The Eilenberg-Steenrod axioms.
Basic theory of topological spaces, homeomorphisms and homotopy equivalence

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Definition

A **topology** on a set X is a collection τ of subsets of X having the following properties:

1. $\emptyset$ and X are in τ.
2. The union of the elements of any subcollection of τ is in τ
3. The intersection of the elements of any finite subcollection of τ is in τ.

(X, τ) is called a **topological space**.

Definition

A topological space is called **connected** if it cannot be written as a disjoint union of two nonempty open subsets.

Topological spaces

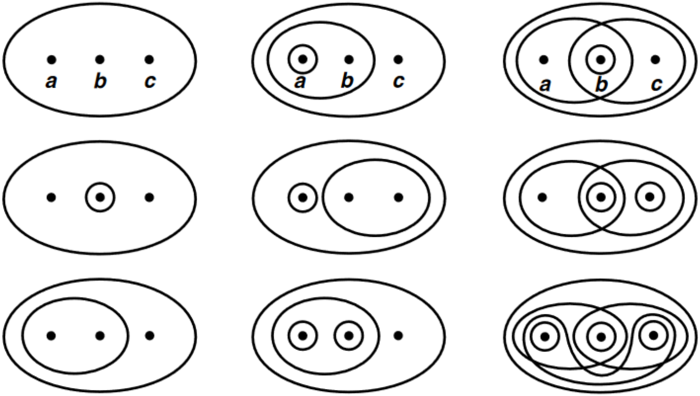


Figure 1: Topologies on the set $X = \{a, b, c\}$

Figure 2: Sets of subsets of X which are not topologies

Example

$\mathbb{N}$, $\tau = \{\mathbb{N}, \emptyset, U | U \text{ finite subset of } \mathbb{N}\}$; τ is not a topology on $\mathbb{N}$

$$\{2\} \cup \{3\} \ldots \cup \{n\} \cup \ldots = \{2, 3, \ldots, n, \ldots\} \notin \tau$$

Definition

Let X and Y be two topological spaces. $f : X \to Y$ is said to be a **continuous mapping** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition

Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then the composite mapping

$$g \circ f : X \to Z$$

is continuous.

Theorem

Let X, Y be two topological spaces and $f : X \to Y$. Then the following are equivalent:

1. f *is continuous*
2. *for any subset A of X, $f(\bar{A}) \subset \overline{f(A)}$*
3. *for any set $B \subset Y$ closed, $f^{-1}(B)$ is closed in X*
4. *for any $x \in X$ and any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.*

Theorem

(The pasting lemma.) *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous map $h : X \to Y$, defined by setting*

$$h(x) = \begin{cases} f(x), \text{ if } x \in A \\ g(x), \text{ if } x \in B. \end{cases}$$

Definition

A **path** in a topological space X from x_0 to x_1 is a continuous map $f : [0,1] \subset \mathbb{R} \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We call these two points x_0 and x_1 the initial and final point, respectively.

Definition

A topological space X is said to be **path-connected** if for any pair of distinct points $a, b \in X$, there exists a path $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$.

Proposition

Every path-connected topological space is connected.

Definition

Two topological spaces X and Y are **homeomorphic** if a mapping between them $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ exist, and they are continuous. In this case f is called a **homeomorphism**.

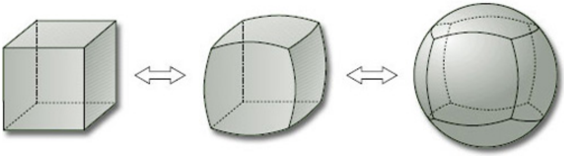


Figure 3: Example of homeomorphic spaces

Definition

Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. A **homotopy** from f to g is a continuous map

$$H : X \times [0,1] \to Y,\ (x,t) \mapsto H(x,t) = H_t(x)$$

such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for $x \in X$, i.e. $f = H_0$ and $g = H_1$.

Remark

The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$.
The equivalence class of f under the homotopy relation is denoted by $[f]$ and is called the **homotopy class** of f. We denote by $[X, Y]$ the set of homotopy classes $[f]$ of maps $f : X \to Y$.

Proposition

If $f, g : X \to Y$, $h : X' \to X$, $k : Y \to Y'$ are continuous maps, then $f \simeq g \Rightarrow f \circ h \simeq g \circ h$ and $k \circ f \simeq k \circ g$.

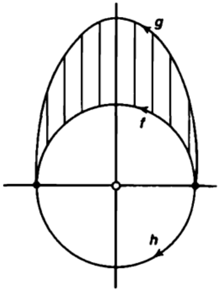
Definition

Two paths f and f' mapping I into X are said to be **path homotopic** if they have the same initial and final points, and if there exists a continuous map $F : I \times I \to X$ such that

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s)$$

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1$$

for any $s, t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.



Example

$$f(s) = (cos\pi s, sin\pi s)$$

$$g(s) = (cos\pi s, 2sin\pi s)$$

$$h(s) = (cos\pi s, -sin\pi s)$$

$$f \simeq_p g$$

$$f \not\simeq_p h$$

Definition

If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the **product** $f * g$ of f and g to be the path h given by

$$h(s) = \begin{cases} f(2s) \text{ for } s \in [0, \frac{1}{2}] \\ g(2s-1) \text{ for } s \in [\frac{1}{2}, 1]. \end{cases}$$

Remark

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by

$$[f] * [g] = [f * g].$$

This is associative and has the properties:
(Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f].$$

(Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Definition

Let X be a topological space and x_0 a point in X. A path in X that begins and ends at x_0 is called a **loop** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the **fundamental group (or first homotopy group)** of X relative to the **base point** x_0. It is denoted by $\pi_1(X, x_0)$.

Definition

Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \text{ by}$$

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

Remark

We have

$$\bar{\alpha} * (f * \alpha) = \begin{cases} \bar{\alpha}(2s) \text{ for } s \in [0, \frac{1}{2}] \\ f(4s-2) \text{ for } s \in [\frac{1}{2}, \frac{3}{4}] \\ \alpha(4s-3) \text{ for } s \in [\frac{3}{4}, 1] \end{cases}$$

Remark

Since α is a path from x_0 to x_1 and $\bar{\alpha}$ is a path from x_1 to x_0 we obtain that $\bar{\alpha} * (f * \alpha)$ is a loop based at x_1. This means that $\hat{\alpha}$ maps $\pi_1(X, x_0)$ into $\pi_1(X, x_1)$.

Theorem

The map $\hat{\alpha}$ is a group isomorphism.

Corollary

If X is path-connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Definition

A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one element) group for some $x_0 \in X$, and hence for every $x \in X$.

Lemma

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Remark

If $h : X \to Y$ is a continuous map that carries $x_0 \in X$ to $y_0 \in Y$, we write

$$h : (X, x_0) \to (Y, y_0).$$

If f is a loop in X based at x_0, then $h \circ f : [0,1] \to Y$ is a loop in Y based at y_0. The correspondence $f \to h \circ f$ gives rise to a map carrying $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$.

Definition

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. The **homomorphism induced by** h, relative to the base point x_0 is the map $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $(h_{x_0})_*([f]) = [h \circ f]$. We also denote $(h_{x_0})_*$ by h_* when no confusion may arise.

Theorem

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Corollary

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.