Characteristic classes
(Prerequisites)

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1 Prerequisites

Definition (Metric space)

Given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y \in X$:

1. $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$
2. $d(x, y) = d(y, x)$, and
3. for all $z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

A **metric space** is a set X together with a metric d.

Definition (ε-neighborhood)

Given $\varepsilon > 0$ and an element x in the metric space (X, d), the ε-neighborhood of x is the set

$$V_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

Definition (Homeomorphism)

Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a **homeomorphism**.

Theorem (Heine-Borel)

Let X be a subset of $\mathbb{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others:

1. *X is compact.*
2. *X is closed and bounded.*
3. *Every open cover for X has a finite subcover.*

Definition (Manifold)

A manifold is a metric space M with the following property: If $x \in M$, then there is some neighborhood U of x and some integer $n \geq 0$ such that U is homeomorphic to $\mathbb{R}^n$.

Examples of manifolds include $\mathbb{R}^n$, an open ball in $\mathbb{R}^n$, and any open subset V of $\mathbb{R}^n$.

Proposition

Anything homeomorphic to a manifold is also a manifold.

Proposition

An open subset of a manifold is also a manifold.

Theorem (Invariance of Domain)

If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is one-to-one (injective) and continuous, then $f(U) \subset \mathbb{R}^n$ is also open.

Proposition

If a continuous function f is injective and defined on an open subset of $\mathbb{R}^n$, it maps open sets in its domain to open sets in its codomain.

Corollary

From the Invariance of Domain theorem, it follows that U in the manifold definition must be open.

Proposition

A disjoint union of manifolds is a manifold.

Definition (Connected spaces)

A space X is said to be **connected** if there does not exist a separation of X, defined as a pair U, V of disjoint nonempty open subsets whose union is X.

Definition (Covering)

A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a **covering** of X if the union of the elements of $\mathcal{A}$ is equal to X.

Definition (Compact space)

A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Definition (σ-compact)

X is said to be σ-compact if there is a countable collection of compact subspaces of X whose interiors cover X.

Theorem

If X is a connected, locally compact metric space, then X is σ-compact.

Definition (Topological Immersion)

A function f that is continuous and **locally one-one**, i.e., every point in the domain has a neighborhood U on which f is one-one, is called a **topological immersion**.

- The quotient topology is not a natural generalization of something we have already studied in analysis.
- One motivation comes from geometry, where one often has the ocasion to use "cut-and-paste" tehniques to construct geometric objects as surfaces.

Definition (Quotient Topology)

If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology T on A relative to which p is a quotient map; it is called the **quotient topology** induced by p.

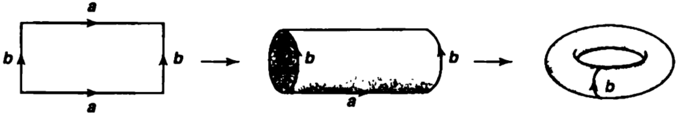

Figure 1: The torus can be constructed by taking a rectangle and "pasting" its edges together appropriately

Definition (Quotient Map)

Let X and Y be topological spaces; let $f : X \to Y$ be a surjective map.
The map p is said to be a **quotient map** provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

- This condition is stronger than continuity; also called "strong continuity" by some mathematicians.
- An equivalent condition is to require that a subset A of Y be closed in Y if and only if $p^{-1}(A)$ is closed in X.
- Equivalence of the two conditions follows from equation

$$f^{-1}(Y - B) = X - f^{-1}(B)$$

Definition (Quotient Space)

Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \rightarrow X^*$ be a surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a **quotient space** of X.

- Given X^*, there is an equivalence relation on X of which the elements of X^* are the equivalence classes.
- One can think of X^* as having been obtained by "identifying" each pair of equivalent points.
- For this reason , the quotient space X^* is ofted called an identification space, or a decomposition space, of the space X.