Symmetric Killing tensors on Riemannian 2-step nilpotent Lie groups

Viviana del Barco
Joint work with Andrei Moroianu

CONICET and UNR, Argentina

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Outline:

1. Symmetric Killing tensors on Riemannian manifolds.
2. Left-invariant Symmetric 2-tensors on 2-step nilpotent Lie groups.
   - PbA. Description of left-invariant symmetric Killing 2-tensors.
   - PbB. (In)decomposability of left-invariant symmetric Killing 2-tensors.
3. Results by dimension.

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A vector field $\xi \in \mathcal{X}(M)$ is a Killing vector field if and only if one of the following equivalent conditions hold:

1. $\mathcal{L}_\xi g = 0$.
2. $g(\nabla_X \xi, X) = 0$ for every $X \in \mathcal{X}(M)$.
3. $\nabla_X \xi = \frac{1}{2} X \lrcorner d\xi$ for every $X \in \mathcal{X}(M)$.

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  Ex: *$T \in \operatorname{Sym}^k TM$ s.t. $\nabla T = 0$.*
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Ex: Killing vector fields.

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$\rightsquigarrow$ The torsion form of the canonical connection in a naturally reductive space is a Killing 3-form.

Examples (symmetric Killing tensors). Cf: [C. Heil, A. Moroianu, U. Semmelmann '16,'17]

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$$p \in M \mapsto (\xi_1 \cdot \ldots \cdot \xi_k)_p(V_1, \dots, V_k) = \sum_{\sigma \in \mathfrak{S}_k} g(\xi_{1p}, V_{\sigma(1)}) \cdots g(\xi_{kp}, V_{\sigma(k)}).$$

For instance $(\xi_1 \cdot \xi_2)(V_1, V_2) = g(\xi_1, V_1)g(\xi_2, V_2) + g(\xi_1, V_2)g(\xi_2, V_1)$.

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$\rightsquigarrow$ Any linear combination of the above:

$$\mathcal{D}^k := \{T + \sum c_{i_1,\ldots,i_k}\xi_1 \cdot \ldots \cdot \xi_k : \nabla T = 0,\ \xi_i \text{ Killing v.f.}, c_{i_1,\ldots,i_k} \in \mathbb{R}\} \subseteq \mathcal{K}^k$$

These are called decomposable symmetric Killing tensors.

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Remarks:

Takeuchi, Thompson ('80): manifolds with constant sectional curvature do not posses indecomposable Killing tensors.

Heil, Moroianu, Semmelmann ('17): They exhibit Riemannian metrics in the torus T^n possessing indecomposable symmetric Killing tensors.

There are no general results, though.

Together with A. Moroianu we studied this question in the context of 2-step nilpotent Lie groups endowed with left-invariant Riemannian metric.

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We considered the question in the following context:

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- Also, if (N, g) is de Rham irreducible every parallel symmetric 2-tensors is necessarily left-invariant, and a multiple of the metric.
- Hence, in the irreducible case, we have $\mathcal{D}^2 = \{\lambda g + \sum c_{ij}\xi_i \cdot \xi_j : \xi_i \in \mathcal{K}^1\}$ which we compare with $\mathcal{K}^2_{inv}$ $\rightsquigarrow$ Pb.B

Pb.A. Description of $\mathcal{K}^2_{inv}$

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$(N, g) \rightsquigarrow (\mathfrak{n}, g)$ its Lie algebra with an inner product.

$$S \in \mathrm{Sym}^2_{inv}\, TN \quad \leftrightsquigarrow \quad S \in \mathrm{End}(\mathfrak{n}) : g(Sx, y) = g(x, Sy), \forall x, y \in \mathfrak{n}.$$

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Being $\mathfrak{n}$ 2-step nilpotent, the center of $\mathfrak{n}$, $\mathfrak{z} = \{z \in \mathfrak{n} : [u, z] = 0\, \forall u \in \mathfrak{n}\}$, verifies $0 \neq \mathfrak{z} \supseteq [\mathfrak{n}, \mathfrak{n}]$. Let $\mathfrak{v} = \mathfrak{z}^\perp$ so that $\mathfrak{n} = \mathfrak{v} \oplus^\perp \mathfrak{z}$ and we obtain $[\mathfrak{v}, \mathfrak{v}] = [\mathfrak{n}, \mathfrak{n}]$.

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$$\mathrm{Sym}^2 \mathfrak{n} = \mathrm{Sym}^2 \mathfrak{v} \oplus (\mathfrak{z} \cdot \mathfrak{v}) \oplus \mathrm{Sym}^2 \mathfrak{z}$$

and, accordingly, we write $S = S^{\mathfrak{v}} + S^{\mathrm{m}} + S^{\mathfrak{z}}$ for $S \in \mathrm{Sym}^2 \mathfrak{n}$.

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If $S = S^{\mathfrak{v}} + S^{\mathrm{m}} + S^{\mathfrak{z}}$, then $S \in \mathcal{K}^2_{inv} \Leftrightarrow S^{\mathfrak{v}}, S^{\mathrm{m}}, S^{\mathfrak{z}} \in \mathcal{K}^2_{inv}$.
Moreover, $S^{\mathfrak{z}} \in \mathcal{K}^2_{inv}$, $S^{\mathfrak{v}} \in \mathcal{K}^2_{inv} \Leftrightarrow [S^{\mathfrak{v}}x, x] = 0, \forall x \in \mathfrak{v}$ and
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Proof. $S \in \mathrm{Sym}^2(\mathfrak{n})$ is a Killing tensor if and only if $g((\nabla_y S)y, y) = 0$ for all $y \in \mathfrak{n}$. By Koszul, $g((\nabla_y S)y, y) = 2g([y, Sy], y)$.

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Write $y = x + z$ with $x \in \mathfrak{v}$ and $z \in \mathfrak{z}$, then $g([y, Sy], y) = 0$ for all $y \in \mathfrak{n}$ if and only if

$$g(z, [Sz, x]) + g(z, [Sx, x]) = 0, \text{ for all } x \in \mathfrak{v},\ z \in \mathfrak{z}.$$

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Hence

$$\mathcal{K}^2_{inv} = \{S \in \mathrm{Sym}^2 \mathfrak{v} : [Sx, x] = 0\} \oplus \{S \in \mathfrak{z} \cdot \mathfrak{v} : \mathrm{ad}_x \circ S \in \mathfrak{so}(\mathfrak{z})\} \oplus \mathrm{Sym}^2 \mathfrak{z}.$$

Pb.A.

Examples: (Elements in $\mathcal{K}^2_{inv}$)

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$$\mathcal{K}^2_{inv} = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & a_2 \\ 0 & a_1 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & a_4 & a_5 \\ 0 & -a_2 & 0 & a_4 & a_7 & a_6 \\ a_2 & 0 & 0 & a_5 & a_6 & a_8 \end{pmatrix} : \quad a_i \in \mathbb{R} \right\}$$

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$\rightsquigarrow$ When N admits a co-compact discrete subgroup Γ, every element in $\mathcal{K}^2_{inv}$ defines a first integral in $(\Gamma \backslash N, g)$.

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Where :

$$\begin{aligned} \mathcal{D}^2 &= \{\lambda g + \sum c_{ij}\xi_i \cdot \xi_j : c_{ij} \in \mathbb{R},\ \xi_i \in \mathcal{K}^1\} \\ \mathcal{K}^2_{inv} &= \{S \in \mathrm{Sym}^2\,\mathfrak{v} :\ [Sx, x] = 0\} \oplus \{S \in \mathfrak{z}\cdot\mathfrak{v} :\ \mathrm{ad}_x \circ S \in \mathfrak{so}(\mathfrak{z})\} \oplus \mathrm{Sym}^2\,\mathfrak{z}. \end{aligned}$$

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$\rightsquigarrow$ When does a symmetric Killing tensor in $\operatorname{Sym}^2 \mathfrak{v}$ is decomposable?

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Define $j : \mathfrak{z} \longrightarrow \mathfrak{so}(\mathfrak{v})$ satisfying

$$g(j(z)x, y) = g(z, [x, y]),\ z \in \mathfrak{z},\ x, y \in \mathfrak{v}.$$

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S is decomposable if and only if there exists some $a \in \mathbb{R}$ such that for every $z \in \mathfrak{z}$, $T_z^{\boldsymbol{\lambda}-a}$ extends to a skew-symmetric derivation of $\mathfrak{n}$.

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In this case, S is a l.c. of g, $\xi_{e_i} \cdot \xi_{e_i}$, $\xi_{z_j} \cdot \xi_{z_j}$ and $\xi_{D_s} \cdot \xi_{z_j}$ where $\{e_i\}_i$, $\{z_j\}_j$ are orthonormal basis of $\mathfrak{v}$ and $\mathfrak{z}$, resp. and $\{D_s\}_s$ a basis of $\mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n})$.

Outline:

1. Symmetric Killing tensors on Riemannian manifolds.
2. Left-invariant Symmetric 2-tensors on 2-step nilpotent Lie groups.
   - PbA. Description of left-invariant symmetric Killing 2-tensors.
   - PbB. (In)decomposability of left-invariant symmetric Killing 2-tensors.
3. Results by dimension.

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THEOREM

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Every left-invariant symmetric Killing tensor defined on a simply connected 2-step nilpotent Lie group (N, g) of dimension ≤ 7 is decomposable.

Pb.B. $S \in \mathrm{Sym}^2 \mathfrak{v}$ decomposable iff $\exists a \in \mathbb{R}$ s.t. $\forall z \in \mathfrak{z}$, $T_z^{\boldsymbol{\lambda}-a}$ extends...

S, λ_i, $\mathfrak{v}_i$ and $j : \mathfrak{z} \longrightarrow \mathfrak{so}(\mathfrak{v})$ as before.

Given $z \in \mathfrak{z}$, $T_z^{\boldsymbol{\lambda}-a} \in \mathfrak{so}(\mathfrak{v})$ extends to a skew-symmetric derivation of $\mathfrak{n}$ if and only if $\exists B_z \in \mathfrak{so}(\mathfrak{z})$ such that for each $i = 1, \ldots, k$ one has

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Proof. If $\dim \mathfrak{z} = 1$ then the right side of $(*)$ is zero, for any a $\rightsquigarrow$ choose $B = 0$, for any tensor S.

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Let $S \in \mathrm{Sym}^2 \mathfrak{v}$ be a Killing tensor and assume there are $i \neq j$ such that $j(\mathfrak{z})|_{\mathfrak{v}_i}$ and $j(\mathfrak{z})|_{\mathfrak{v}_j}$ are not Lie subalgebras of $\mathfrak{so}(\mathfrak{v}_i)$ and $\mathfrak{so}(\mathfrak{v}_j)$, respectively. Then S is indecomposable.

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$\rightsquigarrow$ This can be obtain in any dimension ≥ 8.

Example (Lie algebra with indecomposable Killing tensors).

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Let $\mathfrak{n}$ with basis $e_1, \dots, e_6, z_1, z_2$ with non-zero bracket relations

$$[e_1, e_2] = z_1 = [e_4, e_5], \qquad [e_2, e_3] = z_2 = [e_5, e_6].$$

Let $S \in \mathrm{Sym}^2 \mathfrak{v}$ defined as

$$S = \mathrm{Id}_{\mathfrak{v}_1} + 2\mathrm{Id}_{\mathfrak{v}_2},$$

with $\mathfrak{v}_1 = \mathrm{span}\{e_1, e_2, e_3\}$ and $\mathfrak{v}_2 = \mathrm{span}\{e_4, e_5, e_6\}$. One has $S \in \mathcal{K}^2_{inv}$ because $[Sx, x] = 0$ for all $x \in \mathfrak{v}$.

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Also $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ and for $i = 1, 2$

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$\rightsquigarrow$ If N admits a co-compact discrete subgroup Γ, any left-invariant Killing tensor on (N, g) is induced to the compact manifold $(\Gamma \backslash N, g)$.

Thank you!

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