

Invariant metrics on homogeneous manifolds attached to the Jacobi group

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Abstract

The Jacobi group $G_n^J(\mathbb{R})$ is an interesting object in Math, with many applications in Phys. I have investigated the Jacobi gr. with **the methods of coherent states based on homogeneous Kähler manifolds** attached to $G_n^J(\mathbb{R})$, determining the balanced metric. Here I consider $G_n^J(\mathbb{R})$ embedded in $\mathrm{Sp}(n+1, \mathbb{R})$. I obtain an invariant metric on an odd dimensional homogeneous manifold attached to $G_n^J(\mathbb{R})$. More formulae are presented in the case of $G_1^J(\mathbb{R})$.

Outline

1 Introduction

2 $G_n^J(\mathbb{R})$

- The Heisenberg group
- The symplectic group $\mathrm{Sp}(n, \mathbb{R})$
 - The real symplectic algebra $\mathfrak{sp}(n, \mathbb{R})$
 - Action of $\mathrm{Sp}(n, \mathbb{R})$ on \mathcal{X}_n
 - Pre-Iwasawa decomposition
- $G_n^J(\mathbb{R})$ embedded in $\mathrm{Sp}(n+1, \mathbb{R})$
 - The action
 - The Jacobi group $G_n^J(\mathbb{R})$ as subgroup of $\mathrm{Sp}(n+1, \mathbb{R})$
 - The Lie algebra $\mathfrak{g}_n^J(\mathbb{R})$
 - Invariant one forms on the Jacobi group
- Invariant metrics on \mathcal{X}_n^J and $\tilde{\mathcal{X}}_n^J$

3 The particular case $G_1^J(\mathbb{R})$

- The Heisenberg subgroup of $\mathrm{Sp}(2, \mathbb{R})$
- The $\mathrm{SL}(2, \mathbb{R})$ subgroup of $\mathrm{Sp}(2, \mathbb{R})$
- The Jacobi group $G_1^J(\mathbb{R})$ subgroup of $\mathrm{Sp}(2, \mathbb{R})$
- Invariant metrics on \mathcal{X}_1^J , $\tilde{\mathcal{X}}_1^J$, $G_1^J(\mathbb{R})$
- Natural reductivity of $\mathcal{X}_1^J(\mathbb{R})$

4 Appendix

Notation

Table: Manifolds

Notation	Definition	Dimensions
$\mathrm{Sp}(n, \mathbb{R})$	real symplectic group	$2n^2 + n$
$H_n(\mathbb{R})$	real Heisenberg group	$2n + 1$
$G_n^J(\mathbb{R}) = H_n(\mathbb{R}) \rtimes \mathrm{Sp}(n, \mathbb{R})$	real Jacobi group	$(2n+1)(n+1)$
$\mathcal{X}_n^J = \frac{G_n^J(\mathbb{R})}{U(n) \times \mathbb{R}} \approx \mathcal{X}_n \times \mathbb{R}^{2n}$	Siegel-J. upper half space	$n(n+3)$
$\tilde{\mathcal{X}}_n^J = \frac{G_n^J(\mathbb{R})}{U(n)} \approx \mathcal{X}_n^J \times \mathbb{R}$	extended Sgl-J. upp. h. space	$n(n+3) + 1$
$\mathcal{X}_n = \frac{\mathrm{Sp}(n, \mathbb{R})}{U(n)} \approx \{M(n, \mathbb{C}) \ni v \mid v = s + ir, v = v^t, r > 0\}$	Siegel upper half plane Hermitian symm. sp.	$n(n+1)$
$U(n)$	unitary group	n^2
$\mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C}) \cap U(n, n)$	iso $\mathrm{Sp}(n, \mathbb{R})$	$2n^2 + n$
$G_n^J = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}}$	complex Jacobi group	iso $G_n^J(\mathbb{R})$
$\mathcal{D}_n^J = \mathbb{C}^n \times \mathcal{D}_n$	Siegel-Jacobi ball	iso \mathcal{X}_n^J
$\mathcal{D}_n = \mathrm{Sp}(n, \mathbb{R})_{\mathbb{C}} / U(n) \approx \{M(n, \mathbb{C}) \ni w \mid w = w^t \mid \mathbb{1}_n - w\bar{w} > 0\}$	Siegel ball Hermitian symm. sp.	iso \mathcal{X}_n

$G_n^J(\mathbb{R})$ embedded in $\mathrm{Sp}(n+1, \mathbb{R})$

$g = (M, X, \kappa) \in G_n^J(\mathbb{R}), X = (\lambda, \mu) \in M(1, 2n, \mathbb{R}), \kappa \in \mathbb{R}, (p, q) = XM^{-1}$,

$$\mathrm{Sp}(n+1, \mathbb{R}) \ni g = \begin{pmatrix} a & 0 & b & q^t \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -p^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R}).$$

Relevance:

Math: M. Eichler & D. Zagier 1985; R. Berndt & R. Schmidt 1998; E.

Kähler 1983, ..., 1992, K. Takase, J.-H. Yang...+ :

G_n^J : non-reductive, algebraic gr of Harish-Chandra type, CS-type gr ...

Applications: Jacobi forms, automorphic forms, theta functions, Hecke operators,...

\mathcal{D}_n^J : partially bounded domain, non-symmetric, Lu Qi-Keng manifold, quantizable manifold, projectively induced...

Physics: U. Niederer 1972, C. R. Hagen 1972, K. B. Wolf ...

Applications: quantum mechanics, geometric quantization, nuclear structure, signal processing, quantum optics: Squeezed states, quantum teleportation ...

Applications of the present paper: Berry phase? ...??

$H_n(\mathbb{R}) = (\lambda, \mu, \kappa)$, $\lambda, \mu \in M(1, n, \mathbb{R})$, $\kappa \in \mathbb{R}$. Composition law:

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu'^t - \mu\lambda'^t).$$

$g \in H_n(\mathbb{R})$ embedded in $\mathrm{Sp}(n+1, \mathbb{R})$:

$$g = \begin{pmatrix} 1 & 0 & 0 & \mu^t \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\mu^t \\ -\lambda & 1 & \mu & -\kappa \\ 0 & 0 & 1 & \lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$g^{-1} dg = P\lambda^p + Q\lambda^q + R\lambda^r.$$

$$\lambda^p = d\lambda, \quad \lambda^q = d\mu, \quad \lambda^r = d\kappa - \lambda d\mu^t + \mu d\lambda^t.$$

Left action of the Heisenberg group on itself:

$$\exp(\lambda P + \mu^t Q + \kappa R)(\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda\mu_0^t - \mu\lambda_0^t).$$

Left invariant metric: $g^L(\lambda, \mu, \kappa) = d\lambda^2 + d\mu^2 + (d\kappa - \lambda d\mu^t + \mu d\lambda^t)^2$.

$\mathrm{Sp}(n, \mathbb{K})$ -matrices $M \in M(2n, \mathbb{K})$, \mathbb{K} is \mathbb{R} or \mathbb{C} ,

$$M^t J_n M = J_n; \quad J_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2n, \mathbb{R}),$$

Remark

If $M \in \mathrm{Sp}(n, \mathbb{R})$, then M is similar with M^t and M^{-1} and $\det M = 1$.

$$ab^t - ba^t = 0, \quad ad^t - bc^t = \mathbf{1}_n, \quad cd^t - dc^t = 0; \quad (2.1a)$$

$$a^t c - c^t a = 0, \quad a^t d - c^t b = \mathbf{1}_n, \quad b^t d - d^t b = 0. \quad (2.1b)$$

$$M^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

$M \in \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}_{2n}$ has the expression

$$M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^t a + b^t b = aa^t + bb^t = \mathbb{1}_n, \quad a^t b = b^t a, \quad ba^t = ab^t.$$

$M' := a + ib \in M(n, \mathbb{C})$. The correspondence $M \rightarrow M'$ - group isomorphism and $\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}_{2n} \approx \mathrm{U}(n)$.

$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ = a real form of the simple Lie algebra $\mathfrak{sp}(n, \mathbb{C})$ of type c_n .
 $X \in \mathfrak{sp}(n, \mathbb{R}) \Leftrightarrow X^t J + JX = 0 \Leftrightarrow$

$$X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \quad b = b^t, \quad c = c^t, \quad a, b, c \in M(n, \mathbb{R})$$

$$X = \sum_{ij} a_{ij} H_{ij} + 2 \sum_{i < j} (b_{ij} F_{ij} + c_{ij} G_{ij}) + \sum_{i=j} (b_{ij} F_{ij} + c_{ij} G_{ij}),$$

$$H_{ij} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, 2F_{ij} = \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix}; 2G_{ij} = \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix}.$$

$\mathfrak{sp}(n, \mathbb{R})$ has the $2n^2 + n$ generators: H_{ij} , F_{ij} , G_{ij} , $1 \leq i \leq j \leq n$.

Remark

Transitive action $\mathrm{Sp}(n, \mathbb{R}) \curvearrowright \mathcal{X}_n$

$$v_1 = (av + b)(cv + d)^{-1} = (vc^t + d^t)^{-1}(va^t + b^t),$$

$$\mathcal{X}_n := \{v \in M(n, \mathbb{C}) \mid v = s + ir, s, r \in M(n, \mathbb{R}), r > 0, s^t = s; r^t = r\}.$$

The correspondence

$$\eta : \mathcal{X}_n \rightarrow X_n = \mathrm{Sp}(n, \mathbb{R}) / K, K = \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}_{2n}; v \mapsto M_{x+iy}K,$$

$$M_{x+iy} = \begin{pmatrix} \mathbb{1}_n & x \\ 0 & \mathbb{1}_n \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y^{-1}} \end{pmatrix} = \begin{pmatrix} \sqrt{y} & x\sqrt{y^{-1}} \\ 0 & \sqrt{y^{-1}} \end{pmatrix}.$$

is a 1-1 map; realizes \mathcal{X}_n as homogenous manifold $\mathcal{X}_n = \mathrm{Sp}(n, \mathbb{R}) / \mathrm{U}(n)$.
The subgroup of $\mathrm{Sp}(n, \mathbb{R})$ that stabilizes $i\mathbb{1}_n \in \mathcal{X}_n$ is the subgroup of orthogonal symplectic matrices.

Remark

Unique decomposition, x, y , - symmetric, $y > 0$, $x := ty$, $X + iY \in \mathrm{U}(n)$:

$$\mathrm{Sp}(n, \mathbb{R}) \ni M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbb{1}_n & x \\ 0_n & \mathbb{1}_n \end{pmatrix} \begin{pmatrix} y & 0_n \\ 0_n & y^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix},$$

$$XX^t + YY^t = X^tX + Y^tY = \mathbb{1}_n, \quad X^tY = Y^tX, \quad XY^t = YX^t.$$

$$y = (dd^t + cc^t)^{-\frac{1}{2}}, \quad X - iY = y(d + ic), \quad t = y^2(db^t + ca^t)y^{-1} = (bd^t + ac^t)y,$$

$$x = (dd^t + cc^t)^{-1}(db^t + ca^t) = (bd^t + ac^t)(dd^t + cc^t)^{-1}.$$

The inverse transform:

$$a = yX - xy^{-1}Y, \quad b = yY + xy^{-1}X, \quad c = -y^{-1}Y, \quad d = y^{-1}X.$$

For $SL(2, \mathbb{R})$: $y \rightarrow y^{1/2}$, $X = \cos \theta$, $Y = \sin \theta$. The first factor - “free propagation subgroup”.

Lemma

Action of $M \in \mathrm{Sp}(n, \mathbb{R})$ on $\mathcal{X}_n \approx \frac{\mathrm{Sp}(n, \mathbb{R})}{\mathrm{U}(n)}$, with Pre-Iwasawa dec.:

$$(a, b, c, d) \circ (x', y', X', Y') \rightarrow (x_1, y_1, X_1, Y_1),$$

$$x', y' \in M(n, \mathbb{R}), x' = (x')^t, y' = (y')^t, y' > 0,$$

$$a = y^{1/2}X - xy^{-1/2}Y, \quad b = y^{1/2}Y + xy^{-1/2}X, \quad c = -y^{-1/2}Y, \quad d = y^{-1/2}X,$$

$$\begin{aligned} x_1 + iy_1 = & [c(y' + x'y'^{-1}x')c^t + d(y')^{-1}d^t + cx'(y')^{-1}d^t + d(y')^{-1}x'c^t]^{-1} \\ & \times [c(y' + x'(y')^{-1}x')a^t + cx'(y')^{-1}b^t + d(y')^{-1}x'a^t + d(y')^{-1}b^t + i], \end{aligned}$$

$$\begin{aligned} X_1 - iY_1 = & (y_1)^{1/2} \{ (cx' + d)(y')^{-1/2}X' + c(y')^{1/2}Y' \\ & - i[c(y')^{1/2}X' - (cx' + d)(y')^{-1/2}Y'] \}. \end{aligned}$$

Pre-Iwasawa dec. \equiv Möbius transf.: $M \circ v' \rightarrow v_1 = x_1 + iy_1$

$$x_1 + iy_1 = (\bar{v}'c^t + d^t)^{-1} \left(\frac{B}{2} + iy' \right) (cv' + d)^{-1},$$

$$B = 2\bar{v}'a^t cv' + \bar{v}'(c^t b + a^t d) + (b^t c + d^t a)v' + 2b^t d.$$

Composition law

$G_n^J(\mathbb{R}) = H_n(\mathbb{R}) \rtimes \mathrm{Sp}(n, \mathbb{R})$, with the composition law:

$$(M, (\lambda, \mu, \kappa)) \circ (M', (\lambda', \mu', \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \tilde{\lambda}\mu'^t - \tilde{\mu}\lambda'^t)),$$

$$M, M' \in \mathrm{Sp}(n, \mathbb{R}), (\lambda, \mu, \kappa), (\lambda', \mu', \kappa') \in H_n(\mathbb{R}), (\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'.$$

Restricted real group $G_n^J(\mathbb{R})_0$, $g = (M, X)$, $X = (\lambda, \mu)$.

$$u = pv + q, \quad v = x + iy, \quad v = v^t, \quad y > 0, \quad p, q \in M(1, n, \mathbb{R}).$$

Lemma

a) $\mathcal{X}_n \ni v = x + iy$, $G_n^J(\mathbb{R})_0 \curvearrowright \mathcal{X}_n^J$: $(M, X) \times (v', u') \rightarrow (v_1, u_1)$

$$v_1 = (av' + b)(cv' + d)^{-1} = (v'c^t + d^t)^{-1}(v'a^t + b^t),$$

$$u_1 = (u' + \lambda v' + \mu)(cv' + d)^{-1}.$$

For $\lambda, \mu \in M(1, n, \mathbb{R})$, (p, q) s. t.:

$$(p, q) = (\lambda, \mu)M^{-1} = (\lambda d^t - \mu c^t, -\lambda b^t + \mu a^t),$$

$$(\lambda, \mu) = (p, q)M = (pa + qc, pb + qd), \quad p, q, \lambda, \mu \in M(1, n, \mathbb{R}).$$

Action Lemma - continuation

b) $G_n^J(\mathbb{R})_0 \curvearrowright \mathcal{X}_n^J$: $(M, X) \times (x', y', p', q') \rightarrow (x_1, y_1, p_1, q_1)$, $v = x + iy$,

$$(p_1, q_1) = (p, q) + (p', q') \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (p + p'd^t - q'c^t, q - p'b^t + q'a^t).$$

c) $G_n^J(\mathbb{R}) \curvearrowright \tilde{\mathcal{X}}_n^J \approx \mathcal{X}_n^J \times \mathbb{R}$:

$(M, (\lambda, \mu), \kappa) \times (x', y', p', q', \kappa') \rightarrow (x_1, y_1, p_1, q_1, \kappa_1)$,

$$\kappa_1 = \kappa + \kappa' + \lambda q'^t - \mu p'^t.$$

d) The 1-form -invariant $G_n^J(\mathbb{R}) \curvearrowright \tilde{\mathcal{X}}_n^J$:

$$\lambda^R = d\kappa - p dq^t + q dp^t$$

Let $M \in \mathrm{Sp}(n, \mathbb{R})$. We introduce the matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M(2n+2, \mathbb{R}),$$

$$A = \begin{pmatrix} a_{nn} & O_{n1} \\ \lambda_{1n} & 1_{11} \end{pmatrix}, B = \begin{pmatrix} b_{nn} & q_{n1}^t \\ \mu_{1n} & \kappa_{11} \end{pmatrix}, C = \begin{pmatrix} c_{nn} & O_{n1} \\ O_{1n} & O_{11} \end{pmatrix}, D = \begin{pmatrix} d_{nn} & -p_{n1}^t \\ O_{1n} & 1_{11} \end{pmatrix},$$

Remark

The matrix g is in $\mathrm{Sp}(n+1, \mathbb{R})$.

$$2F_{ij} = \begin{pmatrix} 0 & 0 & E_{ij} + E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$2G_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_{ij} + E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_{ij} = \begin{pmatrix} E_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -E_{ji} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{1p} & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_{p1} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_q = \begin{pmatrix} 0 & 0 & 0 & E_{q1} \\ 0 & 0 & E_{1q} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$X \in \mathfrak{g}_n^J(\mathbb{R})$ as submatrix of $\mathrm{Sp}(n+1, \mathbb{R})$:

$$X = \sum_{i,j=1}^n a_{ij} H_{ij} + 2 \sum_{1 \leq i < j \leq n} (b_{ij} F_{ij} + c_{ij} G_{ij}) \\ + \sum_{1 \leq i=j \leq n} (b_{ij} F_{ij} + c_{ij} G_{ij}) + \sum_{i=1}^n (p_i P_i + q_i Q_i) + r R, \quad b = b^t, \quad c = c^t.$$

$[H_{kl}, F_{ij}] = \delta_{lj} F_{ik} + \delta_{li} F_{kj},$

$[G_{ij}, H_{kl}] = \delta_{ki} G_{lj} + \delta_{kj} G_{li},$

$4[F_{ij}, G_{kl}] = \delta_{li} H_{kj} + \delta_{jl} H_{ik} + \delta_{jk} H_{il} + \delta_{ik} H_{jl},$

$[P_p, Q_q] = 2\delta_{pq} R,$

$2[P_p, F_{ij}] = \delta_{pi} Q_j + \delta_{pj} Q_i,$

$2[Q_q, G_{ij}] = \delta_{iq} P_j + \delta_{jq} P_i,$

$[P_p, H_{ij}] = \delta_{pi} P_j,$

$[H_{ij}, Q_q] = \delta_{jq} Q_i.$

$g = (M, X, \kappa) \in G_n^J(\mathbb{R}),$

$$g = \begin{pmatrix} a & 0 & b & q^t \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -p^t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} d^t & 0 & -b^t & -\mu^t \\ -p & 1 & -q & -\kappa \\ -c^t & 0 & a^t & \lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} g^{-1} \, \mathrm{d} \, g &= \sum_{i,j=1}^n (\lambda^H)_{ij} H_{ij} + \sum_{1 \leq i \leq j \leq n} [(\lambda^F)_{ij} F_{ij} + (\lambda^G)_{ij} G_{ij}] \\ &\quad + \sum_1^n [(\lambda^P)_i P_i + (\lambda^Q)_i Q_i] + \lambda^R R, \end{aligned}$$

$$\lambda^F = d^t \, \mathrm{d} \, b - b^t \, \mathrm{d} \, d = (\lambda^F)^t,$$

$$\lambda^G = -c^t \, \mathrm{d} \, a + a^t \, \mathrm{d} \, c = (\lambda^G)^t,$$

$$\lambda^H = d^t \, \mathrm{d} \, a - b^t \, \mathrm{d} \, c = \mathrm{d} \, b^t c - \mathrm{d} \, d^t a = (\lambda^H)^t,$$

$$\lambda^P = \mathrm{d} \, \lambda - p \, \mathrm{d} \, a - q \, \mathrm{d} \, c = \mathrm{d} \, p a + \mathrm{d} \, q c = \lambda^p - \lambda \lambda^H - \mu \lambda^G,$$

$$\lambda^Q = \mathrm{d} \, q d + \mathrm{d} \, p b = \mathrm{d} \, \mu - p \, \mathrm{d} \, b - q \, \mathrm{d} \, d = \lambda^q - \lambda \lambda^F + \mu \lambda^H,$$

$$\lambda^R = \mathrm{d} \, \kappa - p \, \mathrm{d} \, q^t + q \, \mathrm{d} \, p^t = \lambda^r + \lambda \lambda^F \lambda^t - \mu \lambda^G \mu^t - 2\lambda \lambda^H \mu^t.$$

$$\begin{aligned}\lambda^F &= X^t d Y - Y^t d X + X^t y^{-1} d y Y + (X^t y^{-1} d x + Y^t d y) y^{-1} X, \\ \lambda^G &= -X^t d Y + Y^t d X + Y^t y^{-1} d y X + (-Y^t y^{-1} d x + X^t d y) y^{-1} Y, \\ \lambda^H &= X^t d X + Y^t d Y + X^t y^{-1} d y X - (X^t y^{-1} d x + Y^t d y) y^{-1} Y.\end{aligned}$$

$$\begin{aligned}\lambda^F + \lambda^G &= X^t (y^{-1} d y + d y y^{-1}) Y + Y^t (y^{-1} d y + d y y^{-1}) X \\ &\quad + X^t y^{-1} d x y y^{-1} X - Y^t y^{-1} d x y y^{-1} Y,\end{aligned}$$

$$\begin{aligned}\lambda^F - \lambda^G &= 2(X^t d Y - Y^t d X) + 2X^t (y^{-1} d y - d y y^{-1}) Y \\ &\quad + X^t y^{-1} d x y y^{-1} X + Y^t y^{-1} d x y y^{-1} Y.\end{aligned}$$

$$\begin{aligned}\lambda^F &= \frac{d x}{y} \cos^2 \theta + \frac{d y}{2y} \sin 2\theta + d \theta, \\ \lambda^G &= -\frac{d x}{y} \sin^2 \theta + \frac{d y}{2y} \sin 2\theta - d \theta, \\ \lambda^H &= -\frac{d x}{2y} \sin 2\theta + \frac{d y}{2y} \cos 2\theta.\end{aligned}$$

Differences $G_n^J(\mathbb{R})$, $n > 1, n = 1$

For $G_n^J(\mathbb{R})$, $\forall \mathbb{N} \ni n > 1$:

$\lambda^F + \lambda^G$ does not depend on dX, dY , but λ^H does.

For $G_1^J(\mathbb{R})$:

$X = \cos \theta$, $Y = \sin \theta$, and $\lambda^F + \lambda^G$, λ^H does not depend on $d\theta$.

k indexes the holomorphic discrete series of $\mathrm{Sp}(n, \mathbb{R})$, ν representations of the Heisenberg group:

Lemma

The Kähler two-form on $\mathcal{X}_n^J \approx \mathcal{X}_n \times \mathbb{C}^n$, invariant to the action $G_n^J(\mathbb{R})_0$:

$$-\mathrm{i}\omega_{\mathcal{X}_n^J}(v, u) = \frac{k}{2}\mathrm{Tr}(H \wedge \bar{H}) + \frac{2\nu}{\mathrm{i}}\mathrm{Tr}(G^t D \wedge \bar{G}),$$

$$D = (\bar{v} - v)^{-1}, \quad H = D \, \mathrm{d} \, v, \quad G = \mathrm{d} \, u^t - \mathrm{d} \, v D (\bar{u}^t - u^t).$$

We get for G , $G = v \, \mathrm{d} \, p^t + \mathrm{d} \, q^t$. if $\alpha = \frac{k}{4}$, $\gamma = \nu$

Result

Proposition

The metric on \mathcal{X}_n^J , invariant to the action of $G_n^J(\mathbb{R})_0$:

$$\begin{aligned} d s_{\mathcal{X}_n^J}^2(x, y, p, q) &= \alpha [\text{Tr}(y^{-1} dx)^2 + (y^{-1} dy)^2] \\ &\quad + \gamma \text{Tr}[dp(xy^{-1}x + yy^{-1}y) dp^t + dqy^{-1} dq^t + 2 dp x y^{-1} dq^t] \end{aligned}$$

The three parameter metric on $\tilde{\mathcal{X}}_n^J$, invariant to the action of $G_n^J(\mathbb{R})$:

$$\begin{aligned} d s_{\tilde{\mathcal{X}}_n^J}^2(x, y, p, q, \kappa) &= d s_{\mathcal{X}_n^J}^2(x, y, p, q) + \delta(\lambda^R)^2 \\ &= \alpha [\text{Tr}(y^{-1} dx)^2 + (y^{-1} dy)^2] \\ &\quad + \gamma \text{Tr}[dp(xy^{-1}x + yy^{-1}y) dp^t + dqy^{-1} dq^t + 2 dp x y^{-1} dq^t] \\ &\quad + \delta(d\kappa - p dq^t + q dp^t)^2. \end{aligned}$$

Jacobi group - "in Math" - 1998

The real Jacobi group $G_1^J(\mathbb{R})$ - subgroup of $Sp(2, \mathbb{R})$: 4×4 real matrices
 $g = ((\lambda, \mu, \kappa), M)$, $(\lambda, \mu, \kappa) \in H_1(\mathbb{R})$, $M \in SL(2, \mathbb{R})$

$$H_1 \ni g = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ -\lambda & 1 & -\mu & -\kappa \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

$$Y = (p, q) := XM^{-1} = (\lambda d - \mu c, \mu a - \lambda b), \quad X = (\lambda, \mu)$$

$\mathfrak{g}_1^J(\mathbb{R}) := \langle P, Q, R, F, G, H \rangle_{\mathbb{R}}$, $\mathfrak{h} = \langle P, Q, R \rangle_{\mathbb{R}}$, $\mathfrak{sl}(2, \mathbb{R}) = \langle F, G, H \rangle_{\mathbb{R}}$. P, Q, R, F, G, H are 4×4 matrices of coefficients

$$\begin{aligned} [P, Q] &= 2R, \quad [F, G] = H, \quad [H, F] = 2F, \quad [G, H] = 2G, \\ [P, F] &= Q, \quad [Q, G] = P, \quad [P, H] = P, \quad [H, Q] = Q, \end{aligned}$$

Parametrization of the real Jacobi group

$G_1^J(\mathbb{R})$:

EZ-coordinates: $(x, y, \theta, \lambda, \mu, \kappa)$

S-coordinates: $(x, y, \theta, p, q, \kappa)$, $x + iy \in \mathcal{X}_1^J$

(x, y, θ) - from Iwasawa (=pre-Iwasawa) decomposition of $\mathrm{SL}(2, \mathbb{R})$

Action of $\mathrm{SL}(2, \mathbb{R})$ on \mathcal{X}_1^J compatible with linear fractional transf.

$G_n^J(\mathbb{R})$, $\mathbb{N} \ni n > 1$:

(x, y, X, Y) , $x + iy \in \mathcal{X}_n^J$, $X - iY \in \mathrm{U}(n)$ from pre-Iwasawa decomposition of $\mathrm{Sp}(n, \mathbb{R})$

Action of $\mathrm{Sp}(n, \mathbb{R})$ on \mathcal{X}_n^J compatible with linear fractional transf.

The composition law of the 3-dimensional Heisenberg group $H_1(\mathbb{R})$

$$(\lambda, \mu, \kappa)(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \lambda'\mu).$$

The Lie algebra of H_1 in the space $M(4, \mathbb{R})$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g^{-1} \mathbf{d} g = P\lambda^p + Q\lambda^q + R\lambda^r, \quad \mathbf{d} gg^{-1} = P\rho^p + Q\rho^q + R\rho^r,$$

$$\left\{ \begin{array}{l} \lambda^p = \mathbf{d} \lambda, \\ \lambda^q = \mathbf{d} \mu, \\ \lambda^r = \mathbf{d} \kappa - \lambda \mathbf{d} \mu + \mu \mathbf{d} \lambda \end{array} \right. ;$$

$$\left\{ \begin{array}{l} L^p = \partial_\lambda - \mu \partial_\kappa, \\ L^q = \partial_\mu + \lambda \partial_\kappa, \\ L^r = \partial_\kappa \end{array} , \right.$$

$$\left\{ \begin{array}{l} \rho^p = \mathbf{d} \lambda, \\ \rho^q = \mathbf{d} \mu, \\ \rho^r = \mathbf{d} \kappa - \mu \mathbf{d} \lambda + \lambda \mathbf{d} \mu \end{array} ; \right.$$

$$\left\{ \begin{array}{l} R^p = \partial_\lambda + \mu \partial_\kappa, \\ R^q = \partial_\mu - \lambda \partial_\kappa \\ R^r = \partial_\kappa \end{array} \right.$$

$$g_{H_1}^L(\lambda, \mu, \kappa) = (\lambda^p)^2 + (\lambda^q)^2 + (\lambda^r)^2 = \mathbf{d} \lambda^2 + \mathbf{d} \mu^2 + (\mathbf{d} \kappa - \lambda \mathbf{d} \mu + \mu \mathbf{d} \lambda)^2$$

$$\exp(\lambda P + \mu Q + \kappa R)(\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda \mu_0 - \mu \lambda_0).$$

$$P^* = \partial_\lambda + \mu \partial_\kappa, \quad Q^* = \partial_\mu - \lambda \partial_\kappa, \quad R^* = \partial_\kappa.$$

Boyer & Galicki, 2008; Boyer 2009: Sasakian Geometry of Heisenberg group.

$$\text{SL}(2, \mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow g = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1^J(\mathbb{R}), g^{-1} = \begin{pmatrix} d & 0 & -b \\ 0 & 1 & 0 \\ -c & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathfrak{sl}(2, \mathbb{R}) = \langle F, G, H \rangle_{\mathbb{R}}$ matrices in $M(4, \mathbb{R})$

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\mathfrak{sl}(2, \mathbb{R}) = \langle F, G, H \rangle_{\mathbb{R}}$ matrices in $M(2, \mathbb{R})$

$$F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$g^{-1} dg = F\lambda^f + G\lambda^g + H\lambda^h, \quad dg g^{-1} = F\rho^f + G\rho^g + H\rho^h.$$

$$\left\{ \begin{array}{l} \lambda^f = d \mathbf{d} b - b \mathbf{d} d, \\ \lambda^g = -c \mathbf{d} a + a \mathbf{d} c, \\ \lambda^h = d \mathbf{d} a - b \mathbf{d} c = c \mathbf{d} b - a \mathbf{d} d \end{array} \right. ; \quad \left\{ \begin{array}{l} \rho^f = -b \mathbf{d} a + a \mathbf{d} b, \\ \rho^g = d \mathbf{d} c - c \mathbf{d} d, \\ \rho^h = d \mathbf{d} a - c \mathbf{d} b \end{array} \right..$$

Iwasawa decomposition $M = NAK$

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad y > 0.$$

$$a = y^{1/2} \cos \theta - xy^{-1/2} \sin \theta,$$

$$b = y^{1/2} \sin \theta + xy^{-1/2} \cos \theta,$$

$$c = -y^{-1/2} \sin \theta,$$

$$d = y^{-1/2} \cos \theta,$$

$$x = \frac{ac + bd}{d^2 + c^2}; \quad y = \frac{1}{d^2 + c^2}, \quad \sin \theta = -\frac{c}{\sqrt{c^2 + d^2}}, \quad \cos \theta = \frac{d}{\sqrt{c^2 + d^2}}.$$

$M, M', M_1 \in \text{SL}(2, \mathbb{R}), MM' = M_1$. Action of $M \in \text{SL}(2, \mathbb{R})$ on (x', y', θ')

$$x_1 + i y_1 = \frac{(ax' + b)(cx' + d) + acy'^2 + i y'}{\Lambda}, \quad \Lambda = (cx' + d)^2 + (cy')^2,$$

$$\sin \theta^* = \frac{(cx' + d) \sin \theta' - cy' \cos \theta'}{\sqrt{\Lambda}}, \quad \cos \theta^* = \frac{cy' \sin \theta' + (cx' + d) \cos \theta'}{\sqrt{\Lambda}}$$

Left (right)-invariant one-forms λ -s w. r. action (respectively ρ -s)

$$\left\{ \begin{array}{l} \lambda^f = \frac{dx}{y} \cos^2 \theta + \frac{dy}{2y} \sin 2\theta + d\theta, \\ \lambda^g = -\frac{dx}{y} \sin^2 \theta + \frac{dy}{2y} \sin 2\theta - d\theta, \\ \lambda^h = -\frac{dx}{2y} \sin 2\theta + \frac{dy}{2y} \cos 2\theta \end{array} \right. ; \left\{ \begin{array}{l} \rho^f = dx - \frac{x}{y} dy + \frac{x^2+y^2}{y} d\theta, \\ \rho^g = -\frac{d\theta}{y}, \\ \rho^h = \frac{dy}{2y} - \frac{x}{y} d\theta. \end{array} \right. .$$

$$L^f = y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} + \sin^2 \theta \frac{\partial}{\partial \theta},$$

$$L^g = y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} - \cos^2 \theta \frac{\partial}{\partial \theta},$$

$$L^h = -2y \sin 2\theta \frac{\partial}{\partial x} + 2y \cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta},$$

$$\lambda_1 = \sqrt{\alpha}(\lambda^f + \lambda^g) = \frac{\sqrt{\alpha}}{y}(\cos 2\theta dx + \sin 2\theta dy),$$

$$\lambda_2 = 2\sqrt{\alpha}\lambda^h = \frac{\sqrt{\alpha}}{y}(-\sin 2\theta dx + \cos 2\theta dy),$$

$$\lambda_3 = \sqrt{\beta}(\lambda^f - \lambda^g) = \sqrt{\beta}\left(\frac{dx}{y} + 2d\theta\right).$$

$$\langle \lambda_i | L^j \rangle = \delta_{ij}, \quad i, j = 1, 2, 3.$$

$$[\lambda_1, \lambda_2] = -4 \frac{\alpha}{\sqrt{\beta}} \lambda_3, \quad [\lambda_2, \lambda_3] = 4\sqrt{\beta} \lambda_1, \quad [\lambda_3, \lambda_1] = \sqrt{\beta} \lambda_2,$$

$$L^1 = \frac{1}{2\sqrt{\alpha}}(L^f + L^g) = \frac{1}{\sqrt{\alpha}}(y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} - \frac{1}{2} \cos 2\theta \frac{\partial}{\partial \theta}),$$

$$L^2 = \frac{1}{2\sqrt{\alpha}}L^h = \frac{1}{\sqrt{\alpha}}(-y \sin 2\theta \frac{\partial}{\partial x} + y \cos 2\theta \frac{\partial}{\partial y} + \frac{1}{2} \sin 2\theta \frac{\partial}{\partial \theta}),$$

$$L^3 = \frac{1}{2\sqrt{\beta}}(L^f - L^g) = \frac{1}{2\sqrt{\beta}}\frac{\partial}{\partial \theta}.$$

$$f^* = F_1^* = \frac{\partial}{\partial x}, \quad h^* = H_1^* = 2(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}),$$

$$g^* = G_1^* - y \frac{\partial}{\partial \theta} = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - y\frac{\partial}{\partial \theta},$$

$$v = \sqrt{\alpha}(F + G), \quad h1 = 2\sqrt{\alpha}H, \quad w = \sqrt{\beta}(F - G),$$

$$v^* = \sqrt{\alpha}[(1 - x^2 + y^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - y\frac{\partial}{\partial \theta}],$$

$$h1^* = 4\sqrt{\alpha}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \quad w^* = \sqrt{\beta}[(1 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + y\frac{\partial}{\partial \theta}].$$

Invariant metric on $\text{SL}(2, \mathbb{R})$

Proposition

$$\begin{aligned} d s_{\text{SL}(2, \mathbb{R})}^2(x, y, \theta) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ &= \alpha \frac{dx^2 + dy^2}{y^2} + \beta \left(\frac{dx}{y} + 2d\theta \right)^2 \\ &= \frac{(\alpha + \beta) dx^2 + \alpha dy^2}{y^2} + 4\beta d\theta^2 + 4\frac{\beta}{y} dx d\theta. \end{aligned}$$

$$g_{\text{SL}(2, \mathbb{R})}(x, y, \theta) = \begin{pmatrix} g_{xx} & 0 & g_{x\theta} \\ 0 & g_{yy} & 0 \\ g_{\theta x} & 0 & g_{\theta\theta} \end{pmatrix}, \quad g_{xx} = \frac{\alpha + \beta}{y^2}, \quad g_{yy} = \frac{\alpha}{y^2}, \\ g_{\theta\theta} = 4\beta, \quad g_{x\theta} = 2\frac{\beta}{y}.$$

L^1, L^2, L^3 are orthonormal with respect to the metric. The vector fields $v^*, h1^*, w^*$ are Killing vectors of the invariant metric on $\text{SL}(2, \mathbb{R})$

Proposition - continuation

$$\begin{aligned} & -2(\alpha + \beta)X^2 + 2(\alpha + \beta)y\partial_x X^1 + 4\beta y^2\partial_x X^3 = 0, \\ & \alpha\partial_x X^2 + (\alpha + \beta)\partial_y X^1 + 2\beta y\partial_y X^3 = 0, \\ & -2\beta X^2 + 2\beta y\partial_x X^1 + (\alpha + \beta)\partial_\theta X^1 + 2\beta y\partial_\theta X^3 = 0, \\ & -X^2 + y\partial_y X^2 = 0, \\ & 2\beta y\partial_y X^1 + 4\beta y^2\partial_y X^3 + \alpha\partial_\theta X^2 = 0, \\ & \beta\partial_\theta X^1 + 2\beta y\partial_\theta X^3 + \partial_\theta X^1 + 2\beta y\partial_\theta X^3 = 0. \end{aligned}$$

Proposition - continuation

(L^3, λ_3, Φ') - almost contact structure on $\text{SL}(2, \mathbb{R})$

$$\Phi' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -\frac{1}{2y} & 0 \end{pmatrix}$$

Contact distribution $\mathcal{D} := \text{Ker}(\eta) = \langle V_1, V_2 \rangle = \langle \frac{\partial}{\partial x} - \frac{1}{2y} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y} \rangle$,

$(\text{SL}(2, \mathbb{R})(x, y, \theta), \mathcal{X}_1, d s_{\mathcal{D}}^2)$ is a sub-Riemannian manifold and

$$d s_{\text{SL}(2, \mathbb{R})}^2(x, y, \theta) = d s_{\mathcal{X}_1}^2 + \lambda_3^2,$$

$d s_{\mathcal{X}_1}^2$ is the (Beltrami) Kähler metric

$$d s_{\mathcal{X}_1}^2 = \lambda_1^2 + \lambda_2^2 = \alpha \frac{dx^2 + dy^2}{y^2}$$

The invariant vectors orthonormal orthonormal w.r.t. the metric

$$l_0^1 = \frac{y}{\sqrt{\alpha}} \frac{\partial}{\partial x}, \quad l_0^2 = \frac{y}{\sqrt{\alpha}} \frac{\partial}{\partial y}.$$

Proposition - continuation & end

The manifold $\text{SL}(2, \mathbb{R})$ admits the homogenous contact metric structure $(\lambda_3, L^3, \Phi', g_{\text{SL}(2, \mathbb{R})})$. The group $\text{SL}(2, \mathbb{R})$ has a K-contact structure associated with $\xi = L^3$, and is a **Sasaki manifold** with the Riemann cone $(C(\text{SL}(2, \mathbb{R})), \omega, \bar{g})$ with respect to the metric

$$\bar{g}(r, x, y, \theta) := d r^2 + r^2 g_{\text{SL}(2, \mathbb{R})}(x, y, \theta), \quad \omega := d(r^2 \lambda_3).$$

Comment Explicit invariant metrics on $\text{SL}(2, \mathbb{R})$ in coordinates different of (x, y, θ) appears in Kowalski 1983. A different form of the invariant metric on $\text{SL}(2, \mathbb{R})$ appears in the context of **BCV spaces (1898, 1928, 1962, ...)**. See Patrangenaru 1996 in the context of Milnor approach to invariant metrics on 3-dimensional groups.

Remark The Siegel upper half-plane \mathcal{X}_1 admits a realization as noncompact Hermitian symmetric space

$$\mathcal{X}_1 = \frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)} \approx \frac{\text{SU}(1, 1)}{\text{U}(1)}.$$

\mathcal{X}_1 is a symmetric, naturally reductive space.

Left-invariant one forms on $G_1^J(\mathbb{R})$ in $(x, y, \theta, p, q, \kappa)$

$$g^{-1} \, dg = \lambda^F F + \lambda^G G + \lambda^H H + \lambda^P P + \lambda^Q Q + \lambda^R R.$$

$$\lambda^F = \lambda^f, \quad \lambda^G = \lambda^g, \quad \lambda^H = \lambda^h,$$

$$\lambda^P = d\lambda - p \, da - q \, dc = c \, dq + a \, dp = \lambda^p - \lambda \lambda^h - \mu \lambda^g,$$

$$= -y^{-\frac{1}{2}} \sin \theta \, dq + (y^{\frac{1}{2}} \cos \theta - xy^{-\frac{1}{2}} \sin \theta) \, dp,$$

$$\lambda^Q = d \, dq + b \, dp = \lambda^q - p \, db - q \, dd,$$

$$= y^{-\frac{1}{2}} \cos \theta \, dq + (y^{\frac{1}{2}} \sin \theta + xy^{-\frac{1}{2}} \cos \theta) \, dp,$$

$$\lambda^R = d\kappa - p \, dq + q \, dp = \lambda^r + \lambda^2 \lambda^f - \mu^2 \lambda^g - 2\lambda \mu \lambda^h,$$

Left-invariant vector fields L^α , $\langle \lambda^\beta | L^\alpha \rangle = \delta_{\alpha,\beta}$

Proposition

$$L^F = L^f, \quad L^G = L^g, \quad L^H = L^h, \quad L^P = L_0^P + L_+^P, \quad L^Q = L_0^Q + L_+^Q,$$

$$L_0^P = d \frac{\partial}{\partial p} - b \frac{\partial}{\partial q} = \frac{\cos \theta}{y^{\frac{1}{2}}} \frac{\partial}{\partial p} - \frac{x \cos \theta + y \sin \theta}{y^{\frac{1}{2}}} \frac{\partial}{\partial q},$$

$$L_+^P = -(pb + qd) \frac{\partial}{\partial \kappa} = -\frac{1}{y^{1/2}} [p(x \cos \theta + y \sin \theta) + q \cos \theta] \frac{\partial}{\partial \kappa},$$

$$L_0^Q = -c \frac{\partial}{\partial p} + a \frac{\partial}{\partial q} = \frac{\sin \theta}{y^{\frac{1}{2}}} \frac{\partial}{\partial p} + \frac{y \cos \theta - x \sin \theta}{y^{\frac{1}{2}}} \frac{\partial}{\partial q},$$

$$L_+^Q = (pa + qc) \frac{\partial}{\partial \kappa} = \frac{1}{y^{1/2}} [p(y \cos \theta - x \sin \theta) - q \sin \theta] \frac{\partial}{\partial \kappa},$$

$$L^R = \frac{\partial}{\partial \kappa}$$

$L^F - L^R$ - same comm. relations as $F - R \in \mathfrak{g}_1^J(\mathbb{R})$.

Notation

$$\lambda_4 := \sqrt{\gamma} \lambda^P, \quad \lambda_5 := \sqrt{\gamma} \lambda^Q, \quad \lambda_6 := \sqrt{\delta} \lambda^R,$$

$$L^4 := \frac{1}{\sqrt{\gamma}} L^P, \quad L^5 := \frac{1}{\sqrt{\gamma}} L^Q, \quad L^6 := \frac{1}{\sqrt{\delta}} L^R,$$

$$L_0^1 := \frac{y}{\sqrt{\alpha}} \left(\cos 2\theta \frac{\partial}{\partial x} + \sin 2\theta \frac{\partial}{\partial y} \right), \quad L_0^2 := \frac{y}{\sqrt{\alpha}} \left(-\sin 2\theta \frac{\partial}{\partial x} + \cos 2\theta \frac{\partial}{\partial y} \right),$$

$$L_0^4 := \frac{1}{\sqrt{\gamma}} L_0^P, \quad L_0^5 := \frac{1}{\sqrt{\gamma}} L_0^Q, \quad L_0^6 := \frac{1}{\sqrt{\delta}} L^R.$$

Commutation relations

The vector fields L^i , $i = 1, \dots, 6$ verify the commutations relations

$$\begin{array}{lll} [L^1, L^2] = -\frac{\sqrt{\beta}}{\alpha} L^3 & [L^2, L^3] = \frac{1}{2\sqrt{\beta}} L^1 & [L^3, L^1] = \frac{1}{\sqrt{\beta}} L^2 \\ [L^1, L^4] = -\frac{1}{2\sqrt{\alpha}} L^5 & [L^1, L^5] = -\frac{1}{2\sqrt{\alpha}} L^4 & [L^1, L^6] = 0 \\ [L^2, L^4] = -\frac{1}{2\sqrt{\alpha}} L^5 & [L^2, L^5] = \frac{1}{2\sqrt{\alpha}} L^5 & [L^2, L^6] = 0 \\ [L^3, L^4] = -\frac{1}{2\sqrt{\alpha}} L^5 & [L^3, L^5] = \frac{1}{2\sqrt{\beta}} L^4 & [L^3, L^6] = 0 \\ [L^4, L^5] = \frac{2\sqrt{\delta}}{\gamma} L^6 & [L^4, L^6] = 0 & [L^5, L^6] = 0 \end{array}$$

Invariant metric on \mathcal{X}_1^J

Proposition

Kähler balanced metric, invariant to $G_0^J(\mathbb{R})$

$$ds_{\mathcal{X}_1^J}^2(\tau, z) = -c_1 \frac{d\tau d\bar{\tau}}{(\tau - \bar{\tau})^2} + \frac{2ic_2}{\tau - \bar{\tau}} (dz - p d\tau) \times cc, \quad p = \frac{z - \bar{z}}{\tau - \bar{\tau}}$$

$$\begin{aligned} ds_{\mathcal{X}_1^J}^2(x, y, p, q) &= c_1 \frac{dx^2 + dy^2}{4y^2} + \frac{c_2}{y} [(x^2 + y^2) dp^2 + dq^2 + 2x dp dq] \\ &= c_1 \frac{dx^2 + dy^2}{4y^2} + c_2 \frac{x^2 + y^2}{y} \left[(dp + \frac{x}{x^2 + y^2} dq)^2 + (\frac{y dq}{x^2 + y^2})^2 \right] \end{aligned}$$

$$\begin{aligned} ds_{\mathcal{X}_1^J}^2(x, y, \xi, \eta) &= c_1 \frac{dx^2 + dy^2}{4y^2} + \\ &+ \frac{c_2}{y} \left[d\xi^2 + d\eta^2 + (\frac{\xi}{y})^2 (dx^2 + dy^2) - 2\frac{\eta}{y} (dx d\xi + dy d\eta) \right] \end{aligned}$$

Proposition - continuation

$$\frac{c_1}{4} := \alpha, \ c_2 := \gamma,$$

$$g_{\mathcal{X}_1^J} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 \\ 0 & 0 & g_{pp} & g_{pq} \\ 0 & 0 & g_{pq} & g_{qq} \end{pmatrix}, \quad g_{xx} = g_{yy} = \frac{\alpha}{y^2} \quad g_{pp} = \gamma \frac{x^2 + y^2}{y} \\ g_{qq} = \frac{\gamma}{y} \quad g_{pq} = \gamma \frac{x}{y}.$$

$$ds_{\mathcal{X}_1^J}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_4^2 + \lambda_5^2.$$

The vector fields L_0^j dual orthogonal to the invariant one-forms λ_i ,
 $\langle \lambda_i | L_0^j \rangle = \delta_{ij}$, $i, j = 1, 2, 4, 5$. The metric is orthonormal with respect to
the vector fields $L_0^1, L_0^2, L_0^4, L_0^5$.

Proposition - continuation

Fundamental vector fields are Killing vectors

$$-X^2 + y\partial_x X^1 = 0,$$

$$\partial_x X^2 + \partial_y X^1 = 0,$$

$$c_2[(x^2 + y^2)\partial_x X^3 + x\partial_x X^4] + \frac{c_1}{4y}\partial_p X^1 = 0,$$

$$\frac{c_1}{4y}\partial_q X^1 + c_2(x\partial_x X^3 + \partial_x X^4) = 0,$$

$$-X^2 + y\partial_y X^2 = 0,$$

$$c_2[(x^2 + y^2)\partial_y X^3 + \frac{x}{y}\partial_y X^4] + \frac{c_1}{y}\partial_p X^2 = 0,$$

$$c_2[x\partial_y X^3 + \partial_y X^4] + \frac{c_1}{4y}\partial_q X^2 = 0,$$

$$2xyX^1 + (-x^2 + y^2)X^2 + 2y(x^2 + y^2)\partial_p X^3 + 2xy\partial_p X^4 = 0,$$

$$yX^1 - xX^2 + xy\partial_p X^3 + y\partial_p X^4 + y(x^2 + y^2)\partial_q X^3 + xy\partial_q X^4 = 0,$$

$$-X^2 + 2xy\partial_q X^3 + 2y\partial_q X^4 = 0.$$

"historical" comment

Comment In 1984 Berndt considered the closed two-form $\Omega = d\bar{d}f$ on \mathcal{X}_1^J , $G^J(\mathbb{R})_0$ -invariant, obtained from the Kähler potential

$$f(\tau, z) = c_1 \log(\tau - \bar{\tau}) - i c_2 \frac{(z - \bar{z})^2}{\tau - \bar{\tau}}, \quad c_1, c_2 > 0, \quad (3.12)$$

where $c_1 = \frac{k}{2}$, $c_2 = 2\mu$ comparatively to our formula. Formula (3.12) is presented by Berndt as "communicated to the author by Kähler". In § 36 of his last paper in 1992, Kähler argues how to choose the potential as in (3.12). Yang calculated the metric on \mathcal{X}_n^J , invariant to the action of $G_n^J(\mathbb{R})_0$. The equivalence of the metric of Yang with the metric obtained via coherent states on \mathcal{D}_n^J in S. B. 2012.

Invariant metric on $\tilde{\mathcal{X}}_1^J$ in (x, y, p, q, κ) -coordinates

Proposition

$$ds_{\tilde{\mathcal{X}}_1^J}^2 = ds_{\mathcal{X}_1^J}^2(x, y, p, q) + \lambda_6^2(p, q, \kappa)$$

$$= \frac{\alpha}{y^2}(dx^2 + dy^2) + [\frac{\gamma}{y}(x^2 + y^2) + \delta q^2] dp^2 + (\frac{\gamma}{y} + \delta p^2) dq^2 + \delta d\kappa^2$$

$$+ 2(\gamma \frac{x}{y} - \delta pq) dp dq + 2\delta(q dp d\kappa - p dq d\kappa)$$

$$g_{\tilde{\mathcal{X}}_1^J} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 & 0 \\ 0 & 0 & g'_{pp} & g'_{pq} & g'_{p\kappa} \\ 0 & 0 & g'_{qp} & g'_{qq} & g'_{q\kappa} \\ 0 & 0 & g'_{\kappa p} & g'_{\kappa q} & g'_{\kappa\kappa} \end{pmatrix}, \quad \begin{aligned} g'_{pq} &= g_{pq} - \delta pq, & g'_{p\kappa} &= \delta q, \\ g'_{pp} &= g_{pp} + \delta q^2, & g'_{qq} &= g_{qq} + \delta p^2, \\ g'_{q\kappa} &= -\delta p, & g'_{\kappa\kappa} &= \delta, \end{aligned}$$

Proposition- continuation

The metric is orthonormal w.r.t. the vector fields L_0^i , $i = 1, 2$,
 L^i , $i = 4, 5, 6$.

$\tilde{\mathcal{X}}_1^J$ does not admit an almost contact structure (Φ, ξ, η) with a supposed contact form $\eta = \lambda_6$ and Reeb vector $\xi = \text{Ker}(\eta)$.

Invariant metric on $G_1^J(\mathbb{R})$ in S-coordinates

Theorem

$$\begin{aligned} ds_{G_1^J(\mathbb{R})}^2 &= \sum_{i=1}^6 \lambda_i^2 \\ &= \alpha \frac{dx^2 + dy^2}{y^2} + \beta \left(\frac{dx}{y} + 2d\theta \right)^2 \\ &+ \frac{\gamma}{y} [dq^2 + (x^2 + y^2) dp^2 + 2x dp dq] + \delta (d\kappa - p dq + q dp)^2, \end{aligned}$$

$$g_{G_1^J} = \begin{pmatrix} g_{xx} & 0 & g_{x\theta} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 & 0 & 0 \\ g_{\theta x} & 0 & g_{\theta\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & g'_{pp} & g'_{pq} & g'_{p\kappa} \\ 0 & 0 & 0 & g'_{qp} & g'_{qq} & g'_{q\kappa} \\ 0 & 0 & 0 & g'_{\kappa p} & g'_{\kappa q} & g'_{\kappa\kappa} \end{pmatrix},$$

Theorem - continuation

$$\langle \lambda_i | L^j \rangle = \delta_{ij}, i, j = 1, \dots, 6$$

The vector fields L^i , $i = 1, \dots, 6$ are orthonormal w. r. t. the metric.

- ① if $\beta, \gamma, \delta = 0$ - the Siegel upper half plane \mathcal{X}_1 ,
- ② if $\gamma, \delta = 0, \beta \neq 0$ - the group $SL(2, \mathbb{R})$,
- ③ if $\beta, \delta = 0$ - the Siegel-Jacobi half plane \mathcal{X}_1^J ,
- ④ if $\beta = 0$ - the extended Siegel Jacobi half plane $\tilde{\mathcal{X}}_1^J$,
- ⑤ if $\alpha\beta\gamma\delta \neq 0$ - the Jacobi group G_1^J .

Proposition

$(\mathcal{X}_1^J = \frac{G_1^J(\mathbb{R})}{SO(2) \times \mathbb{R}}, g_{\mathcal{X}_1^J})$ is a reductive non-symmetric manifold, **not naturally reductive w. r. t. the balanced metric.**

\mathcal{X}_1^J is not a g.o. manifold w.r.t. the balanced metric. However, **when expressed in the variables that appear in the FC-transform, it is a naturally reductive space with the metric $g_{\mathcal{X}_1} \times g_{\mathbb{R}^2}$.**

Proposition - continuation- geodesic vectors

$$\mathfrak{g}_1^J \ni X = aL^1 + bL^2 + cL^3 + dL^4 + eL^5 + fL^6$$

Table 1: Components of the geodesic vector

Nr. cr.	a	b	c	d	e	f
1	0	0	c	0	0	f
2	a	b	0	0	0	f
3	rc	0	c	$\pm rc$	0	f
4	a	0	$-a$	0	$\epsilon\sqrt{r}a$	f
5	$\epsilon_1\epsilon_2 \frac{1-r}{\sqrt{r}}e$	$\epsilon_1 e$	$-\frac{\epsilon_1\epsilon_2}{\sqrt{r}}e$	$\epsilon_2\sqrt{r}e$	e	f

$$r = \sqrt{\frac{\alpha}{\beta}}, \quad \epsilon_1^2 = \epsilon_2^2 = \epsilon^2 = 1.$$

The method - Cartan moving frame

$M = G/H$ - reductive homogeneous space. \mathfrak{g} (\mathfrak{h}) – the Lie algebra of G (H), $\rightarrow \exists \mathfrak{m}$ s.t. $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, $\mathfrak{m} \cap \mathfrak{h} = \emptyset$, $T_x M \equiv \mathfrak{m}$, $H = G_x$ — is the isotropy group at x .

$X_i, i = 1, \dots, n$ - a basis \mathfrak{g} :

$$\mathfrak{m} = \langle X_1, \dots, X_m \rangle, \quad \mathfrak{h} = \langle X_{m+1}, \dots, X_n \rangle, \quad \dim \mathfrak{m} = m.$$

Left-invariant one forms λ_i , left-invariant vector fields L^i on G :

$$g^{-1} dg = \sum_{i=1}^n \lambda_i X_i, \quad \langle \lambda_i | L^j \rangle = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Invariant metric on G : $ds_G^2 = \sum_{i=1}^n \lambda_i^2$, $g_M(L^i, L^j) = \delta_{ij}$, $i, j = 1, \dots, n$.

L_0^j - projections on M of the vector fields L^j , $j = 1, \dots, m$.

$$\langle \lambda_i | L_0^j \rangle = \delta_{ij}, \quad g_M = \sum_{i=1}^m \lambda_i^2$$

Fundamental v. f. X_i^* , $i = 1, \dots, m$ are Killing vectors for the metric g_M .

Homogeneous reductive space

Definition

$M = G/H$ homogeneous space is **reductive** $\leftrightarrow \exists \mathfrak{m}, \text{Ad}(H)\text{-invariant subspace}$

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathfrak{h} \cap \mathfrak{m} = 0, \quad (4.1a)$$

$$\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}. \quad (4.1b)$$

$$(4.1b) \implies$$

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad (4.1c)$$

and, conversely, if H is connected, then (4.1c) implies (4.1b).

$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \rightarrow M = \text{symmetric.}$

Naturally reductive spaces

Definition

$(M = G/H, g)$ - homogeneous Riemannian (or pseudo-Riemannian space) is **naturally reductive** if it is reductive and

$$B(X, [Z, Y]_{\mathfrak{m}}) + B([Z, X]_{\mathfrak{m}}, Y) = 0, \quad X, Y, Z \in \mathfrak{m}.$$

B = the non-degenerate symmetric bilinear form on \mathfrak{m} induced by the Riemannian (pseudo-Riemannian) structure on M under the natural identification of the spaces \mathfrak{m} and M_o

$$B(X, Y) = g(X^*, Y^*)_o, \quad X, Y \in \mathfrak{m},$$

$$B(X, Y) = B(\text{Ad}^{G/H}(h)X, \text{Ad}^{G/H}(h)Y), \quad \forall X, Y \in \mathfrak{m}, \quad h \in H.$$

Naturally reductive spaces - continuation

Proposition

(M, g) - homogeneous Riemannian manifold. \implies (M, g) - a naturally reductive Riemannian homogenous space $\leftrightarrow \exists$ a connected Lie subgroup G of $I(M)$ acting transitively and effectively on M and a reductive decomposition (4.1a), such that one of the following equivalent statements hold:

- (i) $g([X, Z]_{\mathfrak{m}}, Y) + g(X, [Z, Y]_{\mathfrak{m}}) = 0 \quad \forall X, Y, Z \in \mathfrak{m},$
- (ii) (*): every geodesic in M = the orbit of a one-parameter subgroup of $I(M)$ generated by some $X \in \mathfrak{m}.$

3-dimensional Naturally reductive spaces - 1983

Theorem

A 3-dimensional complete, simply connected N. R. Riemannian manifold (M, g) is either:

- (a) a symmetric space realized by the real forms: \mathbb{R}^3 , S^3 or the Poincaré half-space \mathcal{H}^3 , and $S^2 \times \mathbb{R}$, $\mathcal{H}^2 \times \mathbb{R}$, or
- (b) a non-symmetric space isometric to one of the following Lie groups with a suitable left-invariant metric:
 - (b1) $SU(2)$,
 - (b2) $\widetilde{SL}(2, \mathbb{R})$ with a special left-invariant metric
 - (b3) the 3-dimensional Heisenberg group H_1 , a left-invariant metric.

The Poincaré half-space $\mathcal{H}^n = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x_1 > 0$, with the metric proportional with $ds^2 := x_1^{-2} \sum_{i=1}^n (dx_i)^2$. Left-invariant metric $H_1 = \mathbb{R}^3[x, y, z]$, $ds_{H_1}^2 = \frac{1}{b}(dx^2 + dz^2 + (dy - x dz)^2)$, $b \in \mathbb{R}_+$.

4-dimensional Naturally reductive spaces

Theorem

(M, g) - four-dimensional simply connected n. r. Riemannian manifold. \Rightarrow

(M, g) is either symmetric or

it is a *Riemannian product of the naturally reductive spaces of dimension 3 of type (b) appearing in Theorem times \mathbb{R} .*

In the last cases, (M, g) is not locally symmetric.

g. o. spaces - 1991 Kowalski, Vanhecke

(**) Each geodesic of $(M, g) = G/H$ is an orbit of a one parameter group of isometries $\{\exp tZ\}$, $Z \in \mathfrak{g}$.

Definition

$X \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector if the curve $\gamma(t) = (\exp tX)(p)$ is a geodesic.

Riemannian homogeneous spaces with property (**) – g. o. spaces.
 (geodesics are orbits). All naturally reductive spaces are g. o. mfd's.

Proposition

(Kowalski and Vanhecke - Geodesic Lemma) $M = G/H$ - homog. Riem. mfd. $X \in \mathfrak{g} \setminus \{0\}$ is geodesic $\rightarrow: B([X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0, \forall Y \in \mathfrak{m}$.

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MANY THANKS