Singularity approach to integrability in discrete dynamical systems

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Overview

Introduction and motivation

Goal: Understand what is integrability of (discrete) dynamical systems and how singularities are useful for understanding it. The first part focuses on the continuous Painlevé property.
Understanding the singularities of a (nonlinear) differential equation can help in explicitly building the solutions of the equation. Integrability of an ODE refers, according to Poincaré, to expressing its solution in a finite expression built from functions, i.e., maps that can be made singlevalued in their whole domain of definition.

Singularities of an ODE

Definition

A singularity is a point where a certain function is not analytical, i.e., it cannot be expanded in a convergent power series centered at that point.

Definition

A function contains a critical singularity if it shows local multivaluedness around the singular point.

Example: $f(x) = \ln(x-1)$ contains a critical singularity at $x = 1$.
A singularity that is not critical is called noncritical.

Definition

A solution of an ODE contains a movable singularity if its position depends on the initial conditions.

A singularity that is not movable is called nonmovable.

Examples of singularities

- Bessel differential equation:

$$x^2 f''(x) + x f'(x) + (x^2 - \alpha^2) f(x) = 0 \quad (1)$$

After dividing with x^2:

$$f''(x) + \frac{1}{x} f'(x) + (1 - \frac{\alpha^2}{x^2}) f(x) = 0 \quad (2)$$

For $\alpha = 0$ the equation has a first order singularity at $x = 0$. Otherwise, it has a second order singularity at $x = 0$.

The need for special functions

Since some ODEs cannot be integrated, we are interested in finding the closed form expression for their solutions. This is how we can introduce some special functions that then occur in other ODEs. One of the motivations behind studying ODEs was to obtain all special functions required to solve ODEs of a certain order. The following were obtained [1]:

- First order ODEs: only one new function, the elliptic function. The Weierstrass elliptic function satisfies the following differential equation:

$$\wp'^2(z) = 4\wp^3(z) - a\wp(z) - b \tag{3}$$

  where a and b are functions obeying certain properties.
- Second order ODEs: six new functions, discussed below
- Third order ODEs: no new functions
- Fourth order ODEs: work in progress regarding the irreducibility of certain eqs.

Painlevé property

Definition

Painlevé property is the requirement that the general solution of an ODE has no movable critical singlarities.

Paul Painlevé studied the second-order differential equations of the form:

$$y'' = R(y', y, t) \tag{4}$$

where R is a rational function. Painlevé realized that through certain transformations, all these equations can be reduced to 50 canonical forms. 44 of these equations could be solved through already existing functions, leaving only six to be solved through newly introduced special functions. Further work in this field was done by Bertrand Gambier, a student of Painlevé.
The objective of Painlevé, as stated by himself is [2]: "Déterminer toutes les équations différentielles algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l'intégrale a ses points critiques fixes."

Painlevé equations

1. (Painlevé):

$$\frac{d^2y}{dt^2} = 6y^2 + t$$

2. (Painlevé):

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

3. (Painlevé):

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y}$$

4. (Gambier):

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2\left(t^2 - \alpha\right)y + \frac{\beta}{y}$$

5. (Gambier):

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \quad + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

6. (R. Fuchs):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}$$

Here $\alpha, \beta, \gamma, \delta$ are complex numbers.

Homographic group

The group of invariance of the Painlevé property is the class of transformations:

$$(u,x) \to (U,X),\ u(x) = \frac{\alpha(x)U(X)+\beta(x)}{\gamma(x)U(X)+\delta(x)},\ X = \xi(x) \tag{5}$$

with $\alpha, \beta, \gamma, \delta, \xi$ functions obeying $\alpha\delta - \beta\gamma \neq 0$.

Painlevé property for Partial Differential Equations

We say that a PDE has Painlevé property when its solutions are single-valued about the movable, singulairty manifolds [3]. Let us consider a condition of the form $\phi(z_1, ..., z_n) = 0$, where ϕ is an analytic function of $z_1, ..., z_n$ in a neighbourhood of the manifold. Then, we assume that a solution $u = u(z_1, ..., z_n)$ takes the form:

$$u = u(z_1, ..., z_n) = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \tag{6}$$

where ϕ and u are analytic functions of $z_1, ..., z_n$ in a neighbourhood of the manifold and α is an integer. By introducing this substitution in the PDE determines the possible values for α and the recursion relation for $u_j, \ j = 0, 1, 2...$

Testing for Painlevé's property

The purpose of Painlevé test is to build necessary conditions for the absence of movable critical points in the general solution. However, there are exception where these conditions are satisfied, yet the system is not integrable. We will present here an introduction to the algorithm behind this test based on [4].
We consider a differential equation of order N, polynomial in $\mathbf{u}$ and its derivatives, and analytic in x.

Step 0: Reduce the number of terms

Performing a transformation on the equation can reduce the number of terms. Consider the following PDE:

$$u_x + u_t + u_{xxt} + u_x u_t = 0 \tag{7}$$

We can transform it as $u = U - x - t$. It becomes:

$$U_{xxt} + U_x U_t + 1 = 0 \tag{8}$$

Step 1: Some very general conditions

Consider the following class of problems:

$$u^{(m)} = R(u^{(m-1)}, u^{(m-2)}, ..., u', u, x) \tag{9}$$

Here R is rational in u and its derivatives. Some of the necessary stability conditions are the following:

- As a rational function of $u^{(m-1)}$, R is a polynomial of degree at most two:

$$u^{(m)} = Au^{(m-1)^2} + Bu^{(m-1)} + C \tag{10}$$

- As a rational function of $u^{(m-2)}$, A has only simple poles a_i with residues $r_i = 1 - 1/n_i$, with n_i nonzero integers, possibly infinite.
- As a rational function of $u^{(m-2)}$, B and C have no other poles than those of A, and these poles are simple.

Step 1: Example

Consider the equation:

$$-3u^2u'u''' + 5u^2u''^2 - uu'^2u'' - u'^4 = 0 \tag{11}$$

Writing $u''' = R(u'', u', u, x)$ the coefficient of u''^2 has a residue $5/3$, which cannot be written as $1 - 1/n$:

$$A = \frac{5/3}{u'} \tag{12}$$

Step 2: ODE satisfied by singular solution

If the degree is greater than one, establish the ODE satisfied by the singular solutions. The algorithm is as follows: compute the discriminant, factorize it, discard the even factors, test each odd factor to check if it defines a solution to the equation.
Example:

$$27uu'^3 - 12xu' + 8u = 0 \tag{13}$$

The zeros of the derivative of the expression with respect to u' are obtained from:

$$27uu'^2 - 4x = 0 \tag{14}$$

Replacing in the equation allows us to compute the singular solution:

$$u^3 = \frac{4}{27}x^3 \tag{15}$$

Step 3: Cauchy theorem

Theorem

Consider an ODE of order N, of degree one in the highest derivative, defined in the canonical form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{x}} = \mathbf{K}[x, \mathbf{u}], x \in \mathcal{C}, \mathbf{u} \in \mathcal{C}^N \tag{16}$$

Let $(x_0, \mathbf{u}_0)$ *be a point in* $\mathcal{C} \times \mathcal{C}^N$ *and D a domain containing* $(x_0, \mathbf{u}_0)$*. If* $\mathbf{K}$ *is holomorphic in D, there exists a unique solution satisfying the initial condition and it is holomorphic in a domain containing* $(x_0, \mathbf{u}_0)$*.*

One needs to find the exceptional points where the theorem does not hold. Then, perform a homographic transformation to allow for the theorem to be applied. For each equation (the initial and the transformed ones), perform step 4.

Step 4: Families of solutions

We seek solutions of the form:

$$\mathbf{u} = \sum_{j=0}^{+\infty} \mathbf{u}_j \chi^{j+\mathbf{p}}, \ \mathbf{u}_0 \neq \mathbf{0}, \chi' = 1 \tag{17}$$

We discard those families which are also solutions of the ODEs for singular solutions obtained at step 2. We discard all families with all components of **p** positive. For each remaining family, perform the next steps.

Step 5: Indicial equation

For each equation, we compute the linear operator $\mathbf{P}(i)$, defined from the following equation:

$$\mathbf{E}_j \equiv \mathbf{P}(j)\mathbf{u}_j + \mathbf{Q}_j(\mathbf{u}_l | l < j) = 0 \tag{18}$$

The indicial equation is obtained as $\det \mathbf{P}(i) = 0$. We compute its zeros (Fuchs indices) and require them to be integers and to satisfy the rank condition: multiplicity of $i = \dim \mathrm{Ker}\, \mathbf{P}(i)$.

Further steps

Depending on the degree of the indicial polynomial, further steps are taken.
Remember that this method only provides necessary conditions.
There are known generalizations of these methods for more general functions.

Applications of ODEs with Painlevé property

- It is to be expected that such equations have physical applications: ODEs with singlevalued solutions

Examples:

- P3 appears in the two-point correlation function for the 2D Ising model [5]
- Elliptic functions are encountered in the study of mathematical pendulum
- The Boussinesq equation of fluid mechanics can be reduced, in the stationary case, either to P1 or to the elliptic equation, depending on the value of one of its first integrals [6].

A physicist's point of view on integrability

- Ususally, the physicists are not interested in finding new functions or establishing a classification of the equations they encounter, but rather to get some pieces of information about those equations.
- Performing Painlevé test to the end, although it fails at some point, and studying each condition separately can lead to finding some global pieces of information, like a first integral or a particular closed form solution.
- Sometimes, the notion of partial integrability is useful. For example, the Kolmogorov–Petrovskii–Piskunov (KPP) equation [4]:

$$E(u) \equiv bu_t - u_{xx} + \gamma u u_x + 2d^{-2}(u - e_1)(u - e_2)(u - e_3) = 0 \qquad (19)$$

fails the Painlevé test, but admits particular solutions which have no movable critical singularities. This equation is encountered in reaction-diffusion systems and prey-predator models.

Conclusions

- Painlevé analysis offers a powerful tool for obtaining necessary conditions for a differential equation to be integrable.
- Painlevé test does not offer a definitive criterion for integrability, but allows us to definitely decide that an equation is not integrable.
- Reducing differential equations to known, classified types allows for easily finding the special functions required to integrate them.

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