarrow{\alpha} H^k(\,\cdot\,,\mathbb{R}/\mathbb{Z}) \xrightarrow{B} H^{k+1}(\,\cdot\,,\mathbb{Z}) \to \ldots \tag{1.18}$$

associated to the coefficient short exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$, and β, d, s are defined as follows. The map β-by

$$H^{k-1}(\,\cdot\,,\mathbb{R}) \xrightarrow{\simeq} H^{k-1}_{\mathrm{dR}}(\,\cdot\,) \hookrightarrow \Omega^{k-1}/\mathrm{d}\Omega^{k-2} \twoheadrightarrow \Omega^{k-1}/\Omega^{k-1}_{\mathbb{Z}}, \tag{1.19}$$

using de Rham Theorem, and the fact that $\mathrm{d}\Omega^{k-2} \subseteq \Omega^{k-1}_{\mathbb{Z}}$. Since $\mathrm{d}\Omega^{k-1}_{\mathbb{Z}} = 0$, the de Rham differential is well-defined on classes in $\Omega^{k-1}/\Omega^{k-1}_{\mathbb{Z}}$. Finally,

$$s : \Omega^k_{\mathbb{Z}} \hookrightarrow \ker \mathrm{d}^k \twoheadrightarrow \ker \mathrm{d}^k/\mathrm{d}\Omega^{k-1} = H^k_{\mathrm{dR}}(\,\cdot\,) \xrightarrow{\simeq} H^k(\,\cdot\,,\mathbb{R}). \tag{1.20}$$

Proposition 1.0.7 ([SS07])**.** *The differential characters $\hat{H}^\bullet$ substituted for $\hat{G}^\bullet$ fit into the character diagram, and, as such, form a character functor together with appropriate natural transformations $i_1, i_2, \delta_1, \delta_2$.*

Proof. Fix a manifold M and let $f \in \hat{H}^k(M, \mathbb{R}/\mathbb{Z})$. We begin by defining $\delta_1(f) \coloneqq \omega_f$. Naturality of this assignment follows from 1.15. For surjectivity, let $\omega \in \Omega^k_{\mathbb{Z}}(M)$ be arbitrary. Define

$$f(c) = \begin{cases} \iota(\omega)(b) & c = \partial b \in B_{k-1}(M) \\ 0 & \text{otherwise.} \end{cases} \tag{1.21}$$

Then, $\delta_1(f) = \omega$. Since $\mathbb{R}/\mathbb{Z}$ is a divisible group, and thus an injective abelian group, Universal Coefficient Theorem asserts

$$H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(H_{k-1}(M), \mathbb{R}/\mathbb{Z}\big). \tag{1.22}$$

Therefore, by the left exactness of the left hom-functor, the canonical projection in the short exact sequence

$$0 \to B_{k-1}(M) \hookrightarrow Z_{k-1}(M) \twoheadrightarrow Z_{k-1}(M)/B_{k-1}(M) = H_{k-1}(M) \to 0 \tag{1.23}$$

induces the inclusion

$$H^{k-1}(M,\mathbb{R}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(H_{k-1}(M),\mathbb{R}/\mathbb{Z}\big) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M),\mathbb{R}/\mathbb{Z}\big). \tag{1.24}$$

This gives the map $i_1 : H^{k-1}(M,\mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{H}^k(M,\mathbb{R}/\mathbb{Z})$. Indeed, by construction for any $[r] \in H^{k-1}(M,\mathbb{R}/\mathbb{Z})$ and any $\partial b \in B_{k-1}(M)$ we have

$$i_1([r])(\partial b) = \delta r(b) = 0. \tag{1.25}$$

Therefore, by Universal Coefficient Theorem and the left exactness of the hom-functor, the image of i_1 corresponds to differential characters with zero curvature. In other words, $\mathrm{im}(i_1) = \ker(\delta_1)$.
The restriction

$$\mathrm{res} : C^{k-1}(M,\mathbb{R}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}\big(C_{k-1}(M),\mathbb{R}/\mathbb{Z}\big) \to \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M),\mathbb{R}/\mathbb{Z}\big) \tag{1.26}$$

composed with ι gives a map

$$\mathrm{res} \circ \iota : \Omega^{k-1}(M) \to \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M),\mathbb{R}/\mathbb{Z}\big), \tag{1.27}$$

whose kernel is composed of those $(k-1)$-forms, which integrate to integers on cycles. This is precisely $\Omega^{k-1}_{\mathbb{Z}}(M)$ (Corollary 1.0.4). Note that, by Stokes Theorem, for any $\partial b \in B_{k-1}(M)$, we have

$$(\mathrm{res} \circ \iota)(\omega)(\partial b) = \iota(\mathrm{d}\omega)(b). \tag{1.28}$$

Thus, $\mathrm{res} \circ \iota$ defines a map to $\hat{H}^k(M,\mathbb{R}/\mathbb{Z})$. We define i_2 as its factorization through $\Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M)$, which is manifestly injective. Moreover, 1.28 proves commutativity of the bottom triangle in 1.17.
In order to define δ_2 consider the following diagram:

$$\begin{tikzcd}
\mathbb{R} \arrow[rr, two heads] & & \mathbb{R}/\mathbb{Z} \\
& C_{k-1}(M) \arrow[ul, dashed, "T"'] & \\
& Z_{k-1}(M). \arrow[u, hook] \arrow[uul, dashed, bend left, "\tilde{f}"] \arrow[uur, bend right, "f"'] &
\end{tikzcd} \tag{1.29}$$

Given $f \in \hat{H}^k(M,\mathbb{R}/\mathbb{Z})$, we use the fact that $Z_{k-1}(M)$ is a free abelian group to lift f to $\tilde{f}$. Then, we use the injectivity of $\mathbb{R}$ as an abelian group to factor $\tilde{f}$ through $C_{k-1}(M)$, and we call this factorization T. Observe that for any $c \in C_k(M)$ we get

$$\delta T(c) \bmod \mathbb{Z} = T(\partial c) \bmod \mathbb{Z} = \tilde{f}(\partial c) \bmod \mathbb{Z} = f(\partial c) = \iota(\omega_f)(c) = \omega_f(c) \bmod \mathbb{Z}, \tag{1.30}$$

where after the last equality we treat ω_f as a cochain given by integration, i.e.,

$$\omega_f \mapsto \Big(c \mapsto \int_c \omega_f\Big). \tag{1.31}$$

Therefore, under this identification, $\omega_f - \delta T \in C^k(M,\mathbb{Z})$ and it is closed, since ω_f is closed. Indeed, by de Rham Theorem, the cochain associated to ω_f is closed if and only if ω_f is d-closed. Moreover, if T' is another map making 1.29 commutative, we compute

$$(T - T')\big|_{Z_{k-1}(M)} = \tilde{f} - \tilde{f} = 0 \quad \Rightarrow \quad T - T' = \delta d,\ d \in C^{k-2}(M,\mathbb{R}). \tag{1.32}$$

Here, we used Universal Coefficient Theorem to infer from $T-T'=0$ in $\mathrm{Hom}_{\mathbb{Z}}\big(H_{k-1}(M),\mathbb{R}\big)$ the equality $[T-T']=0$ in $H^{k-1}(M,\mathbb{R})$. If we pick a different lift $\tilde{f}'$, we get $\tilde{f}'-\tilde{f} \bmod \mathbb{Z} = f-f=0$, so $\tilde{f}-\tilde{f}'=c\big|_{Z_{k-1}(M)}$ for some $c\in C^{k-1}(M,\mathbb{Z})$. We conclude that in general

$$T-T'=\delta d+c \quad\Rightarrow\quad \delta(T-T')=\delta c. \tag{1.33}$$

This means that the cohomology class $[\omega_f-\delta T]\in H^k(M,\mathbb{Z})$ depends only on f. We define $\delta_2(f):=[\omega_f-\delta T]$.
To see that it is surjective, let $[u]\in H^k(M,\mathbb{Z})$ be arbitrary. By Remark 1.0.2, there exists $\omega\in\Omega^k_{\mathbb{Z}}(M)$ with $[\omega]=i_*[u]$ under indentification by de Rham's isomorphism. Then, since $[\omega-i_*u]=0$, for any representative u, the cochain $\omega-i_*u\in C^k(M,\mathbb{R})$ is exact, so we can find $T\in C^{k-1}(M,\mathbb{R})$ with $\delta T=\omega-i_*u$. By postcomposing the restricted cochain $T\big|_{Z_{k-1}(M)}$ with the natural projection $\mathbb{R}\twoheadrightarrow\mathbb{R}/\mathbb{Z}$, we obtain

$$f\in\mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M),\mathbb{R}/\mathbb{Z}\big),\quad f\circ\partial=\iota(\omega), \tag{1.34}$$

a differential character, which satisfies $\delta_2(f)=[u]$. Indeed, we have $\omega-\delta T=i_*u$ by construction.
Now, suppose $\delta_2(f)=0$, that is $[\omega_f-\delta T]=0$. Since δT is exact, using de Rham Theorem, we infer that

$$\exists\theta\in\Omega^{k-1}(M):\mathrm{d}\theta=\omega_f, \tag{1.35}$$

and

$$\exists e\in C^{k-1}(M,\mathbb{Z}):\omega_f-\delta T=\delta e. \tag{1.36}$$

We calculate $\delta(\theta-T-e)=0$, so there exists $\zeta\in Z^{k-1}(M,\mathbb{R})$ such that

$$\theta-T-e=\zeta. \tag{1.37}$$

By de Rham Theorem we can find

$$\phi\in\Omega^{k-1}(M):(\theta-T-e)\big|_{Z_{k-1}(M)}=\phi\big|_{Z_{k-1}(M)}. \tag{1.38}$$

We have $T\big|_{Z_{k-1}(M)}=(\theta-\phi-e)\big|_{Z_{k-1}(M)}$. By postcomposing both sides with the natural projection $\mathbb{R}\twoheadrightarrow\mathbb{R}/\mathbb{Z}$, we obtain $f=\iota(\theta-\phi)\big|_{Z_{k-1}(M)}=\iota(\theta-\phi)\big|_{Z_{k-1}(M)}=i_2(\theta-\phi)$, as e is $\mathbb{Z}$-valued. Hence, $f\in\mathrm{im}(i_2)$. We conclude that both diagonal sequences are exact. We proceed by checking commutativity of the left side of the character diagram.
Since i_2 was defined as a factorization of $\mathrm{res}\circ\iota$ through the quotient by its kernel, let $[r]\in H^{k-1}(M,\mathbb{R})$ and let us pick a representative $\omega\in\Omega^{k-1}(M)$ of $\beta([r])$ within the class in $\Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M)$. Then, $(i_2\circ\beta)([r])=(\mathrm{res}\circ\iota)(\omega)$. Using Universal Coefficient Theorem, we identify α with

$$\begin{array}{ccc} H^{k-1}(M,\mathbb{R}) & \xrightarrow{\quad\alpha\quad} & H^{k-1}(M,\mathbb{R}/\mathbb{Z}) \\ \big\downarrow\simeq & & \big\downarrow\simeq \\ \mathrm{Hom}_{\mathbb{Z}}\big(H_{k-1}(M),\mathbb{R}\big) & \xrightarrow{\pi_*} & \mathrm{Hom}_{\mathbb{Z}}\big(H_{k-1}(M),\mathbb{R}/\mathbb{Z}\big) \end{array} \tag{1.39}$$

Now, it follows from de Rham Theorem that

$$(\mathrm{res}\circ\iota)(\omega)=\mathrm{pr}^*\circ\pi_*([r])=(i_1\circ\alpha)([r]), \tag{1.40}$$

where pr : $Z_{k-1}(M) \twoheadrightarrow H_{k-1}(M)$ of 1.23 was used to define i_1.
We already showed under 1.28 that the bottom triangle commutes. The top triangle commutes by the construction of δ_2 above, where taking $f \in \mathrm{im}(i_1)$ implies $\omega_f = 0$ by exactness of the diagonal sequence in 1.17, and we recover the usual connecting homomorphism $-B$. Finally, the right side of the diagram commutes by Remark 1.0.2, since, by construction, $r(\delta_2(f)) = r([\omega_f - \delta T]) = [\omega_f] \in H^k(M, \mathbb{R})$. The naturality of δ_1 follows from 1.15. The map i_2 is a natural transformation by the change of variables formula for integrals. Pullbacks and pushforwards induce natural transformation for groups of cochains, hence i_1 is a natural transformation. Finally, δ_2 was built using universal constructions, so it is also natural. $\square$

Theorem 1.0.8 ([SS07]). *For any character functor $(\hat{G}^\bullet, i_1, i_2, \delta_1, \delta_2)$ there exists a unique natural isomorphism $\Phi : \hat{G}^\bullet \to \hat{H}^\bullet(\,\cdot\,, \mathbb{R}/\mathbb{Z})$, which commutes with identity transformations on all other functors in the diagram. In other words, the diagram formed by two character diagrams–corresponding to $\hat{G}^\bullet$ and $\hat{H}^\bullet$–connected with identity transformations and Φ is commutative.*

We do not provide the proof, which we are about to generalize in Section 3.1.

Chapter 2

Differential Cohomology with Local Coefficients

There are numerous equivalent approaches to cohomology with local coefficients. In this thesis, we adopt the so called "modular approach". For a broader overview of cohomology with local coefficients and the proofs of equivalence see [Hat02, 3.H.].

Definition 2.0.1 ([DK01, Ch. 5.])**.** Let $\pi_1(M) \coloneqq \pi_1(M, q)$ be the fundamental group of a connected based smooth manifold (M, q). Denote by $\mathbb{Z}\pi_1(M)$ the induced group ring and let A be a left $\mathbb{Z}\pi_1(M)$-module. Consider the universal covering

$$p : \tilde{M} \twoheadrightarrow M \tag{2.1}$$

and a smooth singular chain complex $C_\bullet(\tilde{M})$. Note that $C_\bullet(\tilde{M})$ carries a natural structure of a left $\mathbb{Z}\pi_1(M)$-module: for $\sigma \in C_k(\tilde{M})$ and $g \in \pi_1(M)$, we define $g \cdot \sigma$ as $\Theta(g) \circ \sigma \in C^\infty(\Delta^k, \tilde{M})$, where $\Theta : \pi_1(M) \xrightarrow{\simeq} \mathrm{Deck}(p)$ is the canonical isomorphism [Hat02, 1.3.]. This can be extended to a $\mathbb{Z}$-linear map. Importantly, the action of $\mathbb{Z}\pi_1(M)$ commutes with the boundary operation ∂ on $C_\bullet(\tilde{M})$. This makes it possible to define the cochain complex

$$C^\bullet(M; A) \coloneqq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), A\big). \tag{2.2}$$

The cohomology of this complex is called *the cohomology of M with local coefficients in A* and is denoted by $H^\bullet(M; A)$. We will also denote by $Z^\bullet(M; A)$ the subcomplex of closed cochains, and by $B^\bullet(M; A)$ the subcomplex of exact cochains.

Remark 2.0.2. If we wish to emphasize the homomorphism $\rho : \pi_1(M) \to \mathrm{Aut}_{\mathbb{Z}}(A)$ making the abelian group A into a $\mathbb{Z}\pi_1(M)$-module, we write $H^\bullet(M; A_\rho)$, and call it the *cohomology of M twisted by ρ*. If ρ is the trivial homomorphism, the map $p_\bullet : C_\bullet(\tilde{M}) \to C_\bullet(M)$ induces a chain isomorphism $\mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet(M), A\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), A\big)$, since $\mathrm{Deck}(p)$ acts freely and transitively on the fibers of p. Therefore, in this case, $H^\bullet(M; A_\rho)$ becomes the usual cohomology of M with coefficients the abelian group A.

Lemma 2.0.3. *Let*

$$0 \to \Lambda \xrightarrow{i} A \xrightarrow{p} A/\Lambda \to 0 \tag{2.3}$$

be a short exact sequence in $\mathbb{Z}\pi_1(M)$-Mod. It induces a long exact sequence in cohomology of M with local coefficients

$$\ldots \to H^{k-1}(M; A/\Lambda) \to H^k(M; \Lambda) \to H^k(M; A) \to H^k(M; A/\Lambda) \to H^{k+1}(M; \Lambda) \to \ldots \tag{2.4}$$

Proof. Since the maps i, p are $\mathbb{Z}\pi_1(M)$-maps, they induce chain maps between the respective cochain complexes. Moreover, the chain maps form a short exact sequence, because each $C_k(\tilde{M})$ is a free $\mathbb{Z}\pi_1(M)$-module. We have

$$0 \to C^\bullet(M;\Lambda) \to C^\bullet(M;A) \to C^\bullet(M;A/\Lambda) \to 0. \tag{2.5}$$

Hence, we obtain the long exact sequence in cohomology associated to the sequence of complexes [Wei94, 1.3]. □

2.1. Functoriality

From now on, we assume all manifolds to be connected. In particular we will denote by $\mathbf{Man}_*$ the category of based connected manifolds. One can extend the action deck transformations trivially to other connected components of $\tilde{M}$, but if the homomorphism $\rho : \pi_1(M) \to \mathrm{Aut}_{\mathbb{Z}}(A)$ is nontrivial, there is an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), A\big) \simeq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}_*), A\big), \tag{2.6}$$

where $\tilde{M}_*$ is the universal covering of the connected component of the chosen $q \in M$. This makes the extension to disconnected manifolds uninteresting. In order to describe the functoriality of $H^\bullet$, note that for a fixed based manifold (M, q), any morphism of $\mathbb{Z}\pi_1(M)$-modules $\varphi : A \to B$, induces a chain map

$$\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), A\big) \xrightarrow{\varphi_*} \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), B\big), \tag{2.7}$$

and thus, a morphism in cohomology. Therefore, each based manifold (M, q) corresponds to a functor

$$F_M : \mathbb{Z}\pi_1(M)\text{-}\mathbf{Mod} \to \mathbf{Ab}^{\mathbb{Z}}. \tag{2.8}$$

Let $(M, q_M) \to (N, q_N)$ be a map of based manifolds. By the lifting property of universal coverings [Hat02, 1.3.], there exist functors

$$(\tilde{\cdot}) : \mathbf{Man}_* \to \mathbf{Man}_* : (M, q_M) \mapsto (\tilde{M}, \tilde{q}_M), \tag{2.9}$$

parametrized by the choice of $\tilde{q}_M \in p^{-1}(q_M)$. However, under the composition $C_\bullet \circ (\tilde{\cdot})$, any two such functors are naturally isomorphic via an appropriate deck transformation. Moreover, we can make use of the functor $\mathbb{Z}\pi_1\text{-}\mathbf{Mod} : \mathbf{Man}_*^{\mathrm{op}} \to \mathbf{Cat}$, which sends a map of based manifolds $f : (M, q_M) \to (N, q_N)$ to the functor

$$\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f) : \mathbb{Z}\pi_1(N)\text{-}\mathbf{Mod} \to \mathbb{Z}\pi_1(M)\text{-}\mathbf{Mod} \tag{2.10}$$

defined as the restriction of scalars for the morphism $\mathbb{Z}\pi_1(f) : \mathbb{Z}\pi_1(M) \to \mathbb{Z}\pi_1(N)$. We argue that the map f gives rise to a natural transformation

$$F_N \xRightarrow{H^\bullet_{\mathrm{ch}}(\xi(f))} F_M \circ \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f). \tag{2.11}$$

Pick $A \in \mathbb{Z}\pi_1(N)\text{-}\mathbf{Mod}$. We define

$$\mathrm{Hom}_{\mathbb{Z}\pi_1(N)}\big(C_\bullet(\tilde{N}), A\big) \xrightarrow{\xi(f)_A} \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(A)\big) \tag{2.12}$$

by setting

$$\xi(f)_A(\psi)(\sigma) \equiv \psi(\tilde{f}_*\sigma) \in \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(A) \tag{2.13}$$

for $\sigma \in C_k(\tilde{M})$. To check equivariance we take $g \in \pi_1(M)$ and compute

$$\begin{aligned} &\xi(f)_A(\psi)(\Theta_M(g)_*\sigma) = \psi(\tilde{f}_*\Theta_M(g)_*\sigma) = \psi\big((\tilde{f} \circ \Theta_M(g))_*\sigma\big) \\ &= \psi\big((\Theta_N(\pi_1(f)(g)) \circ \tilde{f}\,)_*\sigma\big) = \psi\big(\Theta_N\big(\pi_1(f)(g)\big)_*\tilde{f}_*\sigma\big) = \pi_1(f)(g) \cdot \psi(\tilde{f}_*\sigma), \end{aligned} \tag{2.14}$$

where we used the lifting property of universal coverings again. By functoriality of $\mathbb{Z}\pi_1$-$\mathbf{Mod}$, for any $\varphi \in \mathrm{Hom}_{\mathbb{Z}\pi_1(N)}(A, B)$ the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}\pi_1(N)}\big(C_\bullet(\tilde{N}), A\big) \arrow[r, "\xi(f)_A"] \arrow[d, "\varphi_*"] & \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(A)\big) \arrow[d, "\left(\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(\varphi)\right)_*"] \\ \mathrm{Hom}_{\mathbb{Z}\pi_1(N)}\big(C_\bullet(\tilde{N}), B\big) \arrow[r, "\xi(f)_B"] & \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(\tilde{M}), \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(B)\big) \end{tikzcd} \tag{2.15}$$

is commutative, so $\xi(f)$ is indeed a natural transformation. It is also clear that each $\xi(f)_A$ is a chain map. Therefore, one can apply the chain cohomology functor[1] $H^\bullet_{\mathrm{ch}}$ to obtain the desired transformation. With the above data, we can construct a functor. Let

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \tag{2.16}$$

be the category of elements of $\mathbb{Z}\pi_1$-$\mathbf{Mod}$. That is, a category whose objects are pairs $\big((M,q),A\big)$ with $(M,q) \in \mathbf{Man}_*^{\mathrm{op}}$ and $A \in \mathbb{Z}\pi_1\text{-}\mathbf{Mod}((M,q)) = \mathbb{Z}\pi_1(M)\text{-}\mathbf{Mod}$, and whose morphisms are pairs

$$(\bar{f}, \varphi) : \big((N, q_N), B\big) \to \big((M, q_M), A\big), \tag{2.17}$$

where $\bar{f} \in \mathrm{Hom}_{\mathbf{Man}_*^{\mathrm{op}}}\big((N,q_N),(M,q_M)\big)$ and $\varphi \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(B), A\big)$. From now on, we will use the bar notation to denote morphisms in the opposite category. The law of composition in $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}$ is the following: for

$$(\bar{h}, \psi) : \big((M, q_M), A\big) \to \big((N, q_N), B\big) \tag{2.18}$$

and

$$(\bar{f}, \varphi) : \big((N, q_N), B\big) \to \big((P, q_P), C\big) \tag{2.19}$$

we set

$$(\bar{f}, \varphi) \circ (\bar{h}, \psi) \equiv \big(\bar{f} \circ \bar{h}, \varphi \circ \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi)\big). \tag{2.20}$$

Finally, let us define the functor

$$H^\bullet_0 : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \to \mathbf{Ab}^{\mathbb{Z}}. \tag{2.21}$$

We already know the definition on objects. We set

$$\begin{aligned} &H^\bullet_0(\bar{f}, \varphi) \equiv \varphi_* \circ H^\bullet_{\mathrm{ch}}(\xi(f)_B) : \\ &H^\bullet_0(N; B) \xrightarrow{H^\bullet_{\mathrm{ch}}(\xi(f)_B)} H^\bullet_0(M; \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f})(B)) \xrightarrow{\varphi_*} H^\bullet_0(M; A). \end{aligned} \tag{2.22}$$

Obviously, $H^\bullet_0(\mathrm{id}_{(M,q)}, \mathrm{id}_A) = \mathrm{id}_{H^\bullet_0(M;A)}$. The preservation of composition can be stated as follows:

$$\big(\varphi \circ \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi)\big)_* \circ H^\bullet_{\mathrm{ch}}\big(\xi(f \circ h)_A\big) = \varphi_* \circ H^\bullet_{\mathrm{ch}}\big(\xi(f)_B\big) \circ \psi_* \circ H^\bullet_{\mathrm{ch}}\big(\xi(h)_A\big). \tag{2.23}$$

[1] Not to confuse with the cohomology functor being described.

This reduces to

$$H^\bullet_{\mathrm{ch}}\big(\xi(f)_B\big) \circ \psi_* = \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi) \circ H^\bullet_{\mathrm{ch}}\big(\xi(f)_{\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{h})(A)}\big), \tag{2.24}$$

which is precisely the naturality condition for $H^\bullet_{\mathrm{ch}}\big(\xi(f)\big)$, which we have already proved. Therefore, we have given the cohomology with local coefficients a functor structure. However, the functor structure described above does not yet exhaust the topological content. Let us fix $\big((M,q),A\big) \in \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}$. It makes sense to define $H^\bullet(U;A)$ for $U \subseteq M$ as cohomology of $\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(p^{-1}(U)),A\big)$, where the action on $\sigma \in C_k(p^{-1}(U))$ is

$$g \cdot \sigma \equiv \Theta(g)\big|_{p^{-1}(U)} \circ \sigma. \tag{2.25}$$

This is well defined, because $p \circ \Theta(g) = p$ for any g. In other words, the action by deck transformations preserves fibers. Moreover, a map $i : V \hookrightarrow U$ gives rise to the inclusion

$$p^{-1}(V) \hookrightarrow p^{-1}(U), \tag{2.26}$$

which induces a $\mathbb{Z}\pi_1(M)$-equivariant chain map

$$C_\bullet(p^{-1}(V)) \xrightarrow{i_*} C_\bullet(p^{-1}(U)). \tag{2.27}$$

Using 2.27 we define the cochain map

$$C^\bullet(U;A) \coloneqq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(p^{-1}(U)),A\big) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet(p^{-1}(V)),A\big) \eqqcolon C^\bullet(V;A). \tag{2.28}$$

Also, it is obvious that for

$$W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U \tag{2.29}$$

we have $(i \circ j)^* = j^* \circ i^*$. Applying $H^\bullet_{\mathrm{ch}}$, we obtain a functor $\mathbf{Top}(M)^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}$, where $\mathbf{Top}(M)$ is the category of open subsets of M and inclusions. We will refer to this structure as the *presheaf* structure. The crucial observation is that the two functor structures are compatible, in the following sense. Let

$$(\bar{f},\varphi) : \big((N,q_N),B\big) \to \big((M,q_M),A\big), \tag{2.30}$$

and take any open $V \subseteq N$. Examining 2.12 and 2.13, we see that $\xi(f)_A$ restricts to

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}\pi_1(N)}\big(C_\bullet(p_N^{-1}(V)),A\big) \xrightarrow{\xi(f)_A|_V} &\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet\big(\tilde{f}^{-1}(p_N^{-1}(V))\big),\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(A)\big)\\ &= \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_\bullet\big(p_M^{-1}(f^{-1}(V))\big),\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(f)(A)\big),\end{aligned} \tag{2.31}$$

where p_M, p_N denote the respective universal covering maps. Moreover, upon restricting the actions of $\mathbb{Z}\pi_1(M)$ and $\mathbb{Z}\pi_1(N)$ as in 2.25, the restrictions $\xi(f)_A\big|_V$ compose a natural transformation in the sense of 2.15, and the restriction of 2.22 to open V preserves compositions. Thus, we have just obtained an extension of the functor $H^\bullet_0$. That is, together with $H^\bullet(U;A)$ for open subsets $U \subseteq M$, a collection of maps

$$\begin{aligned}H^\bullet_0(\bar{f},\varphi)_V : H^\bullet(V;B) \xrightarrow{H^\bullet_{\mathrm{ch}}\left(\xi(f)_B|_V\right)} H^\bullet(f^{-1}(V);\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(B)) \xrightarrow{\varphi_*}&\\ \xrightarrow{\varphi_*} H^\bullet(f^{-1}(V);A),&\end{aligned} \tag{2.32}$$

which preserve compositions in $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}$. This seemingly chaotic structure will be fitted neatly once we establish the following. Let $j_N : V \hookrightarrow U$ and $j_M : f^{-1}(V) \hookrightarrow f^{-1}(U)$ be inclusions of open sets in N and M respectively. Then, the following diagram is commutative:

$$\begin{tikzcd} H^\bullet(U;B) \arrow[r, "j_N^*"] \arrow[d, "H_0^\bullet(\bar{f},\varphi)_U"] & H^\bullet(V;B) \arrow[d, "H_0^\bullet(\bar{f},\varphi)_V"] \\ H^\bullet(f^{-1}(U);A) \arrow[r, "j_M^*"] & H^\bullet(f^{-1}(V);A). \end{tikzcd} \tag{2.33}$$

This simply follows from $f\big|_{f^{-1}(U)} \circ j_M = j_N \circ f\big|_{f^{-1}(V)}$. In consequence, the collection $\{H_0^\bullet(\bar{f},\varphi)_U\}_{U\in\mathbf{Top}(M)}$ forms a natural transformation between the two presheaf structures on N and M respectively. The above is precisely the necessary data to construct a functor

$$H^\bullet : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}, \tag{2.34}$$

where the domain is the category of elements of the product functor $\mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}}$. The functor

$$\mathbf{Top}^{\mathrm{op}} : \mathbf{Man}_*^{\mathrm{op}} \to \mathbf{Cat} \tag{2.35}$$

has the object component $(M,q) \mapsto \mathbf{Top}^{\mathrm{op}}(M)$ and for any $f : (M,q_M) \to (N,q_N)$ the corresponding functor $\mathbf{Top}^{\mathrm{op}}(\bar{f}\,)$ is defined by

$$\forall V \in \mathbf{Top}^{\mathrm{op}}(N) : \mathbf{Top}^{\mathrm{op}}(\bar{f}\,)(V) = f^{-1}(V) \in \mathbf{Top}^{\mathrm{op}}(M), \tag{2.36}$$

and

$$\forall \bar{j} \in \mathrm{Hom}_{\mathbf{Top}^{\mathrm{op}}(N)}(U,V) : \mathbf{Top}^{\mathrm{op}}(\bar{f}\,)(\bar{j}) =: f^{-1}(\bar{j}) \in \mathrm{Hom}_{\mathbf{Top}^{\mathrm{op}}(N)}\big(f^{-1}(U), f^{-1}(V)\big). \tag{2.37}$$

The functoriality is obvious. The objects in $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}\times\mathbf{Top}^{\mathrm{op}}$ are triples $\big((M,q),A,U\big)$ where $\big((M,q),A\big) \in \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}$ and $U \in \mathbf{Top}^{\mathrm{op}}(M,q)$. The morphisms between $\big((N,q_N),B,V\big)$ and $\big((M,q_M),A,U\big)$ are triples $(\bar{f},\varphi,\bar{j})$ such that

$$(\bar{f},\varphi) \in \mathrm{Hom}_{\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}}\Big(\big((N,q_N),B\big),\big((M,q_M),A\big)\Big), \tag{2.38}$$

and

$$\bar{j} \in \mathrm{Hom}_{\mathbf{Top}^{\mathrm{op}}(M)}\big(\mathbf{Top}^{\mathrm{op}}(\bar{f}\,)(V),U\big) = \mathrm{Hom}_{\mathbf{Top}^{\mathrm{op}}(M)}\big(f^{-1}(V),U\big), \tag{2.39}$$

corresponding to an inclusion $j : U \hookrightarrow f^{-1}(V)$ in M. The composition is as follows: for

$$(\bar{h},\psi,\bar{i}) : \big((M,q_M),A,U\big) \to \big((N,q_N),B,V\big) \tag{2.40}$$

and

$$(\bar{f},\varphi,\bar{j}) : \big((N,q_N),B,V\big) \to \big((P,q_P),C,W\big), \tag{2.41}$$

we set

$$\begin{aligned} (\bar{f},\varphi,\bar{j}) \circ (\bar{h},\psi,\bar{i}) &\equiv \big(\bar{f}\circ\bar{h}, \varphi\circ\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi), \bar{j}\circ\mathrm{Top}^{\mathrm{op}}(\bar{f}\,)(\bar{i})\big) \\ &= \big(\bar{f}\circ\bar{h}, \varphi\circ\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi), \bar{j}\circ f^{-1}(\bar{i})\big). \end{aligned} \tag{2.42}$$

We define the object component of $H^\bullet$ by

$$H^\bullet(U;A) \coloneqq H^\bullet\big((M,q),A,U\big) \equiv H^\bullet_{\mathrm{ch}}(C^\bullet(U;A)). \tag{2.43}$$

On morphisms we set

$$H^\bullet(\bar{f},\varphi,\bar{j}) \equiv j^* \circ \varphi_* \circ H^\bullet_{\mathrm{ch}}\big(\xi(f)_B\big|_V\big) = j^* \circ H^\bullet(\bar{f},\varphi)_V : \\ H^\bullet(V;B) \xrightarrow{H^\bullet(\bar{f},\varphi)_V} H^\bullet(f^{-1}(V);A) \xrightarrow{j^*} H^\bullet(U;A). \tag{2.44}$$

Clearly, $H^\bullet(\mathrm{id}_{(M,q)}, \mathrm{id}_A, \mathrm{id}_M) = \mathrm{id}_{H^\bullet(M;A)}$. The preservation of composition follows from 2.33. Indeed,

$$\begin{aligned} &H^\bullet\big(\bar{f}\circ\bar{h}, \varphi\circ\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi), \bar{j}\circ f^{-1}(\bar{i})\big) \\ &= j^*\circ f^{-1}(i)^*\circ H^\bullet_0\big(\bar{f}\circ\bar{h}, \varphi\circ\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(\psi)\big)_U \\ &= j^*\circ f^{-1}(i)^*\circ H^\bullet_0(\bar{f},\varphi)_{h^{-1}(U)}\circ H^\bullet_0(\bar{h},\psi)_U \\ &= j^*\circ H^\bullet_0(\bar{f},\varphi)_V\circ i^*\circ H^\bullet_0(\bar{h},\psi)_U \\ &= H^\bullet_0(\bar{f},\varphi,\bar{j})\circ H^\bullet_0(\bar{h},\psi,\bar{i}). \end{aligned} \tag{2.45}$$

This completes the construction of the functor structure of cohomology with local coefficients.

2.2. Differential Forms Valued in $\mathbb{Z}\pi_1(M)$-$\mathbb{R}$-Bimodules

Note that the differential graded algebra of differential forms $\Omega^\bullet(\tilde{M})$ admits a right action of $\pi_1(M)$ defined by $\omega\cdot g := \Theta(g)^*\omega$, which extends to a right action of $\mathbb{Z}\pi_1(M)$. Suppose A is an $\mathbb{Z}\pi_1(M)$-$\mathbb{R}$-bimodule. In other words, a real vector space equipped with a linear left action of $\mathbb{Z}\pi_1(M)$. Then, the differential graded algebra of A-valued differential forms $\Omega^\bullet(\tilde{M})\otimes_{\mathbb{R}} A$ can de viewed as a $\mathbb{Z}\pi_1(M)$-module in two ways–with respect to the first and the second component. Moreover, these two actions commute. Therefore, it is natural to give

Definition 2.2.1. The graded abelian group of *$\mathbb{Z}\pi_1(M)$-invariant A-valued differential forms* is defined as

$$\Omega^\bullet_{\mathrm{I}}(M;A) := \Omega^\bullet(\tilde{M};A)^{\mathbb{Z}\pi_1(M)} \subseteq \Omega^\bullet(\tilde{M})\otimes_{\mathbb{R}} A \tag{2.46}$$

satisfying

$$\sum_i \Theta(g)^*\omega_i\otimes_{\mathbb{R}} a_i = \sum_i \omega_i\otimes_{\mathbb{R}} g\cdot a_i \tag{2.47}$$

for every $g\in\pi_1(M)$. We define a differential $\mathrm{d}^\bullet : \Omega^\bullet_{\mathrm{I}}(M;A)\to\Omega^{\bullet+1}_{\mathrm{I}}(M;A)$ as a linear extension of

$$\mathrm{d}(\omega\otimes_{\mathbb{R}} a) := \mathrm{d}(\omega)\otimes_{\mathbb{R}} a. \tag{2.48}$$

We check its well-definedness using the fact that the de Rham differential commutes with pullbacks:

$$\Theta(g)^*\mathrm{d}\omega\otimes_{\mathbb{R}} a = \big(\mathrm{d}\Theta(g)^*\omega\big)\otimes_{\mathbb{R}} a = \mathrm{d}\big(\Theta(g)^*\omega\otimes_{\mathbb{R}} a\big) = \mathrm{d}(\omega\otimes_{\mathbb{R}} g\cdot a) = \mathrm{d}\omega\otimes_{\mathbb{R}}(g\cdot a). \tag{2.49}$$

Clearly, $\mathrm{d}^2 = 0$ which makes $\Omega^\bullet_{\mathrm{I}}(M;A)$ into a cochain complex. We will call its cohomology the *twisted de Rham cohomology with values in A*, and denote it by $H^\bullet_{\mathrm{dR}}(M;A)$.

We make invariant module-valued differential forms into a functor $\int_{\mathbf{Man}_*^{\mathrm{op}}}\mathbb{Z}\pi_1\text{-}\mathbf{Mod}\to\mathbf{Ab}^{\mathbb{Z}}$ in a similar fashion. First, we observe the functoriality $\mathbb{Z}\pi_1(M)\text{-}\mathbf{Mod}\ni A\mapsto\Omega^\bullet_{\mathrm{I}}(\tilde{M};A)$ for a fixed M. Then, we check that for each map $f: M\to N$ and every

$$\sum_i \omega_i\otimes_{\mathbb{R}} b_i\in\Omega^\bullet_{\mathrm{I}}(N,B), \tag{2.50}$$

the form

$$\sum_i \tilde{f}^* \omega_i \otimes_{\mathbb{R}} b_i \in \Omega^\bullet(\tilde{M}) \otimes_{\mathbb{R}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(B) \tag{2.51}$$

is invariant under $\mathbb{Z}\pi_1(M)$, because

$$\forall g \in \pi_1(M) : \tilde{f} \circ \Theta_M(g) = \Theta_N(\pi_1(f)(g)) \circ \tilde{f} \tag{2.52}$$

(cf. 2.14). Therefore, by applying $\varphi_* : \Omega^\bullet_{\mathrm{I}}\big(M; \mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f}\,)(B)\big) \to \Omega^\bullet_{\mathrm{I}}(M; A)$, we complete the construction. Since both of the above operations commute with the differential, the functoriality descends to the twisted de Rham cohomology, which gives a functor

$$H^\bullet_{0,\mathrm{dR}} : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \to \mathbf{Ab}^{\mathbb{Z}}. \tag{2.53}$$

Note that again, any choice of $(\tilde{\cdot})$ functor yields the same algebra of differential forms, up to an isomorphism. To introduce the presheaf structure, we define $H^\bullet_{\mathrm{dR}}(U; A)$ as cohomology of

$$\Omega^\bullet_{\mathrm{I}}(U; A) \coloneqq \Omega^\bullet(p^{-1}(U); A)^{\mathbb{Z}\pi_1(M)}, \tag{2.54}$$

where the action on differential forms is $\omega \cdot g \equiv \Theta(g)\big|^*_{p^{-1}(U)}\omega$. For $i : V \hookrightarrow U$, the restriction map i^* is an equivariant cochain map. Indeed, one has

$$\Theta(g)\big|^*_{p^{-1}(V)} i^*\omega = i^*\Theta(g)\big|^*_{p^{-1}(U)}\omega, \tag{2.55}$$

as a consequence of $p \circ \Theta(g) = p$. This way we gave the invariant forms a presheaf structure. Essentially by repeating the reasoning used for cohomology with local coefficients, we obtain the functor structure

$$H^\bullet_{\mathrm{dR}} : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}. \tag{2.56}$$

Note that the constructions of $H^\bullet$ and $H^\bullet_{\mathrm{dR}}$ are practically the same, with cochain maps arising from chain maps being replaced with pullbacks. From now on, we will adopt a shorthand notation $\omega \otimes_{\mathbb{R}} a$ when denoting a general invariant form in $\Omega^\bullet(\tilde{M}) \otimes_{\mathbb{R}} A$.

Lemma 2.2.2. *For a fixed manifold M and a $\pi_1(M)$–$\mathbb{R}$-bimodule A, the invariant A-valued differential forms $\Omega^\bullet_{\mathrm{I}}(\,\cdot\,; A)$ form a sheaf on M.*

Proof. Let $\{O_i\}_{i \in I}$ be an open cover of M. We shall show that $\Omega^\bullet_{\mathrm{I}}$ satisfies locality and gluing axioms. Consider a collection of forms

$$\omega_i \otimes_{\mathbb{R}} a_i \in \Omega^k_{\mathrm{I}}(O_i; A), \tag{2.57}$$

which agree over the intersections $O_{ij} \coloneqq O_i \cap O_j$:

$$\mathrm{res}^{O_i}_{O_{ij}}(\omega_i \otimes_{\mathbb{R}} a_i) = \mathrm{res}^{O_j}_{O_{ij}}(\omega_j \otimes_{\mathbb{R}} a_j), \tag{2.58}$$

where $\mathrm{res}^U_V = j^*$ with $j : V \hookrightarrow U$. Gluing axiom states that there exists a form $\omega \otimes_{\mathbb{R}} a \in \Omega^k_{\mathrm{I}}(M; A)$ which restricts to $\omega_i \otimes_{\mathbb{R}} a_i$ over O_i. The locality axioms states that such a form is unique. But since $\omega_i \otimes_{\mathbb{R}} a_i$ represent finite sums of tensor products of k-forms and vectors from A:

$$\omega_i \otimes_{\mathbb{R}} a_i = \sum_{l=1}^{m_i} \omega_i^l \otimes_{\mathbb{R}} a_i^l \quad \omega_i^l \in \Omega^k(p^{-1}(O_i)), a_i^l \in A, \tag{2.59}$$

we have

$$\sum_{l=1}^{m_i} \mathrm{res}^{p^{-1}(O_i)}_{p^{-1}(O_{ij})}(\omega_i^l) \otimes_{\mathbb{R}} a_i^l = \sum_{l=1}^{m_j} \mathrm{res}^{p^{-1}(O_j)}_{p^{-1}(O_{ij})}(\omega_j^l) \otimes_{\mathbb{R}} a_j^l \tag{2.60}$$

for all $i, j \in I$. It follows that $m_i = m_j$, and up to a permutation

$$\mathrm{res}^{O_i}_{O_{ij}}(\omega_i^l) \otimes_{\mathbb{R}} a_i^l = \mathrm{res}^{O_j}_{O_{ij}}(\omega_j^l) \otimes_{\mathbb{R}} a_j^l. \tag{2.61}$$

Now, using the fact that $\Omega^\bullet$ forms a sheaf on $\tilde{M}$, we know that there exists a unique form

$$\omega \otimes_{\mathbb{R}} a \in \Omega^k(\tilde{M}; A) \tag{2.62}$$

restricting to $\omega_i \otimes_{\mathbb{R}} a_i$ over O_i. What remains to show is that this form is invariant. By assumption, for every $g \in \pi_1(M)$, upon restricting to each O_i, the forms $\Theta(g)^*\omega \otimes_{\mathbb{R}} a$ and $\omega \otimes_{\mathbb{R}} g \cdot a$ agree. But using locality of $\Omega^\bullet$ we infer that

$$\Theta(g)^*\omega \otimes_{\mathbb{R}} a = \omega \otimes_{\mathbb{R}} g \cdot a \tag{2.63}$$

over entire M. □

2.3. Twisted de Rham Theorem

Theorem 2.3.1. *If A is a $\mathbb{Z}\pi_1(M)$–$\mathbb{R}$-bimodule injective as a $\mathbb{Z}\pi_1(M)$-module, the twisted de Rham Theorem holds:*

$$H^\bullet(M; A) \simeq H^\bullet_{\mathrm{dR}}(M; A). \tag{2.64}$$

Proof. Injectivity of A implies the isomorphism $H^\bullet(U; A) \simeq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_\bullet(p^{-1}(U)), A\big)$ in the Universal Coefficient Theorem for any open $U \subseteq M$. Now we are able to define a map

$$H^k_{\mathrm{dR}}(U; A) \ni [\omega \otimes a] \overset{\mathscr{I}}{\longmapsto} \left(H_k(p^{-1}(U)) \ni [c] \mapsto \int_c \omega \otimes_{\mathbb{R}} a \in A \right) \in H^k(U; A). \tag{2.65}$$

We check its well-definedness. The integral is defined as

$$\int_c \omega \otimes_{\mathbb{R}} a := \left(\int_c \omega \right) \otimes_{\mathbb{R}} a \in \mathbb{R} \otimes_{\mathbb{R}} A \simeq A. \tag{2.66}$$

Its value does not depend on the choice of a representative of $[c]$ by Stokes Theorem. For the same reason, it only depends on the class $[\omega \otimes a] \in H^k_{\mathrm{dR}}(U; A)$. It remains to see that $\mathscr{I}([\omega \otimes a])$ is $\mathbb{Z}\pi_1(M)$-equivariant. This is demonstrated in the following calculation:

$$\int_{g \cdot c} \omega \otimes_{\mathbb{R}} a = \int_{\Theta(g) \circ c} \omega \otimes_{\mathbb{R}} a = \int_c \Theta(g)^*\omega \otimes_{\mathbb{R}} a = \int_c \omega \otimes_{\mathbb{R}} g \cdot a. \tag{2.67}$$

In order to prove that $\mathscr{I}$ is an isomorphism, we refer to the proof of the ordinary de Rham's Theorem in [Par17]. We start by proving that both cohomologies considered here fit into Mayer-Vietoris sequences. Consider the sequence

$$C_k(p^{-1}(U \cap V)) \xrightarrow{\alpha} C_k(p^{-1}(U)) \oplus C_k(p^{-1}(V)) \xrightarrow{\beta} C_k(p^{-1}(U \cup V)), \tag{2.68}$$

where $\alpha(\sigma) = (\sigma, -\sigma)$ and $\beta(\sigma_1, \sigma_2) = \sigma_1 + \sigma_2$. It is clear, that α, β are equivariant, and using the property $p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ as well as $p^{-1}(U \cup V) = p^{-1}(U) \cup p^{-1}(V)$, we can apply the standard argument for exactness of the above sequence. This allows us to construct the Mayer-Vietoris sequence

$$\ldots \to H^k(M; A) \to H^k(U; A) \oplus H^k(V; A) \to H^k(U \cap V; A) \to H^{k+1}(M; A) \to \ldots \tag{2.69}$$

Now, we turn to the sequence

$$\Omega^\bullet_{\mathrm{I}}(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^\bullet_{\mathrm{I}}(U) \oplus \Omega^\bullet_{\mathrm{I}}(V) \xrightarrow{j_U^* - j_V^*} \Omega^\bullet_{\mathrm{I}}(U \cap V), \tag{2.70}$$

where $i_T : T \hookrightarrow U \cup V$, $T \in \{U, V\}$ and $j_T : U \cap V \hookrightarrow T$, $T \in \{U, V\}$. Note that 2.70 is a sequence of subcomplexes of vector valued differential forms. We know that on full complexes the sequence is exact, since vector spaces are flat modules. Therefore, the map (i_U^*, i_V^*) is an injection as a restriction of such. For surjectivity of $j_U^* - j_V^*$ we give the explicit form of the preimage. For a smooth distribution of unity ρ_U, ρ_V on $U \cup V$, let $\tilde{\rho}_U := \rho_U \circ p$ and $\tilde{\rho}_V := \rho_V \circ p$ form a smooth partition of unity on $p^{-1}(U \cup V)$. Then we have $\omega \otimes_{\mathbb{R}} a = (j_U^* - j_V^*)(\omega_U \otimes_{\mathbb{R}} a, \omega_V \otimes_{\mathbb{R}} a)$, where ω_U is the extension by zero of $\tilde{\rho}_V \omega \otimes_{\mathbb{R}} a$, and ω_V is the extension by zero of $\tilde{\rho}_U \omega \otimes_{\mathbb{R}} a$. Both of these forms are manifestly invariant, as $\tilde{\rho}_U, \tilde{\rho}_V$ are p-fiberwise constant. We find that 2.70 is exact, so there exists a Mayer-Vietoris sequence for twisted de Rham cohomology. Let $i : V \hookrightarrow U$ be an inclusion of open subsets of M. Then,

$$\begin{tikzcd} H^k_{\mathrm{dR}}(U; A) \arrow[r, "i^*"] \arrow[d, "\mathscr{I}"] & H^k_{\mathrm{dR}}(V; A) \arrow[d, "\mathscr{I}"] \\ H^k(U; A) \arrow[r, "i^*"] & H^k(V; A) \end{tikzcd} \tag{2.71}$$

commutes by

$$\int_{i_* c} \omega \otimes_{\mathbb{R}} a = \int_c i^* \omega \otimes_{\mathbb{R}} a. \tag{2.72}$$

Similarly, by Stokes Theorem, the following diagram commutes:

$$\begin{tikzcd} H^{k-1}_{\mathrm{dR}}(U \cap V; A) \arrow[r, "\delta"] \arrow[d, "\mathscr{I}"] & H^k_{\mathrm{dR}}(U \cup V; A) \arrow[d, "\mathscr{I}"] \\ H^{k-1}(U \cap V; A) \arrow[r, "\xi"] & H^k(U \cup V; A), \end{tikzcd} \tag{2.73}$$

where δ, ξ are connecting homomorphisms in the respective Mayer-Vietoris sequences. Therefore, in order to complete the proof, it suffices to show that

$$H^k_{\mathrm{dR}}(U; A) \xrightarrow{\mathscr{I}} H^k(U; A) \tag{2.74}$$

is an isomorphism for small contractible sets U. Here, small means that $p^{-1}(U)$ is a disjoint union of sets diffeomorphic to U. We rely on p being a covering map to infer that the manifold M has a basis formed by such sets. Let

$$\bigsqcup_{i \in I} U_i = p^{-1}(U) \tag{2.75}$$

denote the decomposition into connected components. Consider a contraction

$$\psi : [0, 1] \times U \to U, \quad \psi(0, m) = m, \quad \psi(1, m) = m_0, \tag{2.76}$$

where $m_0 \in U$ is fixed. We lift it to a map $\Psi : [0,1] \times p^{-1}(U) \to p^{-1}(U)$ as follows: for a point $\tilde{m} \in U_i$ set $\Psi(t, \tilde{m}) = \tilde{n}$, where $\{\tilde{n}\} = p^{-1}\big(\{\psi(t, p(\tilde{m}))\}\big) \cap U_i$. Since, as a covering map, p maps U_i onto U diffeomorphically, Ψ is a well defined smooth map satisfying $\Psi_0(\tilde{m}) \coloneqq \Psi(0, \tilde{m}) = \tilde{m}$ and $\Psi_1(\tilde{m}) \coloneqq \Psi(1, \tilde{m}) = \tilde{m}_0$ with $p(\tilde{m}_0) = m_0$. Now, by the Poincaré Lemma, for any $\omega \in \Omega^k(p^{-1}(U))$ we have

$$\Psi_1^*\omega - \Psi_0^*\omega = \mathrm{d} \int\limits_{[0,1]} \big(\iota_{\partial_t} \Psi^* \omega\big)\, \mathrm{d}t - \int\limits_{[0,1]} \big(\iota_{\partial_t} \Psi^* \mathrm{d}\omega\big)\, \mathrm{d}t. \tag{2.77}$$

$\Psi_1^*\omega - \Psi_0^*\omega$ reduces to $-\omega$ because $p^{-1}(m_0)$ is discrete, and for a closed form $\omega \in \Omega^k_{\mathrm{cl}}(p^{-1}(U))$ we get

$$\omega = -\mathrm{d} \int\limits_{[0,1]} \big(\iota_{\partial_t} \Psi^* \omega\big)\, \mathrm{d}t. \tag{2.78}$$

Note that for any $g \in \pi_1(M)$, the diffeomorphism $\Theta(g)$ commutes with Ψ in the following sense:

$$\Theta(g) \circ \Psi = \Psi \circ (\mathrm{id}_{[0,1]} \times \Theta(g)). \tag{2.79}$$

This is because $\Theta(g)$ preserves p-fibers. Moreover, $\Theta(g)$ maps $U_i \xrightarrow{\simeq} U_{\tau(i)}$ diffeomorphically, were $\tau : I \to I$ is a bijection. Therefore, $\Theta(g)_* \partial_t = \partial_t$ and we compute

$$\begin{aligned} \Theta(g)^*\omega = -\mathrm{d}\Theta^* \int\limits_{[0,1]} \iota_{\partial_t} \Psi^* \omega\, \mathrm{d}t &= -\mathrm{d} \int\limits_{[0,1]} \iota_{\partial_t} (\mathrm{id}_{[0,1]} \times \Theta(g))^* \Psi^* \omega\, \mathrm{d}t \\ &= -\mathrm{d} \int\limits_{[0,1]} \iota_{\partial_t} \Psi^* \Theta(g)^* \omega\, \mathrm{d}t. \end{aligned} \tag{2.80}$$

The homotopy operator is linear, and we conclude that each invariant form

$$\sum_i \omega_i \otimes_{\mathbb{R}} a_i \in \Omega^k_{\mathrm{I}}(U; A) \tag{2.81}$$

has an invariant primitive, so $\forall k > 0 : H^k_{\mathrm{dR}}(M; A) = 0$. Similarly, for $k > 0$

$$H^k(U; A) \simeq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_k(p^{-1}(U)), A\big) = 0. \tag{2.82}$$

Let us focus on the nontrivial case $k = 0$. Denote by τ_g the map $I \to I$ for which $\Theta(g)(U_i) = U_{\tau_g(i)}$ for any $i \in I$. The zeroth twisted de Rham cohomology group is

$$\begin{aligned} H^0_{\mathrm{dR}}(U; A) = \Omega^0_{\mathrm{I,cl}}\big(U; A\big) &\simeq \Omega^0_{\mathrm{cl}}\bigg(\bigsqcup_{i \in I} U_i; A\bigg)^{\mathbb{Z}\pi_1(M)} \\ &\simeq \big\{f : I \to A \mid \forall i \in I : f \circ \tau(i) = g \cdot f(i)\big\}. \end{aligned} \tag{2.83}$$

The analogous calculation for the twisted singular cohomology yields:

$$\begin{aligned} H^0(U; A) \simeq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_0(p^{-1}(U)), A\big) &\simeq \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\bigg(\bigoplus_{i \in I} \mathbb{Z}\,, A\bigg) \\ &\simeq \big\{f : I \to A \mid \forall i \in I : f \circ \tau(i) = g \cdot f(i)\big\}. \end{aligned} \tag{2.84}$$

Finally, let $[\sigma] \in H_0\big(p^{-1}(U)\big)$. The image of any of its representatives is a point, and in some U_j. Let

$$[\omega \otimes_{\mathbb{R}} a] \coloneqq \left[\sum_{i\in I} c_i \otimes_{\mathbb{R}} a_i\right] \in H^0_{\mathrm{dR}}(U;A) \tag{2.85}$$

denote a class in twisted de Rham cohomology. We can assume that each c_i is a locally constant function with $\operatorname{supp} c_i = U_i$. The sum is locally finite, and one can even assume $c_i(m) \in \{0,1\}$. This way, we identify the class with an appropriate $f : I \to A$. We calculate

$$\mathscr{I}\big([\omega \otimes_{\mathbb{R}} a]\big)([\sigma]) = \int_\sigma \sum_{i\in I} c_i \otimes_{\mathbb{R}} a_i = \int_{\Delta_0} \sum_{i\in I} \sigma^* c_i \otimes_{\mathbb{R}} a_i = a_j = f(j), \tag{2.86}$$

and arrive at $\mathscr{I} : H^\bullet_{\mathrm{dR}}(U;A) \xrightarrow{\simeq} H^\bullet(U;A)$. Following [Par17] and using 2.71, 2.73 we easily conclude that

$$\mathscr{I} : H^\bullet_{\mathrm{dR}}(M;A) \xrightarrow{\simeq} H^\bullet(M;A) \tag{2.87}$$

for any M. It is clear, looking at 2.65, that this isomorphism lifts to a natural isomorphism of functors restricted to the full subcategory

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}} \tag{2.88}$$

of $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}}$ consisting only of $\mathbb{Z}\pi_1(\,\cdot\,)$–$\mathbb{R}$-bimodules injective as $\mathbb{Z}\pi_1(\,\cdot\,)$-modules. $\square$

Chapter 3

Twisted Character Functor

To define the twisted character functor, we need to find candidates for every vertex in a diagram analogous to 1.17, but with local coefficients. The remaining vertices are the ones on the bottom of the diagram.

Definition 3.0.1. Let M be a smooth manifold and let A be a $\mathbb{Z}\pi_1(M)$–$\mathbb{R}$-bimodule, injective as a $\mathbb{Z}\pi_1(M)$-module. Consider its sub-$\mathbb{Z}\pi_1(M)$-module Λ. For any open $U \subseteq M$, we will denote by $\Omega^\bullet_{\mathrm{I},\Lambda}(U;A)$ the subalgebra of $\Omega^\bullet_{\mathrm{I}}(U;A)$ consisting of closed invariant forms with periods in Λ. That is, whose integrals over all cycles in $p^{-1}(U)$ lie in Λ. Precisely,

$$\Omega^k_{\mathrm{I},\Lambda}(U;A) \coloneqq \left\{\omega \otimes_{\mathbb{R}} a \in \Omega^k_{\mathrm{I,cl}}(U;A) \,\middle|\, \forall \sigma \in Z_k(p^{-1}(U)) : \int_\sigma \omega \otimes_{\mathbb{R}} a \in \Lambda\right\}. \tag{3.1}$$

Note that

$$\Omega^\bullet_{\mathrm{I},\Lambda}(U;A)/\mathrm{d}\Omega^\bullet_{\mathrm{I}}(U;A) \simeq \mathscr{I}^{-1}\big(i_* H^\bullet(U;\Lambda)\big), \tag{3.2}$$

where $i : \Lambda \hookrightarrow A$.

In order to generalize the character functor, we need to slightly modify the category $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}}$. We define the category

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}} \times \mathbf{Top}^{\mathrm{op}} \tag{3.3}$$

that consists of objects $\big((M,q),A,\Lambda,U\big)$, where $\big((M,q),A,U\big) \in \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}}$ and Λ is a $\mathbb{Z}\pi_1(M)$-submodule of A. The morphisms in this category between $\big((N,q_N),B,\Gamma,V\big)$ and $\big((M,q_M),A,\Lambda,U\big)$ are those

$$(\bar{f},\varphi,\bar{j}) \in \mathrm{Hom}_{\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod} \times \mathbf{Top}^{\mathrm{op}}}\Big(\big((N,q_N),B,V\big),\big((M,q_M),A,U\big)\Big), \tag{3.4}$$

which satisfy

$$\varphi\big(\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f})(\Gamma)\big) = \Lambda. \tag{3.5}$$

Note that we can view $\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f})(\Gamma)$ as a $\mathbb{Z}\pi_1(M)$-submodule of $\mathbb{Z}\pi_1\text{-}\mathbf{Mod}(\bar{f})(B)$. Observe that respectively restricting and factorizing φ in the definition of $H^\bullet$ we can construct functors $H^\bullet_s$ and $H^\bullet_q$ from $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}} \times \mathbf{Top}^{\mathrm{op}}$, mapping

$$H^\bullet_s\big((M,q),A,\Lambda,U\big) \equiv H^\bullet(U;\Lambda), \tag{3.6}$$

and

$$H_q^\bullet\big((M,q),A,\Lambda,U\big) \equiv H^\bullet(U;A/\Lambda). \tag{3.7}$$

From now on, we will treat $H^\bullet$ as a functor from the above category of pairs or its full subcategories. Now, the maps from 2.4 give rise to natural transformations between $H_s^\bullet, H^\bullet$ and $H_q^\bullet$. Note that the assignment

$$\Omega^\bullet_{\mathrm{I,per}} : \big((M,q),A,\Lambda,U\big) \mapsto \Omega^\bullet_{\mathrm{I},\Lambda}(U;A) \tag{3.8}$$

extends to the functor with the same morphism component as in $\Omega^\bullet_{\mathrm{I}}$.
We will denote by

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}} \tag{3.9}$$

the full subcategory of $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}} \times \mathbf{Top}^{\mathrm{op}}$ consisting of objects $\big((M,q),A,\Lambda,U\big)$ such that A is a $\mathbb{Z}\pi_1(M)$–$\mathbb{R}$-bimodule injective as a $\mathbb{Z}\pi_1(M)$-module, and A/Λ is injective as an induced $\mathbb{Z}\pi_1(M)$-module.

Definition 3.0.2. With these constructions in mind, we are ready to define a *twisted character functor* as a 5-tuple $(\hat{\mathscr{G}}^\bullet, i_1, i_2, \delta_1, \delta_2)$, where

$$\hat{\mathscr{G}}^\bullet : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}, \tag{3.10}$$

and $i_1, i_2, \delta_1, \delta_2$ are natural transformations rendering the following *twisted character diagram* commutative, and its diagonal sequences exact for each $k \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H_q^{k-1} \arrow[rr, "-B"] \arrow[dr, "i_1"] & & H_s^k \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1} \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & \hat{\mathscr{G}}^k \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^k \\
& \Omega_{\mathrm{I}}^{k-1}/\Omega_{\mathrm{I,per}}^{k-1} \arrow[rr, "\mathrm{d}"] \arrow[ur, "i_2"] & & \Omega^k_{\mathrm{I,per}} \arrow[ur, "s"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0.
\end{tikzcd} \tag{3.11}$$

Here, all the functors are restricted to the category $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}}$. The maps α, B, r come from 2.4, and the maps β, d, s are defined using Theorem 2.3.1 and fit into a long exact sequence.

Before we state the next definition, we need to yet again restrict the category $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair}}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}}$. Just as in the ordinary case, we need the map

$$\begin{aligned}
&\iota : \Omega^\bullet_{\mathrm{I}} \to C^\bullet : \\
&\Omega^k_{\mathrm{I}}(U;A) \ni \omega \otimes_{\mathbb{R}} a \mapsto \left(c \mapsto \int_c \omega \otimes_{\mathbb{R}} a \bmod \Lambda\right) \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(C_k(p^{-1}(U)), A/\Lambda\big)
\end{aligned} \tag{3.12}$$

to be injective. Therefore, we define $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}}$ as the full subcategory of $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair}} \times \mathbf{Top}^{\mathrm{op}}$ consisting of those objects, which render ι injective.
Note that, under the restriction, we can relax the assumption that invariant forms with periods in respective submodules be closed. The closedness follows fom injectivity of ι just as in the ordinary case (cf. Corollary 1.0.4).

Definition 3.0.3. The *twisted differential characters* form the functor

$$\hat{\mathscr{H}}^\bullet : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}, \tag{3.13}$$

defined on objects by

$$\begin{aligned} \hat{\mathscr{H}}^k(U; A/\Lambda) &:= \hat{\mathscr{H}}^k\big((M,q), A, \Lambda, U\big) \\ = \big\{f \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), A/\Lambda\big) \big| \exists\, \omega_f \otimes_{\mathbb{R}} a \in \Omega^k_{\mathrm{I},\Lambda}(U; A) &: f \circ \partial = \iota(\omega_f \otimes_{\mathbb{R}} a)\big\}. \end{aligned} \tag{3.14}$$

Just like in the standard case, we do not need to assume that $\omega_f \otimes_{\mathbb{R}} a$ has periods in Λ. Moreover, because of injectivity of ι in the domain category, the form $\omega_f \otimes_{\mathbb{R}} a$ is unique for every $f \in \hat{\mathscr{H}}^k(U; A/\Lambda)$. The morphism component of $\hat{\mathscr{H}}^\bullet$ is just the restriction of the morphism component of $C^\bullet$. One only needs to show that for any

$$(\bar{h}, \varphi, \bar{j}) : \big((N, q_N), B, \Gamma, V\big) \to \big((M, q_M), A, \Lambda, U\big) \tag{3.15}$$

the condition

$$\forall f \in \hat{\mathscr{H}}^k(V; B/\Gamma) : \hat{\mathscr{H}}^k(\bar{h}, \varphi, \bar{j})(f) \in \hat{\mathscr{H}}^k(U; A/\Lambda) \tag{3.16}$$

is satisfied. But this follows from the calculation

$$\begin{aligned} \big(j^* \circ \varphi_* \circ \xi(h)\big|_V\big)(f) \circ \partial &= \big(j^* \circ \varphi_* \circ \xi(h)\big|_V\big)(f \circ \partial) \\ = \big(j^* \circ \varphi_* \circ \xi(h)\big|_V\big)\big(\iota(\omega_f \otimes_{\mathbb{R}} b)\big) &= \iota\big(\Omega^\bullet_{\mathrm{I}}(\bar{h}, \varphi, \bar{j})(\omega_f \otimes_{\mathbb{R}} a)\big), \\ \Omega^\bullet_{\mathrm{I,per}}(\bar{h}, \varphi, \bar{j})(\omega_f \otimes_{\mathbb{R}} b) &\in \Omega^k_{\mathrm{I},\Lambda}(U; A), \end{aligned} \tag{3.17}$$

where we restrict j^* and $\xi(h)\big|_V$ to $\mathrm{Hom}_{\mathbb{Z}\pi_1(N)}\big(Z_\bullet(p^{-1}(V)), B/\Gamma\big)$ and use compatibility of the functor structure of $\Omega^\bullet_{\mathrm{I}}$.

Proposition 3.0.4. *Under the restriction of the domain category to*

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}}, \tag{3.18}$$

the twisted differential characters functor $\hat{\mathscr{H}}^\bullet$ substituted for $\hat{\mathscr{G}}^\bullet$ fits into the twisted character diagram and as such, together with appropriate natural transformations $i_1, i_2, \delta_1, \delta_2$, forms a twisted character functor.

Proof. For clarity, we fix $\big((M,q), A, \Lambda, U\big)$. Functoriality will be clear at each step. We begin by defining $\delta_1(f) := \omega_f \otimes_{\mathbb{R}} a$. Just as in the untwisted case, δ_1 is surjective – just pick

$$f(c) = \begin{cases} \iota(\omega_f \otimes_{\mathbb{R}} a)(b) & c = \partial b \in B_{k-1}(p^{-1}(U)) \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

By the left-exactness of the hom-functor we obtain the inclusion

$$\begin{aligned} \wp^* : \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_{k-1}(p^{-1}(U)), A/\Lambda\big) &\hookrightarrow \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), A/\Lambda\big), \\ \wp : Z_{k-1}(p^{-1}(U)) &\twoheadrightarrow H_{k-1}(p^{-1}(U)), \end{aligned} \tag{3.20}$$

which we compose with the isomorphism

$$\phi : H^{k-1}(U; A/\Lambda) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_{k-1}(p^{-1}(U)), A/\Lambda\big) \tag{3.21}$$

to define i_1. Note that we used the injectivity of A/Λ as a $\mathbb{Z}\pi_1(M)$-module. By construction, the map i_2 is injective, and the left-exactness of the hom-functor together with the isomorphism in the Universal Coefficient Theorem imply that $\operatorname{im} i_1 = \ker \delta_1$.
The restriction

$$j^* : C^{k-1}(U; A/\Lambda) \to \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), A/\Lambda\big) \tag{3.22}$$

composed with ι gives a map whose kernel is precisely $\Omega^{k-1}_{\mathrm{I},\Lambda}(U; A)$. We define i_2 as its factorization through $\Omega^{k-1}_{\mathrm{I}}(U; A)/\Omega^{k-1}_{\mathrm{I},\Lambda}(U; A)$, which is manifestly injective. The Stokes Theorem guarantees that for every $\omega \otimes_{\mathbb{R}} a \in \Omega^{k-1}_{\mathrm{I}}(U; A)$, we have $i_2(\omega \otimes_{\mathbb{R}} a) \circ \partial = \iota(\mathrm{d}\omega \otimes_{\mathbb{R}} a)$. Thus, i_2 is a well defined map to $\hat{\mathscr{H}}^k(U; A/\Lambda)$, making the bottom triangle in 3.11 commutative. In order to define δ_2 consider the following diagram in $\mathbb{Z}\pi_1(M)$-**Mod**:

$$\begin{tikzcd}
A \arrow[rr, two heads] & & A/\Lambda \\
& C_{k-1}(p^{-1}(U)) \arrow[ul, dashed, "T"'] & \\
& Z_{k-1}(p^{-1}(U)). \arrow[u, hook] \arrow[uul, dashed, bend left, "\tilde{f}"] \arrow[uur, bend right, "f"'] &
\end{tikzcd} \tag{3.23}$$

Given $f \in \hat{\mathscr{H}}^k(U; A/\Lambda)$, we use the fact that $Z_{k-1}(p^{-1}(U))$ is free as a $\mathbb{Z}\pi_1(M)$-module, to lift f to $\tilde{f}$. Then, we use injectivity of A as a $\mathbb{Z}\pi_1(M)$-module, to factor $\tilde{f}$ through $C_{k-1}(p^{-1}(U))$, and we call this factorization T. Observe that for any $c \in C_k(p^{-1}(U))$ we get

$$\begin{aligned}\delta T(c) \mod \Lambda = T(\partial c) \mod \Lambda = \tilde{f}(\partial c) \mod \Lambda = f(\partial c) &= \iota(\omega_f \otimes_{\mathbb{R}} a)(c) \\ &= \omega_f \otimes_{\mathbb{R}} a \mod \Lambda,\end{aligned} \tag{3.24}$$

where after the last equality we treat $\omega_f \otimes_{\mathbb{R}} a$ as a cochain given by integration.
Therefore, $\omega_f \otimes_{\mathbb{R}} a - \delta T \in C^k(U; \Lambda)$ and it is closed. Moreover, if T' is another map making the above diagram commutative, we compute

$$(T - T')\big|_{Z_{k-1}(M)} = \tilde{f} - \tilde{f} = 0 \quad \Rightarrow \quad T - T' = \delta d,\ d \in C^{k-2}(U; A). \tag{3.25}$$

Here, we used the Universal Coefficient Theorem for A to infer from $T - T' = 0$ in $\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(H_{k-1}(p^{-1}(U)), A\big)$ the equality $[T - T'] = 0$ in $H^{k-1}(U; A)$. If we pick a different lift $\tilde{f}'$, we get $\tilde{f}' - \tilde{f} \mod \mathbb{Z} = f - f = 0$, so $\tilde{f} - \tilde{f}' = c\big|_{Z_{k-1}(p^{-1}(U))}$ for some $c \in C^{k-1}(U; \Lambda)$. We conclude that in general

$$T - T' = \delta d + c \quad \Rightarrow \quad \delta(T - T') = \delta c. \tag{3.26}$$

This means that the cohomology class $[\omega_f \otimes_{\mathbb{R}} a - \delta T] \in H^k(U; \Lambda)$ depends only on f. We define $\delta_2(f) \equiv [\omega_f \otimes_{\mathbb{R}} a - \delta T]$.
To see that it is surjective, let $[u] \in H^k(U; \Lambda)$ be arbitrary. By 3.2, there exists $\omega \otimes_{\mathbb{R}} a \in \Omega^k_{\mathrm{I},\Lambda}(U; A)$ with $[\omega \otimes_{\mathbb{R}} a] = i_*[u]$, identified using Theorem 2.3.1. Then, for any representative u, the cochain $\omega \otimes_{\mathbb{R}} a - i_* u \in C^k(U; A)$ is exact and we can find $T \in C^{k-1}(U; A)$ with

$\delta T = \omega \otimes_{\mathbb{R}} a - i_* u$. By postcomposing the restricted cochain $T\big|_{Z_{k-1}(p^{-1}(U))}$ with the natural projection $A \twoheadrightarrow A/\Lambda$, we obtain

$$f \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), A/\Lambda\big), \quad f \circ \partial = \iota(\omega \otimes_{\mathbb{R}} a), \tag{3.27}$$

making it a twisted differential character, which satisfies $\delta_2(f) = [u]$.
Now, suppose $\delta_2(f) = 0$, that is $[\omega_f \otimes_{\mathbb{R}} a - \delta T] = 0$. Since δT is exact, this means that

$$\exists \theta \otimes_{\mathbb{R}} b \in \Omega^{k-1}_{\mathrm{I}}(U; A) : \mathrm{d}\theta \otimes_{\mathbb{R}} b = \omega \otimes_{\mathbb{R}} a, \tag{3.28}$$

and

$$\exists e \in C^{k-1}(U; \Lambda) : \omega \otimes_{\mathbb{R}} a - \delta T = \delta e. \tag{3.29}$$

We calculate $\delta(\theta \otimes_{\mathbb{R}} b - T - e) = 0$, so there exists $\zeta \in Z^{k-1}(U; A)$ such that

$$\theta \otimes_{\mathbb{R}} b - T - e = \zeta. \tag{3.30}$$

By Theorem 2.3.1 we can find

$$\phi \otimes_{\mathbb{R}} g \in \Omega^{k-1}_{\mathrm{I}}(U; A) : (\theta \otimes_{\mathbb{R}} b - T - e)\big|_{Z_{k-1}(p^{-1}(U))} = (\phi \otimes_{\mathbb{R}} g)\big|_{Z_{k-1}(p^{-1}(U))}. \tag{3.31}$$

We have $T\big|_{Z_{k-1}(p^{-1}(U))} = (\theta \otimes_{\mathbb{R}} b - \phi \otimes_{\mathbb{R}} g - e)\big|_{Z_{k-1}(p^{-1}(U))}$. By postcomposing both sides with the natural projection $A \twoheadrightarrow A/\Lambda$, we obtain

$$f = \iota\big(\theta \otimes_{\mathbb{R}} b - \phi \otimes_{\mathbb{R}} g - e\big)\big|_{Z_{k-1}(p^{-1}(U))} = i_2\big(\theta \otimes_{\mathbb{R}} b - \phi \otimes_{\mathbb{R}} g\big), \tag{3.32}$$

as e is Λ-valued. Hence, $f \in \mathrm{im}\, i_2$. We conclude that both diagonal sequences are exact.
We proceed by checking the commutativity of the left side of the twisted character diagram. It follows from Theorem 2.3.1 and the fact that the map

$$\mathrm{res} \circ \iota\big|_{\ker \mathrm{d}^{k-1}}, \quad \mathrm{res} : C^{k-1}(U; A/\Lambda) \to \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), A/\Lambda\big) \tag{3.33}$$

factorizes as $\hat{\iota}^{k-1}$ through $H^{k-1}_{\mathrm{dR}}(U; A)$, and $\hat{\iota}^{k-1}([\omega \otimes_{\mathbb{R}} a])$ coincides with the image of $\alpha([\omega \otimes_{\mathbb{R}} a])$ under $i_1 : H^{k-1}(U; A/\Lambda) \to \hat{\mathscr{H}}^k(U; A/\Lambda)$. The top triangle commutes by the construction of δ_2 above, with $\omega_f \otimes_{\mathbb{R}} a = 0$ in the image of i_1. Finally, the right side of the diagram commutes by 3.2. □

3.1. Uniqueness Theorem

Before we assert the uniqueness theorem for the twisted character functor, we prove the following important technical lemma, which generalizes the result in [SS07].

Lemma 3.1.1. *Let $\sigma \in C_k(\tilde{M})$. Then every neighborhood U of $p(\mathrm{im}(\sigma)) \subseteq M$ contains a smaller neighborhood U' satisfying $H^{k'}(U'; \Lambda) = 0$ for all $k' > k$. We call such U' a k-good neighborhood of $p(\mathrm{im}(\sigma))$.*

Proof. We start by observing that $p(\mathrm{im}(\sigma))$ is an image of a smooth map

$$p \circ \sigma : \bigsqcup_{i=1}^{l} \Delta^k \to M \tag{3.34}$$

from a finite disjoint union of geometric k-simplices. Since the topological dimension of this union is k, it follows from [SS07, Fact 2.1] that for any open $U \supseteq p(\mathrm{im}(\sigma))$ there exists an open $p(\mathrm{im}(\sigma)) \subseteq U' \subseteq U$ for which

$$H^{k'}(U';\mathbb{Z}) = 0, \quad k' > k, \tag{3.35}$$

where $\mathbb{Z}$ is a trivial $\mathbb{Z}\pi_1(M)$-module. In other words, the ordinary integral cohomology vanishes above k. Since U' has a structure of a smooth manifold, it is homotopy equivalent to a CW-complex. Moreover, it can be realized as a k-dimensional CW-complex. The space $p^{-1}(U')$ is a covering space of U' and as such, can be given a structure of a k-dimensional CW-complex, up to homotopy equivalence [Hat02, 4.1.]. Thus, the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet(p^{-1}(U')), \Lambda\big) \tag{3.36}$$

vanishes above k. This means that for any $k' > k$ and any $c \in \mathrm{Hom}_{\mathbb{Z}}\big(C_{k'}(p^{-1}(U')), \Lambda\big)$ such that $c \circ \partial = 0$ there exists $c' \in \mathrm{Hom}_{\mathbb{Z}}\big(C_{k'-1}(p^{-1}(U')), \Lambda\big)$ satisfying $c = \delta c'$. If we assume that c is equivariant with respect to the restricted action of $\mathbb{Z}\pi_1(M)$, we check that

$$\forall \gamma \in \mathbb{Z}\pi_1(M) \forall s \in C'_k(p^{-1}(U')) : c'(\gamma \cdot \partial s) = c'(\partial(\gamma \cdot s)) = c(\gamma \cdot s) = \gamma \cdot c(s) = \gamma \cdot c'(\partial s), \tag{3.37}$$

where we use the fact that ∂ commutes with the action of $\mathbb{Z}\pi_1(M)$ on chains. Using this fact once again, we argue that one can construct another cochain

$$c'' \in \mathrm{Hom}_{\mathbb{Z}}\big(C_{k'-1}(p^{-1}(U')), \Lambda\big) : c''(s) = \begin{cases} c'(s) & s \in B_{k'-1}(p^{-1}(U')), \\ 0 & \text{otherwise}, \end{cases} \tag{3.38}$$

which is manifestly equivariant and satisfies $c = \delta c''$. Therefore, we conclude that

$$\forall k' > k : H^{k'}(U;\Lambda) = 0. \tag{3.39}$$

□

Theorem 3.1.2. *For any twisted character functor* $(\hat{\mathscr{G}}^\bullet, i_1, i_2, \delta_1, \delta_2)$ *restricted to the category* $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair},\iota}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}}$ *there exists a unique natural isomorphism* $\Phi : \hat{\mathscr{G}}^\bullet \to \hat{\mathscr{H}}^\bullet$ *and a unique natural automorphism* $\Psi : H^\bullet_s \to H^\bullet_s$ *such that* Φ *commutes with* Ψ *on* $H^\bullet_s$ *and with identity transformations on all other functors in the diagram. In other words, the diagram formed by two character diagrams–corresponding to* $\hat{\mathscr{G}}^\bullet$ *and* $\hat{\mathscr{H}}^\bullet$*–connected with identity transformations,* Ψ*, and* Φ*–is commutative.*

Proof. Let $f \in \hat{\mathscr{G}}^k(V; A/\Lambda)$ with $V \subseteq M$ open, and take $\sigma \in Z_{k-1}(p^{-1}(V))$. Let $U \subseteq V$ be a $(k-1)$-good neighborhood of $p(\mathrm{im}(\sigma))$, which exists by Lemma 3.1.1, and let $j : U \hookrightarrow V$ be an inclusion. Since $H^k(U;\Lambda) = 0$, the exactness of a diagonal in the diagram 3.11 guarantees the existence of $[\theta \otimes_{\mathbb{R}} a] \in \Omega^{k-1}_{\mathrm{I}}(U;A)/\Omega^{k-1}_{\mathrm{I},\Lambda}(U;A)$ such that

$$j^* f = i_2([\theta \otimes_{\mathbb{R}} a]). \tag{3.40}$$

We set

$$\Phi_{(V;A/\Lambda)}(f)(\sigma) \coloneqq \iota(\theta \otimes_{\mathbb{R}} a)(\sigma). \tag{3.41}$$

First we show that $\Phi_{(V;A/\Lambda)}$ is a well defined homomorphism

$$\hat{\mathscr{G}}^k(V; A/\Lambda) \to \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(V)), A/\Lambda\big). \tag{3.42}$$

It is clear that 3.41 does not depend on the choice of a representative of $[\theta \otimes_{\mathbb{R}} a]$. The independence of the choice of U follows from Lemma 3.1.1. Indeed, let U' be another open set satisfying the asserted conditions. Then we can find a third such $U'' \subseteq U \cap U'$. Let

$$U'' \xrightarrow{j'} U \xrightarrow{j} V \tag{3.43}$$

be the inclusion maps. Then,

$$(j \circ j')^* f = j'^* j^* f = j'^* (i_2([\theta \otimes_{\mathbb{R}} a]) = i_2(j'^*[\theta \otimes_{\mathbb{R}} a]) = i_2([j'^* \theta \otimes_{\mathbb{R}} a]), \tag{3.44}$$

where we used naturality of i_2. Since

$$\iota(j'^* \theta \otimes_{\mathbb{R}} a)(\sigma) = \iota(\theta \otimes_{\mathbb{R}} a)(\sigma), \tag{3.45}$$

we see that the definitions of $\Phi_{(V;A/\Lambda)}$ using U and U'' agree. Since the same is true for U' and U'', we have shown that the definition of $\Phi_{(V;A/\Lambda)}$ is independent of the choice of U. To show that $\Phi_{(V;A/\Lambda)}(f) \in \operatorname{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(V)), A/\Lambda\big)$ it is enough to check additivity, as $\pi_1(M)$ equivariance follows from invariance of $\theta \otimes_{\mathbb{R}} a$. To see that

$$\forall \sigma_1, \sigma_2 \in Z_{k-1}(p^{-1}(V)) : \Phi_{(V;A/\Lambda)}(f)(\sigma_1 + \sigma_2) = \Phi_{(V;A/\Lambda)}(f)(\sigma_1) + \Phi_{(V;A/\Lambda)}(f)(\sigma_2), \tag{3.46}$$

let σ be a chain such that $\operatorname{im}(\sigma_1) \cup \operatorname{im}(\sigma_2) \subseteq \operatorname{im}(\sigma)$. Pick U, a $(k-1)$-good neighborhood of $p(\operatorname{im}(\sigma))$. Since $\operatorname{im}(\sigma_1 + \sigma_2) \subseteq \operatorname{im}(\sigma_1) \cup \operatorname{im}(\sigma_2)$ we have each of $p(\operatorname{im}(\sigma_1)), p(\operatorname{im}(\sigma_2))$, $p(\operatorname{im}(\sigma_1 + \sigma_2)) \subseteq U$. Choosing $[\theta \otimes_{\mathbb{R}} a]$ as in 3.40 we find

$$\begin{aligned} \Phi_{(V;A/\Lambda)}(f)(\sigma_1 + \sigma_2) &= \iota(\theta \otimes_{\mathbb{R}} a)(\sigma_1) + \iota(\theta \otimes_{\mathbb{R}} a)(\sigma_2) \\ &= \Phi_{(V;A/\Lambda)}(f)(\sigma_1) + \Phi_{(V;A/\Lambda)}(f)(\sigma_2). \end{aligned} \tag{3.47}$$

Moreover, clearly, $\Phi_{(V;A/\Lambda)}(f_1 + f_2) = \Phi_{(V;A/\Lambda)}(f_1) + \Phi_{(V;A/\Lambda)}(f_2)$. Then, we want to show that $\operatorname{im}\big(\Phi_{(V;A/\Lambda)}|_{\hat{\mathscr{G}}^k(V;A/\Lambda)}\big) \subseteq \hat{\mathscr{H}}(V; A/\Lambda)$. Precisely, we will prove that if $\sigma = \partial s$ for some $s \in C_k(p^{-1}(V))$, then

$$\Phi_{(V;A/\Lambda)}(f)(\sigma) = \iota(\delta_1(f))(s). \tag{3.48}$$

Since both sides vanish when $k - 1 = \dim p^{-1}(V)$, we may assume $k - 1 < \dim p^{-1}(V)$. Let U be a $(k-1)$-good neighborhood of $p(\operatorname{im}(\sigma))$. Using [SS07, Fact 2.2] we can find a $(k-1)$-dimensional embedded pseudomanifold $P \subseteq p^{-1}(U)$ and a k-chain w with $p(\operatorname{im}(w)) \subseteq U$ such that

$$\sigma = \partial w + P, \tag{3.49}$$

where we identify P with its fundamental cycle. Letting $[\theta \otimes_R a]$ be defined as in 3.40 we calculate

$$\begin{aligned} \Phi_{(V;A/\Lambda)}(f)(\sigma) &= \iota(\theta \otimes_{\mathbb{R}} a)(\partial w) + \Phi_{(V;A/\Lambda)}(f)(P) \\ &= \iota(\mathrm{d}\theta \otimes_{\mathbb{R}} a)(w) + \Phi_{(V;A/\Lambda)}(f)(P) \\ &= \iota(\delta_1(j^* f))(w) + \Phi_{(V;A/\Lambda)}(f)(P) \\ &= \iota(j^* \delta_1(f))(w) + \Phi_{(V;A/\Lambda)}(f)(P) \\ &= \iota(\delta_1(f))(w) + \Phi_{(V;A/\Lambda)}(f)(P), \end{aligned} \tag{3.50}$$

where we invoke Stokes Theorem, the fact that $\mathrm{d} = \delta_1 \circ i_2$, and naturality of δ_1. Since σ is a boundary in $p^{-1}(V)$ and P is homologous to σ, P is itself a boundary in $p^{-1}(V)$. We use [SS07, Fact 2.3] to find a $(k-1)$-good neighborhood U' of $p(P)$ with the inclusion

$\lambda : U' \hookrightarrow V$, together with $u \in C_k(p^{-1}(V))$ satisfying $p(\mathrm{im}(u)) \subseteq U'$ and $P = \partial u$. Once again $\lambda^* f = i_2([\omega \otimes_{\mathbb{R}} a])$ for some $\omega \otimes_{\mathbb{R}} a \in \Omega_{\mathrm{I}}^{k-1}(U'; A)$. Using the same arguments as above, we find

$$\Phi_{(V;A/\Lambda)}(f)(P) = \iota(\omega \otimes_{\mathbb{R}} a)(\partial u) = \iota(\delta_1(f))(u). \tag{3.51}$$

Together with 3.50 this yields

$$\begin{aligned}\Phi_{(V;A/\Lambda)}(f)(\sigma) &= \iota(\delta_1(f))(w) + \iota(\delta_1(f))(u) \\ &= \big(\iota(\delta_1(f))(w) + \iota(\delta_1(f))(u) - \iota(\delta_1(f))(s)\big) + \iota(\delta_1(f))(s).\end{aligned} \tag{3.52}$$

Note that $\partial(w + u) = \sigma = \partial s$, and $\delta_1(f) \in \Omega_{\mathrm{I},\Lambda}^k(V, A)$, so the first term vanishes, and we managed to prove 3.48. Now, we wish to show that Φ is indeed a natural transformation between two functors $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}$. Consider any morphism

$$(\bar{h}, \varphi, \bar{j}) : \big((N, q_N), B, \Gamma, V\big) \to \big((M, q_M), A, \Lambda, U\big) \tag{3.53}$$

in $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}}$. Then the following diagram should be commutative:

$$\begin{tikzcd} \hat{\mathscr{G}}^k(V; B/\Gamma) \arrow[r, "\hat{\mathscr{G}}^\bullet(\bar{h},\varphi,\bar{j})"] \arrow[d, "\Phi_{(V;B/\Gamma)}"] & \hat{\mathscr{G}}^k(U; A/\Lambda) \arrow[d, "\Phi_{(U;A/\Lambda)}"] \\ \hat{\mathscr{H}}^k(V; B/\Gamma) \arrow[r, "\hat{\mathscr{H}}^\bullet(\bar{h},\varphi,\bar{j})"] & \hat{\mathscr{H}}^k(U; A/\Lambda). \end{tikzcd} \tag{3.54}$$

Take $f \in \hat{\mathscr{G}}^k(V; B/\Gamma)$. Let $\sigma \in C_{k-1}(p_M^{-1}(U))$. Consider a $(k-1)$-good neighborhood $V' \supseteq h(p_M(\mathrm{im}(\sigma)))$ and a $(k-1)$-good neighborhood $p_M(\mathrm{im}(\sigma)) \subseteq U' \subseteq h^{-1}(V')$. Denote

$$l : V' \hookrightarrow V, \quad y : h^{-1}(V') \hookrightarrow U, \quad t : U' \hookrightarrow h^{-1}(V'). \tag{3.55}$$

Note that there are isomorphisms

$$\hat{\mathscr{H}}^k(V', B/\Gamma) \simeq \Omega_{\mathrm{I}}^{k-1}(V'; B)/\Omega_{\mathrm{I},\Gamma}^{k-1}(V'; B) \simeq \hat{\mathscr{G}}^k(V', B/\Gamma), \tag{3.56}$$

and

$$\hat{\mathscr{H}}^k(U', A/\Lambda) \simeq \Omega_{\mathrm{I}}^{k-1}(U'; A)/\Omega_{\mathrm{I},\Lambda}^{k-1}(U'; B) \simeq \hat{\mathscr{G}}^k(U', A/\Lambda). \tag{3.57}$$

Moreover, the map i_2 is a natural transformation in both diagrams–the one around $\hat{\mathscr{G}}^\bullet$ and the one around $\hat{\mathscr{H}}^\bullet$. By the definition of components of Φ, it is clear that it suffices to show that f maps to the same class of invariant forms under the isomorphisms above. Equivalently, that the following diagram is commutative:

$$\begin{tikzcd} \hat{\mathscr{G}}^k(V; B/\Gamma) \arrow[r, "\hat{\mathscr{G}}^\bullet(\bar{h},\varphi,\bar{j})"] \arrow[d, "l^*"] & \hat{\mathscr{G}}^k(U; A/\Lambda) \arrow[d, "y^*"] \\ \hat{\mathscr{G}}^k(V'; B/\Gamma) \arrow[r, "\hat{\mathscr{G}}^\bullet\left(\bar{h},\varphi,\bar{\mathrm{id}}_{h^{-1}(V')}\right)"] \arrow[dd, "\simeq"] & \hat{\mathscr{G}}^k(h^{-1}(V'); A/\Lambda) \arrow[d, "t^*"] \\ & \hat{\mathscr{G}}^k(U'; A/\Lambda) \arrow[d, "\simeq"] \\ \hat{\mathscr{H}}^k(V'; B/\Gamma) \arrow[r, "\hat{\mathscr{H}}^\bullet(\bar{h},\varphi,\bar{t})"] & \hat{\mathscr{H}}^k(U'; A/\Lambda) \end{tikzcd} \tag{3.58}$$

The top square commutes, as $\hat{\mathscr{G}}^\bullet$ preserves compositions in $\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \times \mathbf{Top}^{\mathrm{op}}$. The bottom square commutes by naturality of i_2 and because of the isomorphisms 3.56, 3.57. Since f and σ were chosen arbitrarily and since, by construction,

$$\Phi_{(V;B/\Gamma)}(f) = \Phi_{(V';B/\Gamma)}(l^* f), \tag{3.59}$$

and

$$\Phi_{(U;A/\Lambda)}\big(\hat{\mathscr{G}}^\bullet(\bar{h},\varphi,\bar{\jmath})(f)\big) = \Phi_{(U';A/\Lambda)}\big(t^* y^* \hat{\mathscr{G}}^\bullet(\bar{h},\varphi,\bar{\jmath})(f)\big), \tag{3.60}$$

we conclude that 3.54 is commutative, so Φ is indeed a natural transformation. The next point is to show that, with a slight abuse of notation,

$$\Phi \circ i_1 = i_1, \quad \Phi \circ i_2 = i_2, \quad \delta_1 \circ \Phi = \delta_1. \tag{3.61}$$

Let $\mu \in H^{k-1}(U;A/\Lambda)$ and pick $\sigma \in Z_{k-1}(p^{-1}(U))$. Let U' be a $(k-1)$-good neighborhood of $p(\mathrm{im}(\sigma))$. Since $H^k(U';\Lambda) = 0$, by exactness of of the twisted Bockstein sequence, there exists $x \in H^{k-1}(U;A)$ such that $\mu = \alpha(x)$. By commutativity of the twisted character diagram for $\hat{\mathscr{G}}^\bullet$ we have $i_1(\mu) = i_1(\alpha(x)) = i_2(\beta(x))$. Thus, by 3.41

$$\Phi_{(U;A/\Lambda)}(i_1(\mu))(\sigma) = \iota(\theta \otimes_{\mathbb{R}} a)(\sigma), \tag{3.62}$$

for any $\theta \otimes_{\mathbb{R}} a \in [\beta(x)]$. But, since β is defined using Theorem 2.3.1, and by the definition of i_1 from Proposition 3.0.4 we compute

$$\iota(\theta \otimes_{\mathbb{R}} a)(\sigma) = \wp^* \phi(\alpha(x))(\sigma) = i_1(\mu)(\sigma). \tag{3.63}$$

Thus $\Phi \circ i_1 = i_1$. The identity $\Phi \circ i_2 = i_2$ follows straight from the definition in 3.41, and $\delta_1 \circ \Phi = \delta_1$ follows directly from 3.48. Now, we are in position to show that Φ is an isomorphism. Indeed, consider the following commutative diagram, the rows of which are exact:

$$\begin{tikzcd}
0 \arrow[r] & H^{k-1}(U;A/\Lambda) \arrow[r, "i_1"] \arrow[d, "\simeq"] & \hat{\mathscr{G}}^k(U;A/\Lambda) \arrow[r, "\delta_1"] \arrow[d, "\Phi_{(U;A/\Lambda)}"] & \Omega^k_{\mathrm{I},\Lambda}(U;A) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^{k-1}(U;A/\Lambda) \arrow[r, "i_1"] & \hat{\mathscr{H}}^k(U;A/\Lambda) \arrow[r, "\delta_1"] & \Omega^k_{\mathrm{I},\Lambda}(U;A) \arrow[r] & 0.
\end{tikzcd} \tag{3.64}$$

The hypothesis follows from the Five Lemma. In order to prove that there exists $\Psi \in \mathrm{Aut}(H_s^\bullet)$ satisfying $\delta_2 \circ \Phi = \Psi \circ \delta_2$, we will start by showing that for any twisted character functor $\hat{\mathscr{G}}^\bullet$ the map δ_2 is determined up to a natural automorphism by i_1, i_2 and δ_1. Suppose there exists δ_2' satisfying the same conditions as δ_2. Namely, $\big(\hat{\mathscr{G}}^\bullet, i_1, i_2, \delta_1, \delta_2'\big)$ is a twisted character functor. Since both δ_2 and δ_2' induce natural isomorphisms $\hat{\delta}_2$ and $\hat{\delta}_2'$ from $\hat{\mathscr{G}}^k / i_2\big(\Omega_{\mathrm{I}}^{k-1}/\Omega_{\mathrm{I},\mathrm{per}}^{k-1}\big)$ onto H_s^k, we know that the map

$$\hat{\delta}_2' \circ \hat{\delta}_2^{-1} : H_s^k \to H_s^k \tag{3.65}$$

is a natural automorphism. Moreover, one can see that $\hat{\delta}_2' \circ \hat{\delta}_2^{-1} \circ \delta_2 = \delta_2'$. Now, note that $\delta_2 \circ \Phi$ can be taken as δ_2', since, by previous arguments, $\big(\hat{\mathscr{G}}^\bullet, i_1, i_2, \delta_1, \delta_2 \circ \Phi\big)$ forms a twisted character functor. Thus, to satisfy the commutativity condition $\delta_2 \circ \Phi = \Psi \circ \delta_2$, the natural automorphism Ψ should be given as $(\widehat{\delta_2 \circ \Phi}) \circ \hat{\delta}_2^{-1}$. Finally, we show that Φ is the unique natural isomorphism $\hat{\mathscr{G}}^\bullet \xrightarrow{\simeq} \hat{\mathscr{H}}^\bullet$ satisfying 3.61, and therefore, rendering Ψ the unique automorphism satisfying $\delta_2 \circ \Phi = \Psi \circ \delta_2$. Suppose Φ' is another such isomorphism.

Then $\Phi' \circ \Phi^{-1} : \hat{\mathscr{H}}^\bullet \to \hat{\mathscr{H}}^\bullet$ is a natural automorphism holding fixed all the other terms in the twisted character diagram but $H_s^\bullet$, on which it acts with a natural automorphism $\Psi' \circ \Psi^{-1}$. Thus, for $f \in \hat{\mathscr{H}}^{k-1}(U; A/\Lambda)$, a cycle $\sigma \in Z_{k-1}(p^{-1}(U))$, and a $(k-1)$-good open neighborhood U' of $p(\mathrm{im}(\sigma))$ with $j : U' \hookrightarrow U$, we have $\theta \otimes_{\mathbb{R}} a \in \Omega_{\mathrm{I}}^{k-1}(U'; A)$ such that

$$\begin{aligned}(\Phi' \circ \Phi^{-1})_{(U;A/\Lambda)}(f)(\sigma) &= (\Phi' \circ \Phi^{-1})_{(U';A/\Lambda)}(j^* f)(\sigma) \\ &= (\Phi' \circ \Phi^{-1})_{(U';A/\Lambda)}\big(i_2([\theta \otimes_{\mathbb{R}} a])\big)(\sigma) \\ &= i_2([\theta \otimes_{\mathbb{R}} a])(\sigma) = j^* f(\sigma) = f(\sigma).\end{aligned} \tag{3.66}$$

Thus, $\Phi = \Phi'$. □

3.2. Cohomological Description

In [HS05] Hopkins and Singer provide a cohomological description of ordinary differential characters. They show that differential characters correspond to cohomology classes of a particular presheaf of cochain complexes. We aim to generalize their construction to twisted differential characters.

Definition 3.2.1 ([HS05])**.** Let M be a manifold and let $s \in \mathbb{N}$. The cochain complex $\mathrm{DC}_s^\bullet(M)$ is defined by

$$\mathrm{DC}_s^n(M) = \{(c, h, \omega) : \omega = 0 \text{ for } n < s\} \subseteq C^n(M, \mathbb{Z}) \times C^{n-1}(M, \mathbb{R}) \times \Omega^n(M), \tag{3.67}$$

and

$$\mathrm{d}(c, h, \omega) = (\delta c, \omega - c - \delta h, \mathrm{d}\omega), \tag{3.68}$$

where we identify ω with its real cochain given by integration. Clearly, $\mathrm{DC}_s^\bullet$ forms a presheaf of complexes on **Man**.

Proposition 3.2.2. *For each $n \in \mathbb{N}$ there is a natural isomorphism $H^n(\mathrm{DC}_n^\bullet) \simeq \hat{H}^n(\cdot, \mathbb{R}/\mathbb{Z})$.*

Proof. Let $M \in$ **Man**. We set

$$\psi_M : H^n(\mathrm{DC}_n^\bullet(M)) \ni [(c, h, \omega)] \mapsto h\big|_{Z_{n-1}(M)} \bmod \mathbb{Z} \in \hat{H}^n(M, \mathbb{R}/\mathbb{Z}). \tag{3.69}$$

It is a linear map, so its well-definedness follows from

$$\psi_M(\mathrm{d}(c, h, 0)) = \psi_M(\delta c, -c - \delta h, 0) = (-c - \delta h)\big|_{Z_{n-1}(M)} \bmod \mathbb{Z} = 0, \tag{3.70}$$

and

$$\psi_M([(c, h, \omega)])(\partial b) = \delta h(b) \mod \mathbb{Z} = \omega(b) - c(b) \mod \mathbb{Z} = \omega(b) \mod \mathbb{Z} = \iota(\omega)(b). \tag{3.71}$$

To construct the inverse, given $f \in \hat{H}^n(M, \mathbb{R}/\mathbb{Z})$, we take $T(f) \in C^{n-1}(M, \mathbb{R})$ as in 1.29. For ω simply take $\omega_f = \delta_1(f)$ (**??**). Just as it was shown in the proof of Proposition 1.0.7, the cochain $T(f)$ satisfies

$$\delta T(f) = \omega_f - c, \tag{3.72}$$

where $c \in Z^n(M, \mathbb{Z})$ is a representative of $\delta_2(f)$. Moreover, $T(f)$ is determined up to

$$c' + \delta d,\ c' \in C^{n-1}(M, \mathbb{Z}),\ d \in C^{n-2}(M, \mathbb{R}). \tag{3.73}$$

This makes the class $[(c, T(f), \omega_f)]$ uniquely determined. Indeed, let $(\tilde{c}, \tilde{T}(f), \omega_f)$ be a different choice. Then,

$$\begin{aligned}(\tilde{c}, \tilde{T}(f), \omega_f) - (c, T(f), \omega_f) = \big(\delta(T(f) - \tilde{T}(f)), \tilde{T}(f) - T(f), 0\big) &= \big(-\delta c', c' + \delta d, 0\big) \\ &= \mathrm{d}(-c', -d, 0).\end{aligned} \tag{3.74}$$

We should check that $\psi_M^{-1} \circ \psi_M = \mathrm{id}_{H^n(\mathrm{DC}^\bullet_n(M))}$. But this follows from the fact that the class $[(c, h, \omega)]$ is uniquely determined by $f = h\big|_{Z_{n-1}(M)} \mod \mathbb{Z}$. The equality $\psi_M \circ \psi_M^{-1} = \mathrm{id}_{\hat{H}^n(M)}$ follows directly from the construction of $T(f)$. Naturality of ψ is straightforward. $\square$

It is clear what the generalization should be.

Definition 3.2.3. The *twisted Hopkins-Singer complex of degree* $s \in \mathbb{N}$ is the functor

$$\mathcal{DC}^\bullet_s : \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair},\iota}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ch}^+(\mathbf{Ab}), \tag{3.75}$$

defined by

$$\begin{aligned}\mathcal{DC}^n_s(U, A, \Lambda) &\coloneqq \mathcal{DC}^n_s\big((M, q), A, \Lambda, U\big) \\ &= \{(c, h, \omega \otimes_{\mathbb{R}} a) : \omega \otimes_{\mathbb{R}} a = 0 \text{ for } n < s\} \subseteq C^n(U; \Lambda) \times C^{n-1}(U; A) \times \Omega^n_{\mathrm{I}}(U; A).\end{aligned} \tag{3.76}$$

Clearly, $\mathcal{DC}^\bullet_s = (C^\bullet_s, C^\bullet, \tau_{\geqslant s}\Omega^\bullet_{\mathrm{I}})$, where $C^\bullet_s$ is a functor whose cohomology is, by definition, the functor $H^\bullet_s$ (cf. 3.6), and $\tau_{\geqslant s}\Omega^\bullet_{\mathrm{I}}$ denotes the appropriate truncation. The morphism component of $\mathcal{DC}^\bullet_s$ is therefore apparent. The differential of $\mathcal{DC}^\bullet_s$ is defined by

$$\mathrm{d}(c, h, \omega \otimes_{\mathbb{R}} a) = \big(\delta c, \omega \otimes_{\mathbb{R}} a - c - \delta h, \mathrm{d}(\omega \otimes_{\mathbb{R}} a)\big), \tag{3.77}$$

where we identify $\omega \otimes_{\mathbb{R}} a$ with the A-valued cochain given by integration. The fact that d is a well-defined differential follows from the fact that $C^\bullet_s, C^\bullet, \tau_{\geqslant s}\Omega^\bullet_{\mathrm{I}}$ are cochain complexes with their respective differentials. The cohomology of $\mathcal{DC}^\bullet_s$ provides a functor

$$\int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}^{\mathrm{pair},\iota}_{\mathbb{R},\mathrm{inj}} \times \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}} : \big((M, q), A, \Lambda, U\big) \mapsto H^\bullet(\mathcal{DC}^\bullet_s(U, A, \Lambda)). \tag{3.78}$$

Proposition 3.2.4. *For each $n \in \mathbb{N}$ there is a natural isomorphism of functors:*

$$H^n(\mathcal{DC}^\bullet_n) \simeq \hat{\mathscr{H}}^n. \tag{3.79}$$

Proof. In analogy to the ordinary case, we set (using a shorthand notation)

$$\psi_{(U,A,\Lambda)} : [(c, h, \omega \otimes_{\mathbb{R}} a)] \mapsto h\big|_{Z_{n-1}(p^{-1}(U))} \mod \Lambda. \tag{3.80}$$

We check

$$\psi_{(U,A,\Lambda)}(\mathrm{d}(c, h, \omega \otimes_{\mathbb{R}} a)) = \psi_{(U,A,\Lambda)}(\delta c, -c - \delta h, 0) = (-c - \delta h)\big|_{Z_{n-1}(p^{-1}(U))} \mod \Lambda = 0, \tag{3.81}$$

and

$$\begin{aligned}\psi_{(U,A,\Lambda)}([(c, h, \omega \otimes_{\mathbb{R}} a)])(\partial b) = \delta h(b) \mod \Lambda &= (\omega \otimes_{\mathbb{R}} a)(b) - c(b) \mod \Lambda \\ &= \iota(\omega \otimes_{\mathbb{R}} a)(b).\end{aligned} \tag{3.82}$$

For the inverse, let $f \in \hat{\mathscr{H}}^n(U; A/\Lambda)$. We construct $T(f) \in C^{n-1}(U; A)$ as in 3.23. For $\omega \otimes_{\mathbb{R}} a$ we simply take $\omega_f \otimes_{\mathbb{R}} a = \delta_1(f)$. By the proof of Proposition 3.0.4, the cochain $T(f)$ satisfies

$$\delta T(f) = \omega \otimes_{\mathbb{R}} a - c, \tag{3.83}$$

where $c \in Z^n(U; \Lambda)$ is a representative of $\delta_2(f)$. Moreover, $T(f)$ is determined up to

$$c' + \delta d,\ c' \in C^{n-1}(U; \Lambda),\ d \in C^{n-2}(U; A). \tag{3.84}$$

Consequently, the class $[(c, T(f), \delta_1(f))]$ is uniquely determined. Indeed, let $\big(\tilde{c}, \tilde{T}(f), \omega_f \otimes_{\mathbb{R}} a\big)$. Then,

$$\begin{aligned}(\tilde{c}, \tilde{T}(f), \omega_f \otimes_{\mathbb{R}} a) - (c, T(f), \omega_f \otimes_{\mathbb{R}} a) &= \big(\delta(T(f) - \tilde{T}(f)), \tilde{T}(f) - T(f), 0\big) \\ &= \big(-\delta c', c' + \delta d, 0\big) = \mathrm{d}(-c', -d, 0).\end{aligned} \tag{3.85}$$

The fact that the above constructions are mutually inverse follows from the uniqueness of the class $[(c, T(f), \omega_f \otimes_R a)]$ for a given $T(f)$. The naturality of ψ is straightforward. ☐

3.3. Twisted Differential Characters of Degree-2 as Stacks

In [LM07] E. Lerman and A. Malkin proved that ordinary differential characters of degree 2 form a stack. To be more precise, they showed that the cocycle category ([HS05]) associated to the presheaf $\mathrm{DC}_2^\bullet$ forms a stack over the category **Man**. This should not be surprising, as the cocycle category for this presheaf is equivalent to the category of U(1)-principal bundles with unitary connections. In this thesis, we give a constructive proof of effective descent of degree-2 twisted differential characters. We do not find it necessary to introduce the full language of stacks. Thanks to Proposition 3.2.4, we can make a shortcut, representing the descent data in a double complex on a Čech nerve of a cover of M. We believe that the result should hold for an arbitrary degree $n \in \mathbb{N}$. The proof is likely to be analogous, although more technically involved. The author will attempt to find the proof in his subsequent works.

For an open cover $\mathcal{O} = \bigsqcup_{i \in I} O_i$ of a connected based manifold M we construct the Čech nerve ([GM02, Ch. I]), which is a simplicial manifold $\mathcal{O}_\bullet : \Delta \to \mathbf{Man}$ whose object component is

$$\mathcal{O}_n = \underbrace{\mathcal{O} \times_M \mathcal{O} \times_M \ldots \times_M \mathcal{O}}_{n+1}, \tag{3.86}$$

and the morphism component is fixed by the choice of face and degeneracy maps. The face maps are the canonical projections

$$d_i^{(n)} : \mathcal{O}_n \to \mathcal{O}_{n-1} : (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad 0 \leqslant i \leqslant n. \tag{3.87}$$

The degeneracy maps are

$$s_i^{(n)} : \mathcal{O}_n \to \mathcal{O}_{n+1} : (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x_i, \ldots, x_n) \quad 0 \leqslant i \leqslant n. \tag{3.88}$$

Note that there are diffeomorphisms

$$\mathcal{O}_n \simeq \bigsqcup_{\bar{i} \in I^{n+1}} O_{i_0 i_1 \ldots i_n} \quad \bar{i} = (i_0, \ldots, i_n), \tag{3.89}$$

where $O_{i_0 i_1 \ldots i_n} \coloneqq O_{i_1} \cap O_{i_2} \cap \ldots \cap O_{i_n}$. Denote by j the covering map $\mathcal{O} \to M$, which restricts to inclusions j_i on O_i. Fixing an object

$$\big((M, q), A, \Lambda\big) \in \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \tag{3.90}$$

and using the Čech nerve we construct the double complex:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\tilde{\mathcal{DC}}^0_2(\mathcal{O}_2, A, \Lambda) \arrow[u] \arrow[r, "\mathrm{d}"] & \tilde{\mathcal{DC}}^1_2(\mathcal{O}_2, A, \Lambda) \arrow[u] \arrow[r, "\mathrm{d}"] & \tilde{\mathcal{DC}}^2_2(\mathcal{O}_2, A, \Lambda) \arrow[u] \arrow[r] & \dots \\
\tilde{\mathcal{DC}}^0_2(\mathcal{O}_1, A, \Lambda) \arrow[u, "\delta^1"] \arrow[r, "-\mathrm{d}"] & \tilde{\mathcal{DC}}^1_2(\mathcal{O}_1, A, \Lambda) \arrow[u, "\delta^1"] \arrow[r, "-\mathrm{d}"] & \tilde{\mathcal{DC}}^2_2(\mathcal{O}_1, A, \Lambda) \arrow[u, "\delta^1"] \arrow[r] & \dots \\
\tilde{\mathcal{DC}}^0_2(\mathcal{O}_0, A, \Lambda) \arrow[u, "\delta^0"] \arrow[r, "\mathrm{d}"] & \tilde{\mathcal{DC}}^1_2(\mathcal{O}_0, A, \Lambda) \arrow[u, "\delta^0"] \arrow[r, "\mathrm{d}"] & \tilde{\mathcal{DC}}^2_2(\mathcal{O}_0, A, \Lambda) \arrow[u, "\delta^0"] \arrow[r] & \dots \\
\mathcal{DC}^0_2(M, A, \Lambda) \arrow[u, "j^*"] \arrow[r, "-\mathrm{d}"] & \mathcal{DC}^1_2(M, A, \Lambda) \arrow[u, "j^*"] \arrow[r, "-\mathrm{d}"] & \mathcal{DC}^2_2(M, A, \Lambda) \arrow[u, "j^*"] \arrow[r] & \dots \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{3.91}
$$

where

$$\tilde{\mathcal{DC}}^k_2(\mathcal{O}_n, A, \Lambda) := \prod_{\bar{i} \in I^{n+1}} \mathcal{DC}^k_2(O_{i_0 i_1 \dots i_n}, A, \Lambda), \tag{3.92}$$

and the map j^* is defined as

$$j^* := \left(\mathcal{DC}^\bullet_2(\bar{\mathrm{id}}_M, \mathrm{id}_A, \bar{j}_i)\right)_{i \in I} : \mathcal{DC}^\bullet_2(M, A, \Lambda) \to \tilde{\mathcal{DC}}^\bullet_2(\mathcal{O}_0, A, \Lambda). \tag{3.93}$$

In order to define δ^k, note that via the diffeomorphisms 3.89 one can express the face maps of $\mathcal{O}_\bullet$ as

$$d^{(n)}_l = \bigsqcup_{\bar{i} \in I^{n+1}} d^{(n)}_{l, \bar{i}} : \bigsqcup_{\bar{i} \in I^{n+1}} O_{i_0 i_1 \dots i_n} \to \bigsqcup_{\bar{i} \in I^n} O_{i_0 i_1 \dots i_{n-1}}, \tag{3.94}$$

where

$$d^{(n)}_{l, \bar{i}} : O_{\bar{i}} \hookrightarrow \bigsqcup_{\bar{i} \in I^n} O_{i_0 i_1 \dots i_{n-1}}. \tag{3.95}$$

Using these inclusions, we define

$$\delta^k := \sum_{l=0}^{k+1} (-1)^l \prod_{\bar{i} \in I^{n+2}} \mathcal{DC}^\bullet_2\Big(\bar{\mathrm{id}}_M, \mathrm{id}_A, \bar{d}^{(k+1)}_{l, \bar{i}}\Big). \tag{3.96}$$

We call δ^k the (k-th) Dupont operator. The horizontal differentials of 3.91 are (besides the first row) the respective products of differentials of $\mathcal{DC}^\bullet_2$. Note that j^* and all δ^k are chain maps for cochain complexes $\mathcal{DC}$ and their products. As a general fact ([GM02, Ch. I]), all the columns of 3.91 are exact. For any representative $(c, h, \omega \otimes_{\mathbb{R}} a)$ of a class in $H^2(\mathcal{DC}^\bullet_2(M, A, \Lambda))$, the chain map j^* induces a cocycle in the total complex of the truncation

$$\begin{tikzcd}
0 & 0 & 0 & \\
\tilde{\mathcal{DC}}_2^0(\mathcal{O}_2, A, \Lambda) \arrow[r, "\mathrm{d}"] \arrow[u, "0"] & \tilde{\mathcal{DC}}_2^1(\mathcal{O}_2, A, \Lambda) \arrow[r, "\mathrm{d}"] \arrow[u, "0"] & \tilde{\mathcal{DC}}_2^2(\mathcal{O}_2, A, \Lambda) \arrow[r] \arrow[u, "0"] & \dots \\
\tilde{\mathcal{DC}}_2^0(\mathcal{O}_1, A, \Lambda) \arrow[r, "-\mathrm{d}"] \arrow[u, "\delta^1"] & \tilde{\mathcal{DC}}_2^1(\mathcal{O}_1, A, \Lambda) \arrow[r, "-\mathrm{d}"] \arrow[u, "\delta^1"] & \tilde{\mathcal{DC}}_2^2(\mathcal{O}_1, A, \Lambda) \arrow[r] \arrow[u, "\delta^1"] & \dots \\
\tilde{\mathcal{DC}}_2^0(\mathcal{O}_0, A, \Lambda) \arrow[r, "\mathrm{d}"] \arrow[u, "\delta^0"] & \tilde{\mathcal{DC}}_2^1(\mathcal{O}_0, A, \Lambda) \arrow[r, "\mathrm{d}"] \arrow[u, "\delta^0"] & \tilde{\mathcal{DC}}_2^2(\mathcal{O}_0, A, \Lambda) \arrow[r] \arrow[u, "\delta^0"] & \dots
\end{tikzcd} \tag{3.97}$$

Simply take $j^*(c, h, \omega \otimes_{\mathbb{R}} a) \in \tilde{\mathcal{DC}}_2^2(\mathcal{O}_0, A, \Lambda)$, and $(0,0,0)$ in $\tilde{\mathcal{DC}}_2^1(\mathrm{O}_1, A, \Lambda)$ and $\tilde{\mathcal{DC}}_1^0(\mathcal{O}_2, A, \Lambda)$. This defines a map

$$\varpi : H^2(\mathcal{DC}_2^\bullet(M, A, \Lambda)) \to H^2_{\mathrm{tot}}\big(\tilde{\mathcal{DC}}_2^\bullet(\mathcal{O}_\bullet, A, \Lambda)_{\mathrm{T}}\big), \tag{3.98}$$

where $\tilde{\mathcal{DC}}_2^\bullet(\mathcal{O}_\bullet, A, \Lambda)_{\mathrm{T}}$ is the truncated double complex 3.97.

Theorem 3.3.1. *For every*

$$\big((M, q), A, \Lambda\big) \in \int_{\mathbf{Man}_*^{\mathrm{op}}} \mathbb{Z}\pi_1\text{-}\mathbf{Mod}_{\mathbb{R},\mathrm{inj}}^{\mathrm{pair},\iota} \tag{3.99}$$

and every open cover $\mathcal{O}$ of M, the map $\varpi : H^2(\mathcal{DC}_2^\bullet(M, A, \Lambda)) \to H^2_{\mathrm{tot}}\big(\tilde{\mathcal{DC}}_2^\bullet(\mathcal{O}_\bullet, A, \Lambda)_{\mathrm{T}}\big)$ is an isomorphism.

Proof. Let

$$\begin{array}{cccccc}
(c_{ijk}, 0, 0) & 0 & & & & \\
 & (c_{ij}, h_{ij}, 0) & & 0 & & \\
 & & & (c_i, h_i, \omega_i \otimes_{\mathbb{R}} a) & & 0
\end{array} \tag{3.100}$$

be a general cocycle in the total complex of $\tilde{\mathcal{DC}}_2^\bullet(\mathcal{O}_\bullet, A, \Lambda)_{\mathrm{T}}$. We interpret $x_{i_0,\dots,i_n}$, $x \in \{c, h, \omega\}$ as elements of a product 3.92. We can give the inverse to ϖ as a map from the class of 3.100 to a homomorphism

$$f \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}(Z_1(\tilde{M}), A/\Lambda), \quad f \circ \partial = \iota(\omega_f \otimes_{\mathbb{R}} a). \tag{3.101}$$

We will rely on the Subdivision Theorem [Hat02, Proposition 2.21.], and write every 1-cycle z on $\tilde{M}$ as a sum $z_{p^{-1}\mathcal{O}} + \partial b$, where $z_{p^{-1}\mathcal{O}}$ is a cycle subordinate to $p^{-1}\mathcal{O}$. This means that each simplex of $z_{p^{-1}\mathcal{O}}$ is supported in $p^{-1}(O_i)$ for some $i \in I$. We will denote by $C_\bullet^{p^{-1}\mathcal{O}}(\tilde{M})$ the group of $p^{-1}\mathcal{O}$-subordinate chains on $\tilde{M}$, and by $Z_\bullet^{p^{-1}\mathcal{O}}(\tilde{M})$–the group of $p^{-1}\mathcal{O}$-subordinate cycles on $\tilde{M}$. We can define f by introducing $f' \in \mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_1^{p^{-1}\mathcal{O}}(\tilde{M}), A/\Lambda\big)$, and setting

$$f(z) = f'\big(z_{p^{-1}\mathcal{O}}\big) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b). \tag{3.102}$$

As long as

$$f' \circ \partial = \iota(\omega_f \otimes_{\mathbb{R}} a)\big|_{\partial^{-1} Z_1^{p^{-1}\mathcal{O}}(\tilde{M})}, \tag{3.103}$$

the expression 3.102 does not depend on the choice of a decomposition of z. Indeed, let $z = z'_{p^{-1}\mathcal{O}} + \partial b'$ be another decomposition. We have

$$z_{p^{-1}\mathcal{O}} - z'_{p^{-1}\mathcal{O}} = \partial(b' - b). \tag{3.104}$$

Since both sides are subordinate to $p^{-1}\mathcal{O}$, we can use Lemma 3.3.2 (see below) to find $b_{p^{-1}\mathcal{O}} \in C_2^{p^{-1}\mathcal{O}}(\tilde{M})$ satisfying $\partial b_{p^{-1}\mathcal{O}} = \partial(b' - b)$, and get $z_{p^{-1}\mathcal{O}} - z'_{p^{-1}\mathcal{O}} = \partial b_{p^{-1}\mathcal{O}}$, and thus $\exists d \in Z_2(\tilde{M}) : b' - b - b_{p^{-1}\mathcal{O}} = d$. Applying 3.102 yields

$$\begin{aligned} f'(z_{p^{-1}\mathcal{O}}) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b) = f'(z'_{p^{-1}\mathcal{O}} + \partial b_{p^{-1}\mathcal{O}}) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b) \\ = f'(z'_{p^{-1}\mathcal{O}}) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b_{p^{-1}\mathcal{O}} + b) = f'(z'_{p^{-1}\mathcal{O}}) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b' - d) \\ = f'(z'_{p^{-1}\mathcal{O}}) + \iota(\omega_f \otimes_{\mathbb{R}} a)(b'). \end{aligned} \tag{3.105}$$

We read off the equations affirming cocyclicity of 3.100:

$$(\delta c_i, \omega_i \otimes_{\mathbb{R}} a - c_i - \delta h_i, \mathrm{d}(\omega_i \otimes_{\mathbb{R}} a)) = 0, \tag{3.106}$$

$$(c_j - c_i - \delta c_{ij}, h_j - h_i + c_{ij} + \delta h_{ij}, \omega_j \otimes_{\mathbb{R}} a - \omega_i \otimes_{\mathbb{R}} a) = 0, \tag{3.107}$$

$$(c_{jk} - c_{ik} + c_{ij} + \delta c_{ijk}, h_{jk} - h_{ik} + h_{ij} - c_{ijk}, 0) = 0. \tag{3.108}$$

It follows that local forms $\omega_i \otimes_{\mathbb{R}} a$ agree on double intersections. Since, by Lemma 2.2.2, $\Omega^\bullet_{\mathrm{I}}$ forms a sheaf, this is enough to define $\omega_f \otimes_{\mathbb{R}} a$ as the unique gluing of $\omega_i \otimes_{\mathbb{R}} a$. In order to construct f', let $z_{p^{-1}\mathcal{O}} \in Z_1^{\mathcal{O}}(\tilde{M})$ decompose into individual simplices as

$$z_{p^{-1}\mathcal{O}} = \sum_{j=1}^{m_z} z_j \quad \mathrm{im}(z_j) \subseteq p^{-1}(O_{i_j}). \tag{3.109}$$

This comes with a choice of a function

$$\{1, \ldots, m_z\} \ni j \mapsto i_j \in I. \tag{3.110}$$

We will have to prove that $f'(z_{p^{-1}\mathcal{O}})$ does not depend on this choice. Let $\mathrm{d}_0, \mathrm{d}_1$ denote the face maps defining the boundary ∂:

$$\partial z_j = \mathrm{d}_0 z_j - \mathrm{d}_1 z_j. \tag{3.111}$$

Because $z_{p^{-1}\mathcal{O}}$ is a cycle, for any z_j there exists a $z_{j'}$ such that

$$\mathrm{d}_0 z_j = \mathrm{d}_1 z_{j'}. \tag{3.112}$$

This also involves a choice $j \mapsto j'$, which—as we will show—will not change $f'(z_{p^{-1}\mathcal{O}})$. We set

$$f'(z_{p^{-1}\mathcal{O}}) = \sum_{j=1}^{m_z} h_{i_j}(z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda. \tag{3.113}$$

Note that it is well defined, since $\mathrm{im}(\mathrm{d}_0 z_j) \subseteq p^{-1}(O_{i_{j'} i_j})$. Moreover, 3.113 is linear in the cocycle 3.100. Therefore, in order to prove that f' only depends on the class in cohomology, it is enough to show that 3.113 vanishes on any coboundary

$$\begin{array}{llll} (d_{jk} - d_{ik} + d_{ij}, 0, 0) & & & \\ & (d_{ij}, 0, 0) & (a_j - a_i - \delta d_{ij}, b_j - b_i + d_{ij}, 0) & \\ & & (a_i, b_i, 0) & (\delta a_i, -a_i - \delta b_i, 0). \end{array} \tag{3.114}$$

The expression for f' becomes

$$\begin{aligned} f'(z_{p^{-1}\mathcal{O}}) &= \sum_{j=1}^{m_z} -a_{i_j}(z_j) - \delta b_{i_j}(z_j) + b_{i_j}(\mathrm{d}_0 z_j) - b_{i_{j'}}(\mathrm{d}_0 z_j) + d_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} -b_{i_j}(\mathrm{d}_0 z_j) + b_{i_j}(\mathrm{d}_1 z_j) + b_{i_j}(\mathrm{d}_0 z_j) - b_{i_{j'}}(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} b_{i_j}(\mathrm{d}_1 z_j) - b_{i_{j'}}(\mathrm{d}_1 z_{j'}) \mod \Lambda = 0, \end{aligned} \tag{3.115}$$

where we use the fact that a_i and d_{ij} take values in Λ, and the fact that $z_{p^{-1}\mathcal{O}}$ is a cycle (a sum over j' is just a shifted sum over j). Now, let us consider a different choice function

$$\{1, \dots, m_z\} \ni j \mapsto i'_j \in I. \tag{3.116}$$

Using 3.107 and 3.108 we compute

$$\begin{aligned} &\sum_{j=1}^{m_z} h_{i_j}(z_j) - h_{i'_j}(z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) - h_{i'_{j'} i'_j}(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} \delta h_{i_j i'_j}(z_j) + c_{i_j i'_j}(z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) - h_{i'_{j'} i'_j}(\mathrm{d}_0 z_j) \mod \Lambda \\ &\sum_{j=1}^{m_z} h_{i_j i'_j}(\mathrm{d}_0 z_j) - h_{i_j i'_j}(\mathrm{d}_1 z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) - h_{i'_{j'} i'_j}(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} h_{i_j i'_j}(\mathrm{d}_0 z_j) - h_{i_{j'} i'_{j'}}(\mathrm{d}_0 z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) - h_{i'_{j'} i'_j}(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} \big(h_{i_j i'_j} - h_{i'_{j'} i'_j}\big)(\mathrm{d}_0 z_j) + \big(h_{i_{j'} i_j} - h_{i_{j'} i'_{j'}}\big)(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} \big(h_{i_j i'_{j'}} - c_{i_j i'_{j'} i'_j}\big)(\mathrm{d}_0 z_j) + \big(h_{i'_{j'} i_j} - c_{i_{j'} i'_{j'} i_j}\big)(\mathrm{d}_0 z_j) \mod \Lambda \\ &= \sum_{j=1}^{m_z} \big(h_{i_j i'_{j'}} + h_{i'_{j'} i_j}\big)(\mathrm{d}_0 z_j) \mod \Lambda = \sum_{j=1}^{m_z} \big(c_{i_j i'_{j'} i_j} + c_{i_j i_j i'_{j'}}\big)(\mathrm{d}_0 z_j) \mod \Lambda = 0. \end{aligned} \tag{3.117}$$

That 3.113 is invariant under the change of assignment $j \mapsto j'$ to $j \mapsto j''$ follows by replacing $j \mapsto i_j$ so that $i'_{j''} = i_{j'}$, the invariance under which we just showed. Now, to see that f'

satisfies 3.103, suppose $z_{p^{-1}\mathcal{O}} = \partial(b_{p^{-1}\mathcal{O}})$ for a subordinate 2-chain $b_{p^{-1}\mathcal{O}}$. Note that

$$\partial(b_{p^{-1}\mathcal{O}}) = \partial \sum_{j=1}^{m_b} b_j = \sum_{j=1}^{m_b} \partial b_j, \tag{3.118}$$

so one can decompose $\partial(b_{p^{-1}\mathcal{O}})$ into simplices as follows:

$$\partial(b_{p^{-1}\mathcal{O}}) = \sum_{l=1}^{m_b} \mathrm{d}_0 b_l - \mathrm{d}_1 b_l + \mathrm{d}_2 b_l = \sum_{j=1}^{m_{\partial b}} z_j. \tag{3.119}$$

Importantly, as $\mathrm{im}(\partial b_j) \subseteq p^{-1}(O_{i_j})$ and it is a boundary, we can choose $i_{j'} = i_j$ in the latter decomposition of 3.119. By the linearity of h_i and h_{ij} one can write

$$\begin{aligned} f'(\partial(b_{p^{-1}\mathcal{O}})) &= \sum_{j=1}^{m_b} h_{i_j}(\partial b_j) + h_{i_j i_j}(\mathrm{d}_0 \partial b_j) \mod \Lambda \\ = \sum_{j=1}^{m_b} \delta h_{i_j}(b_j) + c_{i_j i_j i_j}(\mathrm{d}_0 \partial b_j) \mod \Lambda &= \sum_{j=1}^{m_b} \big(\omega_{i_j} \otimes_{\mathbb{R}} a - c_{i_j}\big)(b_j) \mod \Lambda \\ &= \iota(\omega_f \otimes_{\mathbb{R}} a)(b_{p^{-1}\mathcal{O}}), \end{aligned} \tag{3.120}$$

where we use $c_{iii} = h_{ii} - h_{ii} + h_{ii} = h_{ii}$. This completes the proof of the well-definedness of 3.102. It remains to show that the assignment of f is an isomorphism. First, let $f_M \in \hat{\mathscr{H}}^2(M; A/\Lambda)$ be a character represented by a cocycle $(c, h, \omega \otimes_{\mathbb{R}} a) \in \mathcal{DC}_2^2(M, A, \Lambda)$. In particular, $f_M = h\big|_{Z_1(\tilde{M})} \mod \Lambda$. On the other hand, the character assigned to $\varpi(f_M)$ can be written as

$$\begin{aligned} f(z_{p^{-1}\mathcal{O}} + \partial b) &= \sum_{j=1}^{m_z} h_{i_j}(z_j) \mod \Lambda + \iota(\omega \otimes_{\mathbb{R}} a)(b) \\ &= \sum_{j=1}^{m_z} h\big|_{p^{-1}(O_{i_j})}(z_j) \mod \Lambda + \iota(\omega \otimes_{\mathbb{R}} a)(b) \\ &= \sum_{j=1}^{m_z} h(z_j) \mod \Lambda + \iota(\omega \otimes_{\mathbb{R}} a)(b) \\ &= f_M(z_{p^{-1}\mathcal{O}} + \partial b). \end{aligned} \tag{3.121}$$

It follows that our map $H^2_{\mathrm{tot}}\big(\tilde{\mathcal{DC}}_2^{\bullet}(\mathcal{O}_\bullet, A, \Lambda)_{\mathrm{T}}\big) \to H^2(\mathcal{DC}_2^{\bullet}(M, A, \Lambda))$ is surjective. It is now enough to show injectivity. Suppose that for a cocycle 3.100 the corresponding character f is zero. That is, in particular

$$f'(z_{p^{-1}\mathcal{O}}) = \sum_{j=1}^{m_z} h_{i_j}(z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda = 0 \tag{3.122}$$

for each $z_{p^{-1}\mathcal{O}} \in Z_1^{p^{-1}\mathcal{O}}(\tilde{M})$. Note that for $z = z_{p^{-1}\mathcal{O}}$, we may take $b = 0$. By considering a cycle supported in $p^{-1}(O_i)$, picking $i_{j'} = i_j$, and using the fact that $h_{ii} \in C^0(O_{ii}; \Lambda)$ we find that $h_i(z_{p^{-1}(O_i)})$ for any such $z_{p^{-1}(O_i)}$. Therefore, by the Universal Coefficient Theorem, and by the injectivity of A as a $\mathbb{Z}\pi_1(M)$-module, there exist $b_i \in C^1(O_i; A)$ and $c'_i \in C^1(O_i; \Lambda)$

such that $h_i = -\delta b_i + c_i'$. The expression for f' becomes

$$\begin{aligned}
f'(z_{p^{-1}\mathcal{O}}) &= \sum_{j=1}^{m_z} b_{i_j}(\mathrm{d}_1 z_j) - b_{i_j}(\mathrm{d}_0 z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda \\
&= \sum_{j=1}^{m_z} b_{i_{j'}}(\mathrm{d}_1 z_{j'}) - b_{i_j}(\mathrm{d}_0 z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda \\
&= \sum_{j=1}^{m_z} b_{i_{j'}}(\mathrm{d}_0 z_j) - b_{i_j}(\mathrm{d}_0 z_j) + h_{i_{j'} i_j}(\mathrm{d}_0 z_j) \mod \Lambda \\
&= \sum_{j=1}^{m_z} \big(-(\delta^0 b)_{i_{j'} i_j} + h_{i_{j'} i_j}\big)(\mathrm{d}_0 z_j) \mod \Lambda.
\end{aligned} \tag{3.123}$$

Since z_j are arbitrary, it follows that $h_{ij} = (\delta^0 b)_{ij} + d_{ij}$ for $d_{ij} \in C^0(O_{ij}; \Lambda)$. Moreover, by the cocyclicity of $(c_i, h_i, \omega_i \otimes_{\mathbb{R}} a)$, we obtain

$$\mathrm{d}(c_i, -\delta b_i + c_i', \omega_i \otimes_{\mathbb{R}} a) = (\delta c_i, \omega_i - c_i - \delta c_i', \mathrm{d}(\omega_i \otimes_{\mathbb{R}} a)) = 0, \tag{3.124}$$

so, by the injectivity of ι, we infer $\omega_i = 0$. Combining all the above, we find that $c_i = -\delta c_i'$, and then $(c_i, -\delta b_i + c_i', \omega_i \otimes_{\mathbb{R}} a) = \mathrm{d}(-c_i', b_i, 0)$, so the cocycle 3.100 is a coboundary 3.114. Hence, the proof is complete. □

Lemma 3.3.2. *Let M be a manifold and let $b \in C_k(M)$ be such that $\partial b \in C^{\mathcal{O}}_{k-1}(M)$. Then, there exists $b' \in C^{\mathcal{O}}_k(M)$ satisfying $\partial b' = \partial b$.*

Proof. We write down the homotopy

$$\rho \circ \iota - \mathrm{id}_{C^{\mathcal{O}}_\bullet} = \partial \circ h + h \circ \partial. \tag{3.125}$$

Now, define $b' \coloneqq \rho(b) - h(\partial b)$. This lies in $C^{\mathcal{O}}_k(M)$, since $\partial b \in C^{\mathcal{O}}_{k-1}(M)$. We compute

$$\begin{aligned}
\partial b' = \partial\rho(b) - \partial(h(\partial b)) &= \rho(\partial b) - \big((\rho \circ \iota - \mathrm{id}_{C^{\mathcal{O}}_\bullet})(\partial b) - h(\partial(\partial b))\big) \\
&= \rho(\partial b) - \rho(\iota(\partial b)) + \partial b = \rho(\partial b) - \rho(\partial b) + \partial b = \partial b,
\end{aligned} \tag{3.126}$$

where we used the fact that ρ is a chain map, and that $\iota(\partial b) = \partial b$. □

Chapter 4

Examples and applications to weakly abelian gauge theory

4.1. Topologically trivial coefficient systems

The simplest situation in our theory arises when the coefficient system is topologically trivial in the sense that it corresponds to a trivial representation $\rho = \mathrm{id}_{\mathrm{Aut}_{\mathbb{R}}(A)}$ of $\pi_1(M)$ in an $\mathbb{R}$-vector space A. We indicate this using an underline under the coefficient system symbol. In this case, the $\mathbb{Z}\pi_1(M)$-module structure of A coincides with the $\mathbb{Z}$-module structure of the underlying abelian group of A, which is divisible and hence injective. A $\mathbb{Z}\pi_1(M)$-submodule Λ is an arbitrary subgroup of the underlying additive abelian group of A. A ι-subgroup Λ is then such a subgroup of A that

$$\Omega^\bullet(M, A) := \Omega^\bullet(M) \otimes_{\mathbb{R}} A \ni \omega \otimes_{\mathbb{R}} a \overset{\iota}{\mapsto} \Big(c \mapsto \int_c \omega \otimes_{\mathbb{R}} a \mod \Lambda\Big) \in C^\bullet(M, A/\Lambda) \tag{4.1}$$

is an injection. Moreover, we define

$$\Omega^k_\Lambda(M, A) := \Big\{\omega \in \Omega^k(M, A) : \mathrm{d}(\omega \otimes_{\mathbb{R}} a) = 0 \ \wedge \ \forall c \in Z_k(M) : \int_c \omega \otimes_{\mathbb{R}} a \in \Lambda\Big\}. \tag{4.2}$$

Note that for any ι-subgroup Λ of A we can define a simple extension of ordinary differential characters. Namely,

$$\hat{H}^k(M, A/\Lambda) := \Big\{f \in \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(M), A/\Lambda\big) \Big| \exists\, \omega_f \otimes_{\mathbb{R}} a \in \Omega^k_\Lambda(M, A) : f \circ \partial = \iota(\omega_f \otimes_{\mathbb{R}} a)\Big\}. \tag{4.3}$$

Proposition 4.1.1. *For a topologically trivial local system as above, we have*

$$\hat{\mathscr{H}}^k(U; \underline{A/\Lambda}) \simeq \hat{H}^k(U, A/\Lambda). \tag{4.4}$$

Proof. Following the arguments from Remark 2.0.2, for any open $U \subseteq M$, we have

$$\Omega^k_{\mathrm{I}}(p^{-1}(U); \underline{A}) \simeq \Omega^k(U) \otimes_{\mathbb{R}} A = \Omega^k(U, A) \tag{4.5}$$

and

$$\Omega^k_{\mathrm{I},\Lambda}(p^{-1}(U); \underline{A}) \simeq \Omega^k_\Lambda(U, A). \tag{4.6}$$

Using the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}\big(Z_{k-1}(p^{-1}(U)), \underline{A/\Lambda}\big) \simeq \mathrm{Hom}_{\mathbb{Z}}\big(Z_{k-1}(U), A/\Lambda\big) \tag{4.7}$$

completes the proof. □

4.2. Application to weakly abelian gauge theories

4.2.1. Weakly abelian Lie groups

Definition 4.2.1. A (generally disconnected) Lie group G is called *weakly abelian* if its Lie algebra $\mathfrak{g}$ is abelian, i.e. if the Lie bracket of $\mathfrak{g}$ is identically zero.

Let G be a weakly abelian Lie group. Then it is easy to see that the connected component[1] $A := G_0$ of the identity is an abelian Lie group which coincides with the image of the exponential map $\exp_G : \mathfrak{g} \to G$. The Baker-Campbell-Hausdorff formula implies that the latter is a group epimorphism from the underlying abelian group of the vector space $\mathfrak{g}$ to A. We will use additive notation for A and multiplicative notation for G. By a standard classification theorem for connected abelian Lie groups, A is a direct product of a torus group with a translation group. It is a torus group if and only if A is compact. Let $\Gamma := G/A \simeq \pi_0(G)$ be the group of connected components of G. We have an exact sequence of Lie groups:

$$1 \to A \xrightarrow{i} G \xrightarrow{q} \Gamma \to 1 \ , \tag{4.8}$$

where i and q respectively are the inclusion and projection and Γ is endowed with the discrete topology and zero-dimensional manifold structure. The conjugation action of G preserves A and hence induces a morphism of groups $\mathrm{Ad}_G^A : G \to \mathrm{Aut}(A)$ (called the *restricted conjugation action*), where $\mathrm{Aut}(A)$ is the group of Lie automorphisms of A. This factors through q to the so-called *characteristic morphism* $\rho : \Gamma \to \mathrm{Aut}(A)$ of G:

$$\mathrm{Ad}_G^A = \rho \circ q \ . \tag{4.9}$$

This makes A into an abelian smooth module over Γ, which we denote by A_ρ and call the *characteristic module* of G. The extension 4.8 is central if and only if the characteristic module is trivial. Since A is abelian, its group of inner automorphisms is trivial. This implies that equivalent Lie group extensions of Γ by A have the same characteristic morphism ρ (see Sec. 18 in [HN12]).

Since A is abelian, the adjoint representation of $\mathrm{ad}_G : G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ factors through q to the *reduced adjoint representation* $\bar{\rho} : \Gamma \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$:

$$\mathrm{ad}_G = \bar{\rho} \circ q \ . \tag{4.10}$$

Note that $\bar{\rho}$ determines ρ via the exponential map. The *exponential lattice of* G is the abelian group:

$$\Lambda := \ker(\exp_G) \subseteq \mathfrak{g} \ . \tag{4.11}$$

This is a (generally non-full) lattice in $\mathfrak{g}$ which is stabilized by the adjoint and reduced adjoint representations. Since $\exp_G : \mathfrak{g} \to A$ is the universal covering of A, and by $\mathfrak{g}/\Lambda \simeq A$, we find that Λ (as an additive group) is naturally isomorphic to the abelian group $\pi_1(G) := \pi_1(A, 1_G)$. The restriction $\mathrm{ad}_0 : G \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ of the adjoint representation ad_G is called the *restricted conjugation action* of G while the restriction $\bar{\rho}_0 : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ of the reduced adjoint representation $\bar{\rho}$ is called the *coefficient morphism* of G. We have:

$$\mathrm{ad}_0 = \bar{\rho}_0 \circ q \ . \tag{4.12}$$

[1] The $\mathbb{R}$-vector space denoted by A in previous sections corresponds to the vector space denoted by $\mathfrak{g}$ in the present section, while the abelian group A of the present section corresponds to the quotient A/Λ in the notation of previous sections.

The coefficient morphism makes Λ into a discrete Γ-module which we denote by Λ_{ρ_0} and call the *coefficient module* of G. It is easy to see that we have:

$$A \simeq (\Lambda \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda \times (\mathbb{R}^{\dim A - \mathrm{rk}\Lambda}, +) . \tag{4.13}$$

The translation group on the right is trivial if and only if Λ is a full lattice in $\mathfrak{g}$ (which happens if and only if A is compact). In this case, A is a torus group and we say that that the weakly abelian Lie group G is *full*.

4.2.2. Weakly abelian principal bundles

By definition, a *weakly abelian* classical gauge theory defined on a manifold M is a classical gauge theory defined on M and whose structure group is weakly abelian. Such a theory describes principal connections on a weakly abelian principal bundle P defined over M, i.e. a principal bundle over M whose structure group G is weakly abelian. Notice that such principal bundles have *disconnected* structure group unless $\pi_0(G) = \Gamma$ is the trivial group.

Let G be a weakly abelian Lie group characterized by an extension sequence 4.8 and P be a principal G-bundle defined over a connected manifold M. We define the *discrete remnant* of P to be the principal Γ-bundle:

$$\Gamma(P) := P \times_q \Gamma \tag{4.14}$$

associated to P through the group epimorphism $q : G \to \Gamma$. The discrete remnant map is the surjective based morphism of principal bundles $\Phi_P : P \to \Gamma(P)$ above the morphism of groups $q : G \to \Gamma$ which sends a point $p \in P$ to the equivalence class $[p, 1_\Gamma] \in \Gamma(P)$, thus giving a reduction of the structure group (in the general sense of that term) of $\Gamma(P)$ from Γ to G. Notice that $\Gamma(P)$ identifies with the principal Γ-bundle P/A whose fiber at $m \in M$ is the right Γ-space P_m/A of A-orbits of the right G-space P_m.

Since $\Gamma(P)$ is a bundle with discrete fiber, it carries a natural flat Ehresmann connection. Accordingly, the following bundles associated to $\Gamma(P)$ carry natural flat Ehresmann connections:

1. The *adjoint bundle* of P, which we denote by:

$$\mathfrak{g}(P) := P \times_{\mathrm{ad}_G} \mathfrak{g} = \Gamma(P) \times_{\bar{\rho}} \mathfrak{g} . \tag{4.15}$$

2. The *Dirac system*:

$$\Lambda(P) := P \times_{\mathrm{Ad}_0} \Lambda = \Gamma(P) \times_{\bar{\rho}_0} \Lambda \tag{4.16}$$

3. The bundle of connected abelian Lie groups:

$$A(P) := P \times_{\mathrm{Ad}_G^A} A = \Gamma(P) \times_\rho A . \tag{4.17}$$

The flat connections induced from $\Gamma(P)$ are compatible with the fiber structures of the bundles $\mathfrak{g}(P)$, $\Lambda(P)$ and $A(P)$. Thus, the induced connection on $\mathfrak{g}(P)$ is a flat *linear* connection, the parallel transport of the flat connection of $\Lambda(P)$ proceeds through morphisms of abelian groups between the fibers (which makes $\Lambda(P)$ into a local system of discrete abelian groups) and the parallel transport of the flat connection of $A(P)$ proceeds through morphisms of Lie groups (which makes $A(P)$ into a flat bundle of abelian Lie groups). Note that $\Lambda(P)$ is a discrete flat fiber sub-bundle of the vector bundle $\mathfrak{g}(P)$, i.e. a fiber sub-bundle which is preserved by the parallel transport of the linear flat connection of $\mathfrak{g}(P)$. The exponential short exact sequence of abelian groups:

$$0 \to \Lambda \to \mathfrak{g} \stackrel{\exp_G}{\to} A \to 0 \tag{4.18}$$

induces a short exact sequence of bundles of abelian Lie groups:

$$0 \to \Lambda(P) \to \mathfrak{g}(P) \stackrel{\exp_G}{\to} A(P) \to 0\ . \tag{4.19}$$

4.2.3. Twisted differential characters associated with weakly abelian principal bundles

Denote by D the flat linear connection of $\mathfrak{g}(P)$ and let $\mathrm{d}_D : \Omega^\bullet(M, \mathfrak{g}(P)) \to \Omega^{\bullet+1}(M, \mathfrak{g}(P))$ be the de Rham differential twisted by D [KN63, II. 5.]. Then $\mathrm{d}_D^2 = 0$ and the the fact that sheaves of smooth local sections of vector bundles are acyclic implies that the de Rham cohomology of $\mathfrak{g}(P)$ twisted by D coincides with the sheaf cohomology of the sheaf $\mathcal{S}(P)$ of flat local sections of $\mathfrak{g}(P)$:

$$H^\bullet_{\mathrm{d}_D}(M, \mathfrak{g}(P)) = H^\bullet(M, \mathcal{S}(P)). \tag{4.20}$$

It is natural to consider the differential cohomology $\hat{\mathscr{H}}^\bullet(M; \mathfrak{g}(P)/\Lambda(P))$ with local coefficients given by $\mathfrak{g}(P)$ and integrality conditions imposed along the Dirac system $\Lambda(P) \subseteq \mathfrak{g}(P)$. This provides natural secondary characteristic classes for $\mathfrak{g}(P)$-valued differential forms. A twisted differential character $f \in \hat{\mathscr{H}}^k(M; \mathfrak{g}(P)/\Lambda(P))$ associates an element of $\mathfrak{g}/\Lambda$ to any $(k-1)$-cycle $c_{k-1} \in Z_{k-1}(\tilde{M})$ in such a way that, for any k-chain $c_k \in C_k(\tilde{M})$, we have $f(\partial c_k) = \iota(\omega_f)$ for some d_D-closed form $\omega_f \in \Omega^2_{\mathrm{d}_D-\mathrm{cl}}(M, \mathfrak{g}(P))$ which satisfies the *Dirac integrality condition* with respect to the Dirac system $\Lambda(P)$. The latter states that ω_f has Λ-valued periods when the notion of period is considered in the sense explained in Chapter 2. The translation from the language of flat bundles on M to $\mathbb{Z}\pi_1(M)$-modules is straightforward upon trivializing the pullbacks $p^*\mathfrak{g}(P)$ and $p^*\Lambda(P)$ over the contractible $\tilde{M}$. From this perspective, the parallel transport amounts to the action of $\mathbb{Z}\pi_1(M)$ on fibers.

4.2.4. The Dirac quantization condition for weakly abelian gauge theories

An obvious application to weakly abelian gauge theory is provided by degree-1 twisted differential characters of this type. The adjoint curvature of a principal connection $A \in \mathrm{Conn}(P)$ is an $\mathfrak{g}(P)$-valued 2-form $F_A \in \Omega^2(M, \mathfrak{g}(P))$. This form is d_D-closed by the Bianchi identity and hence we have:

$$F_A \in \Omega^2_{\mathrm{d}_D-\mathrm{cl}}(M, \mathfrak{g}(P))\ . \tag{4.21}$$

The Wilson loop of a principal connection $A \in \mathrm{Conn}(P)$ along a 1-cycle $c_1 \in Z_1(P)$ defines an element $f_A(c_1)$ of $A = \exp(\mathfrak{g}) \simeq \mathfrak{g}/\Lambda$. By the nonabelian Stokes theorem [KMR99], this satisfies

$$f_A(\partial c_2) = \int_{c_2} F_A \tag{4.22}$$

for all $c_2 \in C_2(M)$, when the integral in the right hand side is interpreted appropriately. It follows that the Wilson loop of A is a twisted differential character if and only if the curvature 2-form F_A is integral with respect to the Dirac system $\Lambda(P)$, i.e. if and only if A satisfies the Dirac quantization condition relative to the Dirac system $\Lambda(P)$. In this case, A a *semiclassical* (rather than merely classical) principal connection. This implies the following:

Proposition 4.2.2. *The elements of $\hat{\mathscr{H}}^2(M;\mathfrak{g}(P)/\Lambda(P))$ correspond to the Wilson loops of semiclassical principal connections defined on P, i.e. those principal connections which obey the Dirac quantization condition relative to the Dirac system $\Lambda(P)$.*

The study of principal bundles with weakly abelian structure group of associated gauge theories is of interest for self-dual formulations of theories containing various extended versions of electromagnetism, such as $N = 1$ supergravity in four dimensions (see [LS22]). In that case, the group extension 4.8 is split [2] and one has

$$A = \mathrm{U}(1)^{2n} \text{ and } \Gamma = \mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) \tag{4.23}$$

for some natural number $n \in \mathbb{Z}_{>0}$, where the *modular exponent* $\mathfrak{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n_{>0}$ is a sequence of positive integers which satisfy the division conditions:

$$t_1|t_2|\ldots|t_n, \tag{4.24}$$

and

$$\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z}) = \{\sigma \in \mathrm{Sp}(2n, \mathbb{R})|\sigma(\Lambda_{\mathfrak{t}}) = \Lambda_{\mathfrak{t}}\} \tag{4.25}$$

is the *modified Siegel modular group* in dimension $2n$. Here $\Lambda_{\mathfrak{t}}$ is the integral symplectic lattice spanned inside $\mathbb{R}^{2n}$ by the vectors $e_1, \ldots, e_n, t_1 e_{n+1}, \ldots, t_n e_{2n}$, where $e_1, \ldots e_{2n}$ is the canonical basis of $\mathbb{R}^{2n}$. In this case, we have $\mathfrak{g} = \mathbb{R}^{2n}$ and $\Lambda = \Lambda_{\mathfrak{t}} \simeq_{\mathbb{Z}} \mathbb{Z}^{2n}$. We refer the reader to [LS21] for a brief explanation of the physics origin and meaning of this choice. Much more detail can be found in [LS18].

The Dirac quantization condition for the situation relevant to $N = 1$ supergravity was studied in detail in [LS22]. The results above give an interpretation of those results through the degree two twisted differential cohomology of the pair $(\mathfrak{g}(P), \Lambda(P))$.

4.3. On computations of twisted differential cohomology when Λ is an arithmetic group

The computation of twisted differential cohomology can be highly nontrivial when the corresponding $\mathbb{Z}\pi_1(M)$-$\mathbb{R}$-bimodule has a complicated structure. For example, in weakly abelian gauge theories relevant to N=1 supergravity, the representation of $\pi_1(M)$ in the underlying vector space factors through a representation [3] of the discrete group Γ, which in such theories is the arithmetic group $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. This relates the problem of computing the relevant twisted differential cohomology groups to the study of linear representations of the modified Siegel modular groups $\mathrm{Sp}_{\mathfrak{t}}(2n, \mathbb{Z})$. This relates the twisted differential cohomology relevant for such theories to the theory of Siegel modular forms. As explained for example in [LS18], modified Siegel modular groups arise naturally in the theory of non-principally polarized abelian varieties and their representations play a crucial role in the theory of Siegel modular varieties.

[2]The precise semidirect product structure of the split extension 4.8 for this case is given in loc. cit.
[3]This can be described explicitly using the universal model provided by the classifying space of G or via universal Chern-Weyl theory.

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