Lattice super-KdV equation:integrability, reductions and singularities

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Bucharest Conference on Geometry and Physics

IMAR, Sept, 2019

Outline

Basics of the Lax integrabiliy

What is integrability in classical nonlinear ODE/PDEs or OΔE/PΔEs? Existence of a huge quantity of internal symmetry which gives enough invariants to compute everything. Usually is related to the hamiltonian structure, existence of invariants

However there are the following main approaches to integrability:

- Lax integrability: the nonlinear equation itself is related to an isospectral deformation of a *linear* operator. All the information of the invariants is included in the spectrum.
- Bilinear (Hirota) integrability: related to the behaviour of localised solutions (solitons). Qualitatively, one can think that counterbalance of dispersion and nonlinearity can support a kind of stability. Existence of multi-soliton solution with arbitrary parameters guarantees the integrability.
- Singularity structure: Kowalwskaya had the first idea that a "nice" dynamics imposes the solutions of the equations to have singularities in the complex plane at most poles. This requirements eventually became the famous Painleve property. Not very clear at the continuous level. At the discrete level is more clear.
- multi-hamiltonian structure: Existence of many simplectic operators with many hamiltonians for the same equation is a strong indicator for integrability since these simplectic operators can be used to generate the invariants recursively

We will discuss the extension of Lax integrability and singularity structure for a model consisting of a nonlinearly coupled classical bosonic and fermionic fields on a 2D lattice.

General formulation of zero curvature representation

Main idea is to write the nonlinear PDE as a compatibility of two linear operators containing the dependent variable and some parameter.

Compatibility = key of Lax integrability

Consider two matrices $U(x,t|\lambda), V(x,t|\lambda)$ containing the dependent variable:

$$\partial_x \psi(x,t) = U(x,t|\lambda)\psi(x,t)$$
$$\partial_t \psi(x,t) = V(x,t|\lambda)\psi(x,t)$$

From the compatibility $\partial^2_{x,t}\psi(x,t) = \partial^2_{t,x}\psi(x,t)$ one gets:

$$\partial_t U - \partial_x V + [U,V] = 0$$

which is called zero-curvature representation having a differential geometric meaning for "connection" coefficients $U(x,t|\lambda), V(x,t|\lambda)$
Example:

$$U(x,t|\lambda) = \frac{i}{2}\begin{pmatrix} u_x & -\lambda \\ -\lambda & -u_x \end{pmatrix}$$

$$V(x,t|\lambda) = \frac{i}{2}\begin{pmatrix} 0 & e^{iu}/\lambda \\ e^{-iu}/\lambda & 0 \end{pmatrix}$$

Compatibility gives the celebrated sine-Gordon equation

$$\partial_t U - \partial_x V + [U,V] \equiv \partial_x\partial_t u(x,t) = \sin u$$

Advantages:

- computation of multisoliton solution and nonlinear dispersive-wave asymptotics using Riemann-Hilbert approach or inverse scattering theory.
- a particular version of the zero-curvature representation is the Lax version

$$\partial_x \psi = U(x|\lambda)\psi \Longrightarrow L\psi(x,t) = \lambda\psi(x,t)$$

- it allows computation of invariants $I_n = \mathrm{trace}(L^n)$
- in the zero-curvature reprezentation finding the invariants is more complicatedand it is based that zero-curvature gives a trivial parallel transport on a loop. It involves the monodromy matrix

$$\psi(l,t) = \tau(l,0|\lambda)\psi(0,t)$$

  and assuming periodic boundary condition on the interval $x \in (0,l)$:

$$\lim_{\lambda \to i\infty} \left(e^{i\lambda l/2} \mathrm{tr}\tau(l,0\lambda)\right) = \sum_{n \geq 0} \lambda^{-n} I_n$$

  where I_n are the conservation laws (Fadeev-Takhtajan 1985)
- multihamiltonian structure another key of integrability using the recursion operator.
- zero-curvature representation allows quantisation by means of the monodromy matrix and R-matrix using the Yang-Baxter equation: it gives the eigenvalues for **all** I_n

Zero-curvature representation in the discrete world:

The equations we are dealing with are the quad-graph lattice equations having the form:

$$Q(u_{n,m}, u_{n+1,m}, u_{n,m+\delta}, u_{n+1,m+\delta}|p_1, p_2) = 0$$

They are essentialy nonlinear (birational) recursion relations of defined on the corners of a square and one vertex is given by the nonlinear equation containg the other three. Notation:

$$T_1 u_{n,m} = u_{n+1,m} \equiv \bar{u}, \quad T_2 u_{n,m} = u_{n,m+\delta} \equiv \tilde{u}$$

$$T_{-1} u_{n-1,m} \equiv \underline{u}, \quad T_1 T_2 u_{n+1,m+\delta} = \tilde{\bar{u}}$$

Examples:

- ($H1$ =discrete KdV equation)

$$(\bar{u} - \tilde{u})(u - \tilde{\bar{u}}) = p_1 - p_2 \Longleftrightarrow \tilde{\bar{u}} = u - \frac{p_1 - p_2}{\bar{u} - \tilde{u}}$$

- (Hirota equation or discrete generalized mKdV/sine-Gordon equation)

$$u(p_1\bar{u} - p_2\tilde{u}) - \tilde{\bar{u}}(p_2\bar{u} - p_1\tilde{u}) = 0$$

- p_1, p_2 are parameters related to the axis n, m (discretization steps). They are fixed.

- We are interested in the integrability of such nonlinear lattice equations

Formulation of zero-curvature representation

$$\psi(n+1, m) = U(n, m|\lambda)\psi(n, m)$$
$$\psi(n, m+\delta) = V(n, m|\lambda)\psi(n, m)$$

General compatibility of these discrete equations:

$$\bar{\tilde{\psi}} = \tilde{\bar{\psi}} \Longleftrightarrow \tilde{U}V - \overline{V}U = 0$$

Example

$$U(n, m|\lambda) = \begin{pmatrix} -\bar{u} & 1 \\ -\lambda - u\bar{u} + p_1 & u \end{pmatrix}$$
$$V(n, m|\lambda) = \begin{pmatrix} -\tilde{u} & 1 \\ -\lambda - u\tilde{u} + p_2 & u \end{pmatrix}$$

Compatibility gives the H1 (discrete KdV equation)

$$\tilde{U}V - \overline{V}U = 0 \Longrightarrow \tilde{\bar{u}} = u - \frac{p_1 - p_2}{\bar{u} - \tilde{u}}$$

Adler, Bobenko and Suris found a crucial property that gives the possibility of classifying integrable quad-graph equations together with their zero-curvature reprezentation

Discrete zero-curvature reprezentation vs continous one

- allows construction of multisoliton solution using inverse scattering of discrete operatos - much harder procedure. Only H1 in fact has been worked out
- computing invariants only recently the method was given (D. J. Zhang, 2006)
- hamiltonian formalism unclear; multi-hamiltonian structure as well
- Lie point symmetries only recently
- quantisation; although it goes on the same procedure as in the continous case it is much harder because of the unclear hamiltonian formalism. Only Schwartzian-H1 equation has been done. In the case of differential-discrete setting (continuos time, discrete space) everything works simple and even better than continuos because there are no ultraviolet divergences.

Consistency around cube

The idea is to extend the quad-graph equation in a 3D space on any square face of a cube - it corresponds to a hierarchy of commuting flows

So one adjoint a third direction $u_{n,m} \to u_{n,m,k}$ and construct a cube in (n, m, k). We are using the same equation in all the planes and consider 3 parameters q, p, r. So in the case of H1 equation we have:

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) + q - p = 0$$

$$(\bar{u} - \bar{\hat{\tilde{u}}})(\bar{\tilde{u}} - \bar{\hat{u}}) + q - p = 0$$

$$(u - \hat{\bar{u}})(\bar{u} - \hat{u}) + q - r = 0$$

$$(\tilde{u} - \tilde{\hat{\bar{u}}})(\tilde{\bar{u}} - \tilde{\hat{u}}) + q - r = 0$$

$$(u - \bar{\tilde{u}})(\tilde{u} - \bar{u}) + r - p = 0$$

$$(\tilde{u} - \hat{\bar{\tilde{u}}})(\hat{\tilde{u}} - \hat{\bar{u}}) + r - p = 0$$

Integrability = **all red u's must be equal**. And indeed they are equal to

$$\bar{\hat{\tilde{u}}} = \tilde{\hat{\bar{u}}} = \hat{\bar{\tilde{u}}} = \frac{\tilde{u}\hat{u}(p - q) + \bar{u}\hat{u}(q - r) + \tilde{u}\bar{u}(r - p)}{\tilde{u}(r - q) + \hat{u}(p - r) + \bar{u}(q - p)}$$

- Moreover the red u's are not depending on u but only on shifted u's = *tetrahedron property*
- Why integrability is related to this cubic consistency? Because immediately gives zero-curvature representation and Backlund transformations

Effective construction of Lax pairs: the main idea is taking the third direction to be the spectral parameter

Write again the equations on the faces of the cube:

$$(u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) + q - p = 0$$

$$(u - \hat{\bar{u}})(\bar{u} - \hat{u}) + q - r = 0$$

$$(u - \bar{\tilde{u}})(\tilde{u} - \bar{u}) + r - p = 0$$

Lets call $\bar{u} = W$ and $r = \lambda$ spectral parameter. We get:

$$\tilde{W} = \frac{uW + (\lambda - p - u\tilde{u})}{W - \tilde{u}} \quad \hat{W} = \frac{uW + (\lambda - p - u\hat{u})}{W - \hat{w}}$$

Now crucial substitution $W = F/G$ and define de column vector $\psi = (F, G)^T$. Identifying numetrators and denominators we get immediately the zero curvature representation

$$\tilde{\psi} = U\psi \Longleftrightarrow \begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \gamma \begin{pmatrix} u & \lambda - p - u\tilde{u} \\ 1 & -\tilde{u} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\hat{\psi} = V\psi \Longleftrightarrow \begin{pmatrix} \hat{F} \\ \hat{G} \end{pmatrix} = \gamma' \begin{pmatrix} u & \lambda - p - u\hat{u} \\ 1 & -\hat{u} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

Compatibility condition gives:

$$\hat{\gamma}\gamma' \det(\hat{U}) \det(V) = \tilde{\gamma}' \gamma \det(\tilde{V}) \det(U)$$

But $\det(U) = p - \lambda \quad \det(V) = q - \lambda$ so we can take $\gamma = \gamma' = 1$

Adding fermions

The basic idea is to consider coupled partial difference (lattice) equations containing two dependent variables (fields) $u(n,m) \equiv u_{n,m}$ and $\psi(n,m) \equiv \psi_{n,m}$ such that $u : \mathbb{Z}^2 \to \Lambda_0, \psi : \mathbb{Z}^2 \to \Lambda_1$ where Λ_0 and Λ_1 are the even (bosonic) and odd (fermionic) sectors of an infinite dimensional Grassmann algebra $\Lambda = \Lambda_0 \oplus \Lambda_1$; it is a graduate modulo 2 algebra i.e. $\Lambda_0\Lambda_0 \subset \Lambda_0, \Lambda_1\Lambda_1 \subset \Lambda_0, \Lambda_0\Lambda_1 \subset \Lambda_1$. Also when $\psi_{n,m} = 0$ then we recover the usual lattice (in our case H1) equation in the variable $u_{n,m}$.
Let us consider the following coupled system:

$$\psi_{n+1,m+1} - \psi_{n,m} = \frac{2(p_1+p_2)(\psi_{n+1,m} - \psi_{n,m+1})}{2(p_2-p_1) + u_{n+1,m} - u_{n,m+1}} \tag{1}$$

$$\begin{aligned} u_{n+1,m+1} - u_{n,m} &= \frac{2(p_1+p_2)(u_{n+1,m} - u_{n,m+1})}{2(p_2-p_1) + u_{n+1,m} - u_{n,m+1}} \\ &- \frac{(p_1+p_2)(4(p_2-p_1) + u_{n+1,m} - u_{n,m+1})}{(2(p_2-p_1) + u_{n+1,m} - u_{n,m+1})^2}(\psi_{n+1,m} - \psi_{n,m+1})(\psi_{n,m+1} - \psi_{n,m}), \end{aligned} \tag{2}$$

The above system was obtained by Xue, Levi and Liu in 2013 (JPA, 2013) and is the first integrable discretization of the supersymmetric integrable KdV equation. In the continuum limit

$$x \to \epsilon(n+2m), \quad t \to m\epsilon^3/3, \quad u_{n,m} = \epsilon u(x,t), \quad \psi_{n,m} = \epsilon^{1/2}\xi(x,t)$$

it goes in the order ϵ^4 to the "susy" KdV (Manin, 1985) written in the superspace formalism

$$N = 1 super - KdV : \quad \begin{cases} u_{n,m}, \psi_{n,m} \to \Phi(x,t,\theta) = \xi(x,t) + \theta u(x,t), D = \partial_\theta + \theta\partial_x, \\ \Phi_t + \Phi_{xxx} + 3(\Phi D\Phi)_x = 0 \end{cases}$$

Remarks

When the fermions go to zero ($\psi_{n,m} = 0$) we obtain the following lattice equation

$$u_{n+1,m+1} - u_{n,m} = \frac{2(p_1 + p_2)(u_{n+1,m} - u_{n,m+1})}{2(p_2 - p_1) + u_{n+1,m} - u_{n,m+1}}$$

which is equivalent with the H1 equation via the transformation

$$u_{n,m} \to u_{n,m} + a(p_1, p_2)n + b(p_1, p_2)m$$

- $u_{n,m}$ and $\psi_{n,m}$ are functions with values in the commuting (bosonic) and anti-commuting (fermionic) sector of an infinite dimensional Grassmann algebra, i.e. $u_{n,m}u_{n',m'} = u_{n',m'}u_{n,m}$, $\psi_{n,m}\psi_{n',m'} = -\psi_{n',m'}\psi_{n,m}$, $u_{n,m}\psi_{n,m} = \psi_{n,m}u_{n,m}$; also p_1, p_2 are parameters of the lattice which for simplicity we consider to be ordinary complex numbers (not even grassmann).

QUESTIONS

- Is there any super-zero curvature representation for these type of equations?
- Is there any kind of consistency around the cube?
- How is the multisoliton solution?
- How arethe invariants and singularities?

Multisoliton solution is constructed using Hirota bilinear formalism adapted to the grassmann algebra and the main idea is to start from the Hirota form of the H1-type equation and add fermions; then impose the existence of at least 3-soliton solution $\Longrightarrow$ INTEGRABILITY.

- bilinear H1 ($u_{n,m} = g_{n,m}/f_{n,m}$ with this sublstitution the H1-type equation is split into the system for g and f)

$$\tilde{g}f - g\tilde{f} = h(\bar{\tilde{g}}\underline{f} - \bar{\tilde{f}}\underline{g})$$

$$(\bar{\tilde{g}}\underline{f} - \bar{\tilde{f}}\underline{g}) = (\bar{\tilde{f}}\underline{f} - \tilde{f}f)$$

- add fermions (and after that we impose $\psi_{n,m} = \gamma_{n,m}/f_{n,m}$) and we *add* some anticommuting terms (expressed by functions $\gamma(n,m) \in \Lambda_1$) keeping the Hirota-gauge invariance (conservation of number of "tildes" and "bars"),

$$\tilde{\gamma}f - \gamma\tilde{f} = (\bar{\tilde{\gamma}}\underline{f} - \underline{\gamma}\bar{\tilde{f}})$$

$$\tilde{g}f - g\tilde{f} - h(\bar{\tilde{g}}\underline{f} - \bar{\tilde{f}}\underline{g}) - m_1\gamma\tilde{\gamma} + hm_2\bar{\tilde{\gamma}}\underline{\gamma} = 0$$

$$h(\bar{\tilde{g}}\underline{f} - \bar{\tilde{f}}\underline{g}) - h(\bar{\tilde{f}}\underline{f} - \tilde{f}f) + m_3\gamma\tilde{\gamma} + hm_4\bar{\tilde{\gamma}}\underline{\gamma} = 0$$

  where $m_i, i = 1, \ldots, 4$ are arbitrary constants.

- impose the existence of 3-soliton solution

Take the case $m_1 = m_2, m_3 = m_2 + m_4, m_4 = 1, m_2 = -1$.

$$\tilde{\psi} - \psi = h \frac{\tilde{\underset{\sim}{\psi}} - \underline{\psi}}{1 - \tilde{\underset{\sim}{u}} - \underline{u}}$$

$$\tilde{u} - u = h \frac{\tilde{\underset{\sim}{u}} - \underline{u}}{1 - \tilde{\underset{\sim}{u}} - \underline{u}} + h \frac{2 - \tilde{\underset{\sim}{u}} + \underline{u}}{(1 - \tilde{\underset{\sim}{u}} + \underline{u})^2} (\tilde{\underset{\sim}{\psi}} - \underline{\psi})(\psi - \underline{\psi})$$

The N-super-soliton solution (ASC, JPA 2015):

$$\gamma = \sum_{\mu_i} \left(\sum_{i=1}^{N} \mu_i \zeta_i \prod_{m \neq i} \alpha_{im} \right) \exp \left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} \mu_i \mu_j \left(\ln a_{ij} + \beta_{ij} \zeta_i \zeta_j \prod_{k \neq i,j} \alpha_{ik} \alpha_{jk} \right) \right)$$

$$g = \sum_{\mu_i} \left(\sum_{i=1}^{N} b_i \mu_i \right) \exp \left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} \mu_i \mu_j \left(\ln a_{ij} + \beta_{ij} \zeta_i \zeta_j \prod_{k \neq i,j} \alpha_{ik} \alpha_{jk} \right) \right)$$

$$f = \sum_{\mu_i} \exp \left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} \mu_i \mu_j \left(\ln a_{ij} + 2\beta_{ij} \zeta_i \zeta_j \prod_{k \neq i,j} \alpha_{ik} \alpha_{jk} \right) \right)$$

where

$$a_{ij} = \left(\frac{e^{k_i} - e^{k_j}}{e^{k_i + k_j} - 1} \right)^2, \alpha_{ij} = \left(\frac{e^{k_i + k_j} - 1}{e^{k_i} - e^{k_j}} \right), \beta_{ij} = \frac{\alpha_{ij}}{b_i + b_j}$$

$$\eta_i = k_i n + \omega_i h t, b_i = (1 + h)(e^{k_i} - 1)/(e^{k_i} + 1), e^{h\omega_i} = (h - e^{k_1})/e^{k_1}(h e^{k_1} - 1)$$

Zero-curvature representation

The associated spectral problems:

$$\Phi_{n,m+1} = B_{n,m}\Phi_{n,m}, \quad \Phi_{n+1,m} = A_{n,m}\Phi_{n,m}, \tag{3}$$

where

$$A_{n,m} = \begin{pmatrix} p_1 & 1 & -\eta & 0 \\ \lambda^2 & p_1 & -2p_1\eta + \eta g & -\eta \\ 0 & \eta & p_1 - g & 1 \\ \lambda^2\eta & 2p_1\eta - \eta g & \lambda^2 - 2p_1 g + g^2 & p_1 - g \end{pmatrix}.$$

$$B_{n,m} = \begin{pmatrix} p_2 & 1 & -\sigma & 0 \\ \lambda^2 & p_2 & -2p_2\sigma + \sigma f & -\sigma \\ 0 & \sigma & p_2 - f & 1 \\ \lambda^2\sigma & 2p_2\sigma - \sigma f & \lambda^2 - 2p_2 f + f^2 & p_2 - f \end{pmatrix}$$

with

$$\eta = (\psi_{n+1,m} - \psi_{n,m})/2, \quad g = (u_{n+1,m} - u_{n,m})/2, \sigma = (\psi_{n,m+1} - \psi_{n,m})/2, \quad f = (u_{n,m+1} - u_{n,m})/2$$

and λ is the spectral parameter which is a commuting invertible number (grassmann commuting number with nonzero body)

Remarks

- the nonlinear super-H1 system is a pair of quad-graph equations. Indeed in the corners of the square we have $(u,\psi),(\bar{u},\bar{\psi}),(\tilde{u},\tilde{\psi}),(\tilde{\bar{u}},\tilde{\bar{\psi}})$
- if we impose the third direction one can prove that the system is consistent around the cube
- the matrices associated with the zero-curvature representation are *grassmann even matrices* the computation of them through the third direction imposes the constants of super-Lax matrices. Taking the berezinians (super-determinants) the conditions become extremely complicated. Computation of Lax super-matrices is for the moment an open problem.
- there is another completely integrable lattice version of super-H1 equation (ASC, JPA 2015)

$$\tilde{\psi}-\psi=h\frac{\tilde{\bar{\psi}}-\underline{\psi}}{1-\tilde{\bar{u}}-\underline{u}}$$

$$\tilde{u}-u=h\frac{\tilde{\bar{u}}-\underline{u}}{1-\tilde{\bar{u}}-\underline{u}}+h\frac{(1-\tilde{\bar{u}}+\underline{u})\psi(\tilde{\bar{\psi}}-\underline{\psi})+\tilde{\bar{\psi}}\underline{\psi}}{(1-\tilde{\bar{u}}+\underline{u})^2}$$

but we don not know the Lax pair. The multi-soliton solution has a more complicated interaction

Consider the so called (p,q) reduction or "travelling wave reduction" i.e. the system becomes a coupled system of nonlinear ordinary discrete equations with bosonic and fermionic fields So we take for simplicity $(p,q)=(-1,2)$, namely $\nu=2m-n$. So in this case $u_{n+1,m}\to u_{\nu-1}, u_{n+1,m+1}\to u_{\nu+1}$, etc. Our system will turn into:

$$\psi_{\nu+1}-\psi_\nu = \frac{2(p_1+p_2)(\psi_{\nu-1}-\psi_{\nu+2})}{2(p_2-p_1)+u_{\nu-1}-u_{\nu+2}} \tag{4}$$

$$\begin{aligned} u_{\nu+1}-u_\nu &= \frac{2(p_1+p_2)(u_{\nu-1}-u_{\nu+2})}{2(p_2-p_1)+u_{\nu-1}-u_{\nu+2}} \\ &- \frac{(p_1+p_2)(4(p_2-p_1)+u_{\nu-1}-u_{\nu+2})}{(2(p_2-p_1)+u_{\nu-1}-u_{\nu+2})^2}(\psi_{\nu-1}-\psi_{\nu+2})(\psi_{\nu+2}-\psi_\nu) \end{aligned} \tag{5}$$

This system is of order six. We can reduce the order by "integrating" once each equation in the system. Defining $x_\nu=u_{\nu+1}-u_\nu$ and $\zeta_\nu=\psi_{\nu+1}-\psi_\nu$, we obtain the 4D system:

$$\zeta_\nu = \frac{2(p_1+p_2)(-\zeta_{\nu-1}-\zeta_\nu-\zeta_{\nu+1})}{2(p_2-p_1)-x_{\nu-1}-x_\nu-x_{\nu+1}} \tag{6}$$

$$\begin{aligned} x_\nu &= \frac{2(p_1+p_2)(-x_{\nu-1}-x_\nu-x_{\nu+1})}{2(p_2-p_1)-x_{\nu-1}-x_\nu-x_{\nu+1}} \\ &+ (p_1+p_2)(\zeta_{\nu-1}+\zeta_\nu+\zeta_{\nu+1})(\zeta_{\nu+1}+\zeta_\nu)\frac{4(p_2-p_1)-x_{\nu-1}-x_\nu-x_{\nu+1}}{(2(p_2-p_1)-x_{\nu-1}-x_\nu-x_{\nu+1})^2}, \end{aligned} \tag{7}$$

where x_ν is a bosonic variable and ζ_ν is a fermionic one.

Using the formula $\sqrt{a+b\zeta_1\zeta_2} = \sqrt{a}(1+\frac{1}{2}\zeta_1\zeta_2 b)$ and solving for $(\bar{x}+x+\underline{x})$ we find two solutions but the first one turns out to give a linear trivial system. However we can further simplify it; multiplying ζ_ν to the first equation from the left, we obtain $\zeta_\nu\zeta_{\nu+1}+\zeta_\nu\zeta_{\nu-1}=0$, and hence

$$\zeta_\nu\zeta_{\nu+1} = \zeta_{\nu-1}\zeta_\nu. \tag{8}$$

Also, using

$$\begin{aligned}\zeta_{\nu-1}(\zeta_\nu+\zeta_{\nu+1}) =& (\zeta_{\nu-1}+\zeta_\nu+\zeta_{\nu+1})(\zeta_\nu+\zeta_{\nu+1}) \\ =& \left(\frac{x_{\nu-1}+x_\nu+x_{\nu+1}}{2(p_1+p_2)} - \frac{p_2-p_1}{p_2+p_1}\right)\zeta_\nu\zeta_{\nu+1}\end{aligned}$$

and replacing $x_{\nu-1}+x_\nu+x_{\nu+1}$ becomes

$$x_{\nu-1}+x_\nu+x_{\nu+1} = \frac{2x_\nu(p_1-p_2)}{2(p_2+p_1)-x_\nu} + (p_1-p_2)\frac{4p_1+4p_2-x_\nu}{(2p_1+2p_2-x_\nu)^2}\zeta_\nu\zeta_{\nu+1}. \tag{9}$$

The equation implies $\gamma_\nu = \zeta_\nu\zeta_{\nu+1}$ to be a constant of motion($\gamma_{\nu+1}=\gamma_\nu, \forall\nu$), which is the *first conservation law* and we call it γ Hence, our system can be reduced to a single second order mapping depending on γ:

$$x_{\nu-1}+x_\nu+x_{\nu+1} = \frac{2x_\nu(p_1-p_2)}{2(p_2+p_1)-x_\nu} + (p_1-p_2)\frac{4p_1+4p_2-x_\nu}{(2p_1+2p_2-x_\nu)^2}\gamma \tag{10}$$

Rescaling we find the simpler form (h is an ordinary complex number):

$$x_{\nu+1}+x_\nu+x_{\nu-1} = \frac{hx_\nu}{1-x_\nu} + \frac{2-x_\nu}{(1-x_\nu)^2}\gamma \tag{11}$$

We assume that the Grassmann algebra is generated by two generators $\{\xi_1, \xi_2\}$. In this base, $x_\nu = x_\nu^{(0)} + x_\nu^{(3)}\xi_1\xi_2$ (with ordinary complex functions $x_\nu^{(i)}$'s) and, for simplicity, we consider $\gamma = \xi_1\xi_2$, h an ordinary complex parameter. Then we have

$$(x_{\nu+1}^{(0)} + x_\nu^{(0)} + x_{\nu-1}^{(0)}) + (x_{\nu+1}^{(3)} + x_\nu^{(3)} + x_{\nu-1}^{(3)})\gamma = \frac{hx_\nu^{(0)}}{1-x_\nu^{(0)}} + \frac{2 - x_\nu^{(0)} + hx_\nu^{(3)}}{(1-x_\nu^{(0)})^2}\gamma.$$

Setting as $x_0 = x_{\nu-1}^{(0)}$, $x_1 = x_{\nu-1}^{(3)}$, $x_2 = x_\nu^{(0)}$, $x_3 = x_\nu^{(3)}$ and $\bar{x}_0 = x_\nu^{(0)}$, $\bar{x}_1 = x_\nu^{(3)}$, $\bar{x}_2 = x_{\nu+1}^{(0)}$, $\bar{x}_3 = x_{\nu+1}^{(3)}$, this equation becomes a four dimensional system as

$$\begin{aligned}
\overline{x_0} &= x_2 \\
\overline{x_1} &= x_3 \\
\overline{x_2} &= -x_2 - x_0 + \frac{hx_2}{1-x_2} \\
\overline{x_3} &= -x_1 - x_3 + \frac{2 - x_2 + hx_3}{(1-x_2)^2}
\end{aligned}$$

Let us consider system on the projective space $\mathbb{P}^2 \times \mathbb{P}^2$. In the following, we aim to obtain a four-dimensional rational variety by blowing-up procedure such that the birational map is lifted to a *algebraically stable* map on the variety. A birational mapping φ from an N-dimensional rational variety X to itself is said to be algebraically stable if $(\varphi^*)^n(\mathcal{D}) = (\varphi^n)^*(\mathcal{D})$ holds for any divisor class $\mathcal{D}$ on X and an arbitrary positive integer n. Moreover, let $I(\varphi)$ denote the indeterminacy set of φ. It is known that the mapping φ is algebraically stable if and only if there does not exist a positive integer k and a divisor D on $\mathcal{X}$ such that $\varphi(D \setminus I(\varphi)) \subset I(\varphi^k)$, i.e. the image of the generic part of a divisor by φ^n is included in the indeterminate set.

Thye mapping can be solved by means of 17 blow-ups given by the following expressions

$$\begin{aligned}
C_1 :& (x_0, x_1, z_2, z_3) = (1, P, 0, 0) \\
&\leftarrow (s_1, t_1, u_1, v_1) := (x_0 - 1, x_1, z_2(x_0 - 1)^{-1}, z_3(x_0 - 1)^{-1}), \\
C_2 :& (s_1, t_1, u_1, v_1) = (0, P, Q, 0) \\
&\leftarrow (s_2, t_2, u_2, v_2) := (s_1, t_1, u_1, v_1 s_1^{-1}), \\
C_3 :& (s_2, t_2, u_2, v_2) = (0, P, -h(1 + hP)^{-1}, Q) \\
&\leftarrow (s_3, t_3, u_3, v_3) := (s_2, t_2, (u_2 + h(1 + ht_2)^{-1})s_2^{-1}, v_2), \\
C_4 :& (s_3, t_3, u_3, v_3) = (0, P, Q, (1 + hP)^{-1}) \\
&\leftarrow (s_4, t_4, u_4, v_4) := (s_3, t_3, u_3, (v_3 - (1 + ht_3)^{-1})s_3^{-1}), \\
C_5 :& (s_4, t_4, u_4, v_4) = (0, P, Q, (1 + hP)^{-2}) \\
&\leftarrow (s_5, t_5, u_5, v_5) := (s_4, t_4, u_4, (v_4 - (1 + ht_4)^{-2})s_4^{-1}),
\end{aligned}$$

where only one of the coordinate systems is written for each blowup.

$$
\begin{aligned}
C_6 :&(z_0, z_1, x_2, x_3) = (0, 0, 1, P)\\
&\leftarrow (s_6, t_6, u_6, v_6) := (x_2 - 1, x_3, z_0(x_2 - 1)^{-1}, z_1(x_2 - 1)^{-1}),\\
C_7 :&(s_6, t_6, u_6, v_6) = (0, P, Q, 0)\\
&\leftarrow (s_7, t_7, u_7, v_7) := (s_6, t_6, u_6, v_6 s_6^{-1}),\\
C_8 :&(s_7, t_7, u_7, v_7) = (0, P, -h(1 + hP)^{-1}, Q)\\
&\leftarrow (s_8, t_8, u_8, v_8) := (s_7, t_7, (u_7 + h(1 + ht_7)^{-1})s_7^{-1}, v_7),\\
C_9 :&(s_8, t_8, u_8, v_8) = (0, P, Q, (1 + hP)^{-1})\\
&\leftarrow (s_9, t_9, u_9, v_9) := (s_8, t_8, u_8, (v_8 - (1 + ht_8)^{-1})s_8^{-1}),\\
C_{10} :&(s_9, t_9, u_9, v_9) = (0, P, Q, (1 + hP)^{-2})\\
&\leftarrow (s_{10}, t_{10}, u_{10}, v_{10}) := (s_9, t_9, u_9, (v_9 - (1 + ht_9)^{-2})s_9^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
C_{11} :&(z_0, z_1, z_2, z_3) = (0, 0, 0, 0)\\
&\leftarrow (s_{11}, t_{11}, u_{11}, v_{11}) := (z_0, z_1 z_0^{-1}, z_2 z_0^{-1}, z_3 z_0^{-1}),\\
C_{12} :&(s_{11}, t_{11}, u_{11}, v_{11}) = (P, 0, 1, 0)\\
&\leftarrow (s_{12}, t_{12}, u_{12}, v_{12}) := (s_{11}, t_{11}, (u_{11} - 1)t_{11}^{-1}, v_{11} t_{11}^{-1}),\\
C_{13} :&(s_{12}, t_{12}, u_{12}, v_{12}) = (P, 0, Q, -1)\\
&\leftarrow (s_{13}, t_{13}, u_{13}, v_{13}) := (s_{12}, t_{12}, u_{12}, (v_{12} + 1)t_{12}^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
C_{14}:&(s_{13},t_{13},u_{13},v_{13})=(0,0,1+h,0)\\
&\leftarrow (s_{14},t_{14},u_{14},v_{14}):=(s_{13}t_{13}^{-1},t_{13},(u_{13}-1-h)t_{13}^{-1},v_{13}t_{13}^{-1}),\\
C_{15}:&(s_{14},t_{14},u_{14},v_{14})=(P,0,-2Q-Ph^{-1},Q)\\
&\leftarrow (s_{15},t_{15},u_{15},v_{15}):=(s_{14},t_{14},v_{14},(u_{14}+2v_{14}+s_{14}h^{-1})t_{14}^{-1}),\\
C_{16}:&(s_{15},t_{15},u_{15},v_{15})=(P,0,-Ph^{-1},Q)\\
&\leftarrow (s_{16},t_{16},u_{16},v_{16}):=(s_{15},t_{15},(u_{15}+s_{15}h^{-1})t_{15}^{-1},v_{15}),\\
C_{17}:&(s_{16},t_{16},u_{16},v_{16})=(P,0,Q,2^{-1}Q+(1+h)h^{-1}P)\\
\leftarrow &(s_{17},t_{17},u_{17},v_{17}):=(s_{16},t_{16},u_{16},(v_{16}-2^{-1}u_{16}-(1+h)h^{-1}s_{16})t_{16}^{-1}).
\end{aligned}
$$

Let us denote the total transform (with respect to blowups) of the divisors (hyper-surfaces) $c_0x_0 + c_1x_1 + a = 0$ and $c_2x_2 + c_3x_3 + b$ by H_a and H_b respectively, where $(c_0 : c_1 : a)$ and $(c_2 : c_3 : b)$ are constant $\mathbb{P}^2$ vectors. We also denote the total transform of the i-th exceptional divisor by E_i. Let us write their classes modulo linear equivalence as H_a, H_b and E_i. Then, the Picard group of this variety X becomes a $\mathbb{Z}$-module:

$$\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}H_a \oplus \mathbb{Z}H_b \oplus \bigoplus_{i=1}^{16} \mathbb{Z}E_i.$$

The pull-back φ^* of our equaton is a linear action on the Picard group given by

$$\begin{aligned}
&H_a \to H_b, \\
&H_b \to H_a + 3H_b - 2E_1 - 3E_{11} - E_{6,7,9,10,12,13,14}, \\
&E_1 \to H_b - E_{1,10,11}, \quad E_2 \to H_b - E_{1,9,11}, \quad E_3 \to H_b - E_{1,7,9,11} + E_8, \\
&E_4 \to H_b - E_{1,7,11}, \quad E_5 \to H_b - E_{1,6,11}, \\
&E_6 \to E_{14}, \quad E_7 \to E_{14}, \quad E_8 \to E_{15}, \quad E_9 \to E_{16}, \quad E_{10} \to E_{17}, \\
&E_{11} \to E_{1,11} - E_{14}, \quad E_{12} \to H_b - E_{1,11,13}, \quad E_{13} \to H_b - E_{1,11,12}, \\
&E_{14} \to E_2, \quad E_{15} \to E_3, \quad E_{16} \to E_4, \quad E_{17} \to E_5,
\end{aligned}$$

where $E_{i_1,\ldots,i_k}$ denotes $E_{i_1} + \cdots + E_{i_k}$.

The set of irreducible hyper-surfaces whose class is

$$2H_a + 2H_b - 2E_1 - 2E_6 - 4E_{11} - E_{2,4,7,9,12,13,14,16}$$

are $C_0 + C_1 I_1 = 0$, where $(C_0 : C_1) \in \mathbb{P}^1$ and $C_1 \neq 0$.
The set of irreducible hyper-surfaces whose class is

$$2H_a + 2H_b - 3E_{11} - E_{1,2,4,5,6,7,9,10,12,13,14,16,17}$$

are $C_0 + C_1 I_1 + C_2 I_2 = 0$, where $(C_0 : C_1 : C_2) \in \mathbb{P}^2$ and $C_2 \neq 0$. from where we get the invariants

$$I_1 = -hx_0^2 - hx_0x_2 + h^2x_0x_2 + hx_0^2x_2 - hx_2^2 + hx_0x_2^2 \quad (12)$$

$$\begin{aligned} I_2 =& 2hx_0 + x_0^2 - 2hx_0x_1 + 2hx_2 + x_0x_2 - hx_1x_2 + h^2x_1x_2 + 2hx_0x_1x_2 \\ &+ x_2^2 + hx_1x_2^2 - hx_0x_3 + h^2x_0x_3 + hx_0^2x_3 - 2hx_2x_3 + 2hx_0x_2x_3 \end{aligned} \quad (13)$$

One can obtian the two invariants also from the so called "staircase method" of Papageorgiou, Nijhoff, Capel (1992) applied to super-zero-curvature representation In the (p, q) traveling-wave reduction we consider that our dependent variables will depend only on one variable $\nu = pm + qn$, and accordingly, our system will turn into a system of nonlinear ordinary discrete super-equations. The method relies on the fact that (p, q) traveling reduction perform a periodic problem on the staircase in the (n, m) lattice (the lattice system being quadrilateral and the initial conditions defined on a *stair*). In this periodic problem, one of the Lax operators is transformed into a monodromy matrix (L_ν) and the other one (M_ν) will be related to evolution on ν. These new Lax even super-matrices are given by the following relations:

$$L_\nu = \prod_{j=0}^{p-1}{}' B_{n+q,m+j} \prod_{i=0}^{q-1}{}' A_{n+i,m} \tag{14}$$

$$M_\nu = \prod_{j=0}^{c-1}{}' B_{n+c,m+j} \prod_{i=0}^{d-1}{}' A_{n+i,m} \tag{15}$$

where $\prod'$ follows the order of the stair (in our subsequent case the order will be just reversible order) and (c, d) are related to the shift $\nu \to \nu + 1$ in the way $(n, m) \to (n + c, m + d)$. However in our applications we will consider the simplest case $c = d = 1$.
Now the compatibility condition

$$A_{n,m+1}B_{n,m} - B_{n+1,m}A_{n,m} = 0, L_{\nu+1}M_\nu - M_\nu L_\nu = 0$$

which is indeed the compatibility of discrete Lax representation

$$L_\nu \phi_\nu = \lambda \phi_\nu \quad \phi_{\nu+1} = M_\nu \phi_\nu$$

In our case applying Nijhoff theorem we get the following Lax representation for our equation:

$$L_\nu = B_{\nu-2}A_{\nu-1}A_\nu, \quad M_\nu = B_{\nu-1}A_\nu$$

where A_ν and B_ν are depending on the combination $\nu = 2m - n$ through

$$A_\nu = \begin{pmatrix} p_1 & 1 & -\alpha_1 & 0 \\ \lambda^2 & p_1 & -\beta_1 & -\alpha_1 \\ 0 & \alpha_1 & a_1 & 1 \\ \lambda^2\alpha_1 & \beta_1 & b_1 & a_1 \end{pmatrix}, \quad B_\nu = \begin{pmatrix} p_2 & 1 & -\alpha_2 & 0 \\ \lambda^2 & p_2 & -\beta_2 & -\alpha_2 \\ 0 & \alpha_2 & a_2 & 1 \\ \lambda^2\alpha_2 & \beta_2 & b_2 & a_2 \end{pmatrix}.$$

with

$$\begin{aligned} \alpha_1 =& \frac{\zeta_{\nu-1}}{2} \\ \beta_1 =& p_1\zeta_{\nu-1} + \frac{1}{4}x_{\nu-1}\zeta_{\nu-1} \\ a_1 =& p_1 + \frac{x_{\nu-1}}{2} \\ b_1 =& \lambda^2 - p_1(x_\nu + x_{\nu+1}) + \frac{1}{4}(x_\nu + x_{\nu+1})^2 \end{aligned}$$

and

$$\begin{aligned} \alpha_2 =& \frac{1}{2}(\zeta_\nu + \zeta_{\nu+1}) \\ \beta_2 =& \frac{1}{2}p_2(\zeta_\nu + \zeta_{\nu+1}) - \frac{1}{4}(x_\nu + x_{\nu+1})(\zeta_\nu + \zeta_{\nu+1}) \\ a_2 =& p_2 - \frac{1}{2}(x_\nu + x_{\nu+1}) \\ b_2 =& \lambda^2 - p_2(x_\nu + x_{\nu+1}) + \frac{1}{4}(x_\nu + x_{\nu+1})^2. \end{aligned}$$

The compatibility condition of the Lax supermatrices L_ν and M_ν is $L_{\nu+1}M_\nu = M_\nu L_\nu \Leftrightarrow L_{\nu+1} = M_\nu L_\nu M_\nu^{-1}$. So applying the supertrace we have:

$$\mathrm{str} L_{\nu+1} = \mathrm{str}(M_\nu L_\nu M_\nu^{-1})$$

. From the property of supertraces we have $\mathrm{str}(M_\nu L_\nu M_\nu^{-1}) = \mathrm{str}(L_\nu M_\nu^{-1} M_\nu) = \mathrm{str} L_\nu$. So the supertrace of L_ν is invariant at the evolution $\nu \to \nu + 1$. Expanding str(L_ν) in powers of the spectral parameter the corresponding coefficients will be the conservation laws. After long, but straightforward computations we obtain the second conservation law:

$$\begin{aligned} I =& \frac{1}{8}(2p_1(x_\nu^2 + x_{\nu-1}^2) + (x_\nu + x_{\nu-1})(-x_\nu x_{\nu-1} + 2p_2(x_\nu + x_{\nu-1}))) \\ &+ \gamma(p_1(p_1 + 2p_2) - \frac{1}{16}((4p_1 - 4p_2 + x_\nu)(x_\nu + x_{\nu-1}) + x_{\nu-1}^2)), \end{aligned} \tag{16}$$

Further, by the following rescalings,

$$x_\nu \to 2x_\nu(p_1 + p_2), \quad \gamma \to 4\gamma\frac{(p_1 + p_2)^2}{(p_1 - p_2)}, \quad p_2 \to p_1\frac{(1 - h)}{(1 + h)}$$

the invariant is simplified

$$I(x, y) = h(x^2(y - 1) - y^2 + xy(y + h - 1)) + \gamma(x^2 + xy + y^2 + 2h(x + y)) \tag{17}$$

where $x_\nu \equiv x, x_{\nu-1} \equiv y$ are grassmann commuting function, and h is an ordinary complex number. One can see that the this invariant is expressed by an elliptic super-curve with coefficients in the commuting sector of the grassmann algebra.

One can see immediately that for a Grassmann algebra with two generators ξ_1 and ξ_2 the above invariant is written as:

$$I = I_1 + \xi_1\xi_2 I_2$$

where

$$I_1 = -hx_0^2 - hx_0x_2 + h^2x_0x_2 + hx_0^2x_2 - hx_2^2 + hx_0x_2^2$$

$$I_2 = 2hx_0 + x_0^2 - 2hx_0x_1 + 2hx_2 + x_0x_2 - hx_1x_2 + h^2x_1x_2 + 2hx_0x_1x_2 +$$
$$+ x_2^2 + hx_1x_2^2 - hx_0x_3 + h^2x_0x_3 + hx_0^2x_3 - 2hx_2x_3 + 2hx_0x_2x_3$$

so we recover the particular case studied by means of algebraic geometric methods.
Main references:

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