Geometrisation in supersymmetry-invariant cohomology – supergerbes, κ-symmetry, and all that

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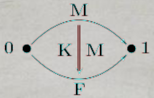

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Geometrisation of de Rham cocycles (with integer periods) in the form of so-called abelian gerbes has long been known as not only necessary for a rigorous definition of the lagrangean (σ-model) dynamics of charged pointlike particles, strings and branes and its geometric (pre-)quantisation but also exceptionally useful in the canonical description of its symmetries and dualities, classification and field-theoretic realisation of the corresponding defects, as well as in a cohomological description of obstructions against their gauging and classification of the ensuing gauged σ-models.

In the talk, a natural geometrisation scheme for cocycles in the supersymmetry-invariant refinement of the de Rham cohomology on (a class of) homogeneous spaces of supersymmetry Lie supergroups shall be postulated, in close structural analogy with its Graßmann-even ancestor, and illustrated on examples motivated by superstring-theoretic considerations. The geometrisation, based on the classical correspondence between the Cartan–Eilenberg cohomology of the supersymmetry Lie supergroup and the Chevalley–Eilenberg cohomology of its tangent Lie superalgebra, in conjunction with the cohomological description of (equivalence classes of) Lie-superalgebra extensions and a moment-map criterion for their integrability, gives rise to higher-(super)geometric objects termed supergerbes. Various (anticipated) equivariance properties of the supergerbes shall be indicated, including the physically fundamental κ-symmetry, and – time permitting – a higher-geometric variant of the İnönü–Wigner contraction mechanism (for the supersymmetry Lie superalgebras) shall be outlined.

Goal:

Extending the **gerbe-theoretic approach** of the bosonic two-dimensional σ-model to (super-)σ-models with **homogeneous spaces of Lie supergroups** as target spaces, in a manner consistent with **rigid and local supersymmetry**.

Discussion based upon

1. arXiv:1706.05682
2. arXiv:1808.04470
3. arXiv:1810.00856
4. arXiv:1905.05235
5. arXiv:1909.xxxxx (in writing)

Part I

Learning from life without spin

Point of departure: The non-linear σ-model

Given a *closed* m_fold Ω_p of $\dim \Omega_p = p + 1$ (the **worldvolume**) & a metric m_fold (M, g) with $\underset{(p+2)}{\mathrm{H}} \in Z^{p+2}_{\mathrm{dR}}(M)$ (the **target space**), consider the theory of maps determined by (the PLA for) the **Dirac–Feynman amplitudes**

$$\mathcal{A}_{\mathrm{DF}} \equiv \exp\left(\tfrac{\mathrm{i}}{\hbar}\, S^{(\mathrm{NG})}_{\sigma}[\cdot]\right) \; : \; [\Omega_p, \mathcal{M}] \longrightarrow \mathrm{U}(1)$$

$$S^{(\mathrm{NG})}_{\sigma}[x] = \int_{\Omega_p} \sqrt{|\det x^* \mathrm{g}|} + \int_{\Omega_p} x^* \left(\mathrm{d}^{-1} \underset{(p+2)}{\mathrm{H}}\right),$$

describing minimal embeddings deformed by Lorentz-type forces sourced by a Maxwell-type $(p+2)$-form field $\underset{(p+2)}{\mathrm{H}}$.

Applications: mainly the critical bosonic string (and membrane) theory, but also the effective FT of (certain slow) collective excitations of spin chains

Problem: May need $[\underset{(p+2)}{\mathrm{H}}]_{\mathrm{dR}} \neq 0$ (*e.g.*, for conformality), and so

$$\neg\exists_{\underset{(p+1)}{\mathrm{B}} \in \Omega^{p+1}(M)} : \mathsf{d}\underset{(p+1)}{\mathrm{B}} = \underset{(p+2)}{\mathrm{H}}$$

E.g., $(M, \mathrm{g}) = \big(\mathrm{G}, \kappa_{\mathfrak{g}} \circ (\theta_{\mathrm{L}} \otimes \theta_{\mathrm{L}})\big) \implies \underset{(3)}{\mathrm{H}} = \lambda\, \kappa_{\mathfrak{g}} \circ \big(\theta_{\mathrm{L}} \wedge \theta_{\mathrm{L}} \wedge \theta_{\mathrm{L}}\big)$

and the Cartan 3-form $\underset{(3)}{\mathrm{H}}$ generates $H^3_{\mathrm{dR}}(\mathrm{G})$ for G 1-connected

But QM à la Dirac & Feynman requires that we compare amplitudes for cobordant trajectories!

Conclusion: Need $S_\sigma^{(\mathrm{NG})}$ with critical points (the EL eq^{n}s) as for $[\underset{(p+2)}{\mathrm{H}}]_{\mathrm{dR}} = 0$ but s.t. $\mathcal{A}_{\mathrm{DF}}$ is well-defined $\forall\, x(\Omega_p) \in \ker \partial_M$.

This calls for the use of **Cheeger–Simons differential characters**

$$\mathrm{Hol}_{\mathcal{G}^{(p)}} \in \mathrm{Hom}(Z_{p+1}, \mathrm{U}(1)) \text{ s.t. } \mathrm{Hol}_{\mathcal{G}^{(p)}} \circ \partial_M(\cdot) = \exp\Big(\tfrac{\mathrm{i}}{\hbar} \int_{(\cdot)} \underset{(p+2)}{\mathrm{H}}\Big)$$

Solution: Fix an arbitrary *good* open cover $\mathcal{O}_M = \{\mathcal{O}_i\}_{i\in I}$ & a tesselation $\triangle_{\Omega_p} = \mathfrak{T}_{p+1} \sqcup \mathfrak{T}_p \sqcup \cdots \sqcup \mathfrak{T}_0$ of Ω_p subordinate to it for $x \in [\Omega_p, M]$, *i.e.*, s.t.

$$\exists_{\iota.\in \mathrm{Map}(\triangle_{\Omega_p}, I)}\ \forall_{\tau\in\triangle_{\Omega_p}}\ :\ x(\tau) \subset \mathcal{O}_{\iota_\tau}\ ,$$

and pull back, along x, a resolution/trivialisation of $\underset{(p+2)}{\mathrm{H}}$ over $\mathcal{O}_M$.

E.g., use $\underset{(2)}{b} = (B_i, A_{ij}, g_{ijk}) \in \Omega^2(\mathcal{O}_i) \times \Omega^1(\mathcal{O}_{ij}) \times \mathrm{U}(1)_{\mathcal{O}_{ijk}}$ s.t.

$$\underset{(3)}{\mathrm{H}}\restriction_{\mathcal{O}_i} = \mathrm{d}B_i\ , \qquad (B_j - B_i)\restriction_{\mathcal{O}_{ij}} = \mathrm{d}A_{ij}\ , \qquad (A_{jk} - A_{ik} + A_{ij})\restriction_{\mathcal{O}_{ijk}} = \mathrm{i}\,\mathrm{d}\log g_{ijk}$$

to write (for $x_\tau \equiv x\restriction_\tau$)

$$S_\sigma^{(\mathrm{NG}),\mathrm{top}}[x] = \sum_{p\in\mathfrak{T}_2} \Big[\int_p x_p^* B_{\iota_p} + \sum_{e\in\partial p} \Big(\int_e x_e^* A_{\iota_p \iota_e} - \mathrm{i} \sum_{v\in\partial e} \varepsilon_{ev} \log g_{\iota_p\iota_e\iota_v}(x(v)) \Big) \Big]\ ,$$

with $\mathcal{A}_{\mathrm{DF}}$ well-defined iff $\check{\delta} g_{ijkl} = 1$, so that $D\underset{(2)}{b} = (\underset{(3)}{\mathrm{H}}\restriction_{\mathcal{O}_i}, 0)$ and $\mathrm{Per}(\underset{(3)}{\mathrm{H}}) \subset 2\pi\mathbb{Z}$ (Dirac's quantisation of charge)

Upshot: As in the Clutching Theorem, the DB $(p+1)$-cocycle $\underset{(p+1)}{b}$ geometrises as an abelian bundle p-gerbe $\mathcal{G}_p$ of $\mathrm{curv}\,(\mathcal{G}_p) = \underset{(p+2)}{\mathrm{H}}$:

$$
\begin{tikzcd}
\delta_{\mathrm{Y}}\mathcal{G}_{-1} = 1 \arrow[d] & \cdots & \mathcal{G}_{p-3} : \delta_{\mathrm{Y}}\mathcal{G}_{p-2} \cong \mathcal{I}_0^{p-2} \arrow[d] & \mathcal{G}_{p-2} : \delta_{\mathrm{Y}}\mathcal{G}_{p-1} \cong \mathcal{I}_0^{p-1} \arrow[d] & \mathcal{G}_{p-1},\ \mathrm{curv}\,(\mathcal{G}_{p-1}) = \delta_{\mathrm{Y}}\mathrm{B} \arrow[d] & \mathcal{I}_{\mathrm{B}}^p \arrow[d] \\
\mathrm{Y}^{[p+3]}M & \cdots & \mathrm{Y}^{[4]}M \arrow[r, "\mathrm{pr}_{ijk}"] & \mathrm{Y}^{[3]}M \arrow[r, "\mathrm{pr}_{ij}"] & \mathrm{Y}^{[2]}M \arrow[r, "\mathrm{pr}_i"] & \mathrm{Y}M \arrow[d, "\pi_{\mathrm{Y}M}"'] \\
 & & & & \pi_{\mathrm{Y}M}^* \underset{(p+2)}{\mathrm{H}} = \mathrm{dB} & M
\end{tikzcd}
$$

and $\mathcal{A}_{\mathrm{DF}}^{(\mathrm{NG}),\mathrm{top}}[x] \equiv \mathrm{Hol}_{\mathcal{G}_p}(x(\Omega_p)) = \iota_p([x^*\mathcal{G}_p])$ for a canonical group isomorphism $\iota_p \ : \ \mathcal{W}^{p+2}(\Omega_p;0) \xrightarrow{\cong} \mathrm{U}(1)$.

E.g., an abelian bundle 1-gerbe $\mathcal{G}_1$ [Murray & Stevenson '94-'99]

$$
\begin{tikzcd}
\mu_L \;:\; \mathrm{pr}_{1,2}^*L\otimes\mathrm{pr}_{2,3}^*L \xrightarrow{\;\cong\;} \mathrm{pr}_{1,3}^*L \arrow[d] & & \mathbb{C}^\times \arrow[r] & L \arrow[d, "\pi_L"] & \mathcal{I}^1_{\mathrm{B}} \arrow[d] \\
\mathsf{Y}^{[3]}M \arrow[rrr, shift left=2, "\mathrm{pr}_{1,2}"] \arrow[rrr, "\mathrm{pr}_{2,3}" description] \arrow[rrr, shift right=2, "\mathrm{pr}_{1,3}"'] & & & \mathsf{Y}^{[2]}M \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & \mathsf{Y}M \arrow[d, "\pi_{\mathsf{Y}M}"] \\
(\mathrm{pr}_2^* - \mathrm{pr}_1^*)\mathrm{B} = \mathrm{curv}(\nabla_L) & & & \pi_{\mathsf{Y}M}^*\mathrm{H}_{(3)} = \mathrm{dB} & M
\end{tikzcd}
$$

with the groupoid product μ_L on fibres of L associative.

The origin of Species:

la *gerbe* [fr.] – spray, sheaf, wreath *etc.* . . . [Giraud ’71]

Upshot & spin-off:

- geometric quantisation *via* cohomological transgression [Gawędzki '87, rrS '11]

$$\tau_p\ :\ \mathbb{H}^{p+1}\big(M,\mathcal{D}(p+1)^\bullet\big)\longrightarrow\mathbb{H}^1\big(\mathcal{C}_pM,\mathcal{D}(1)^\bullet\big)\,,\qquad \mathcal{C}_pM\equiv[\mathcal{C}_p,M]$$

yields a (pre)quantum bundle $\mathcal{H}_\sigma=\Gamma_{(\mathrm{pol})}(\mathcal{L}_\sigma)$, where

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r] & \pi^*_{\mathsf{T}^*\mathcal{C}_pM}\mathcal{L}_{\mathcal{G}_p}\otimes\mathcal{I}^0_{\vartheta_{\mathsf{T}^*\mathcal{C}_pM}}\equiv\mathcal{L}_\sigma\,,\ \nabla_{\mathcal{L}_\sigma} \arrow[d] \\ & \mathsf{P}_\sigma\equiv\mathsf{T}^*\mathcal{C}_pM\,,\ \Omega_\sigma=\delta\vartheta_{\mathsf{T}^*\mathcal{C}_pM}+\pi^*_{\mathsf{T}^*\mathcal{C}_pM}\int_{\mathcal{C}_p}\mathrm{ev}^*\underset{(p+2)}{\mathrm{H}}\equiv\mathrm{curv}(\nabla_{\mathcal{L}_\sigma}) \end{tikzcd}$$

and hence, classification of σ-models;

- geometrisation and cohomological classification of duality defects [Fuchs *et al.* '07, Runkel & rrS '08, rrS '11-'12];
- explicit constructions for the 'all' 2d RCFTs *via*...

…The Universal Gauge Principle (*morally*, $M \longrightarrow M//\mathrm{G}_\sigma$)
[Gawędzki & Reis '02-'03, Gawędzki, Waldorf & rrS '07-'13, rrS '11-'13]

Let G_σ be a Lie group with $\mathrm{Lie}(\mathrm{G}_\sigma) \equiv \mathfrak{g}_\sigma = \bigoplus_{A=1}^N \langle t_A \rangle$. An action $\lambda_\cdot : \mathrm{G}_\sigma \times M \longrightarrow M$ with the fundamental $\mathcal{K}_A \equiv \mathcal{K}_{t_A} \in \Gamma(\mathrm{T}M)$ is

- a **global symmetry** of the σ-model if

$$\forall_{(g,A)\in \mathrm{G}_\sigma \times \overline{1,N}} \; : \; \lambda_g^* \mathrm{g} = \mathrm{g} \quad \wedge \quad \iota_{\mathcal{K}_A} \underset{(p+2)}{\mathrm{H}} = -\mathsf{d} \underset{(p)}{\kappa}_A \quad \wedge \quad \lambda_g^* \mathcal{G}_p \cong \mathcal{G}_p \,.$$

- can be **gauged** *via* the 'minimal coupling' of $\mathcal{A} \equiv \mathcal{A}^A \otimes t_A \in \Omega^1(\mathrm{P}_{\mathrm{G}}) \otimes \mathfrak{g}_\sigma$ iff

1. [Gawędzki, Waldorf & rrS '07-'13, rrS '11-'13, '19]

the **small gauge anomaly** vanishes

$$\Longleftrightarrow \qquad \left(\bigoplus_{A=1}^N C^\infty(M,\mathbb{R}) \, (\mathcal{K}_A, \underset{(p)}{\kappa}_A), [\cdot,\cdot]_{\underset{(p+2)}{\mathrm{H}}} \right) \cong \mathfrak{g}_\sigma \ltimes_{\lambda_\cdot} M \,,$$

where $[\cdot,\cdot]_{\underset{(p+2)}{\mathrm{H}}}$ is a $\underset{(p+2)}{\mathrm{H}}$-twisted (à la Ševera–Weinstein) Vinogradov-type bracket on $\Gamma(\mathrm{T}M \oplus_{M,\mathbb{R}} \wedge^p \mathrm{T}^* M)$;

2. [Gawędzki, Waldorf & rrS '07-'13]

the **large gauge anomaly** vanishes

$\Longleftrightarrow$ $\mathcal{G}_p$ admits a G_σ-equivariant structure relative to curvature

$$\underset{(p+1)}{\varrho} := \sum_{k=1}^{p+1} \frac{(-1)^{p-k}}{k!}\, \mathrm{pr}_2^* \underset{(p+1-k)}{\alpha_{A_1 A_2 \ldots A_k}} \wedge \mathrm{pr}_1^* \big(\theta_{\mathrm{L}}^{A_1} \wedge \theta_{\mathrm{L}}^{A_2} \wedge \cdots \wedge \theta_{\mathrm{L}}^{A_k}\big)$$

on $\mathrm{G}_\sigma \times M$, with the $\underset{(p+1-k)}{\alpha_{A_1 A_2 \ldots A_k}}$ determined by the $(\mathcal{K}_A, \underset{(p)}{\kappa}{}_A)$,

i.e., in particular,

$$\Upsilon_p \ :\ \lambda_\cdot^* \mathcal{G}_p \xrightarrow{\cong} \mathrm{pr}_2^* \mathcal{G}_p \otimes \mathcal{I}^p_{\underset{(p+1)}{\varrho}} .$$

NB: $\mathcal{G}_p$ *descends* to $M//\mathrm{G}_\sigma$ iff $\underset{(p+1)}{\varrho} = 0$.

We shall call the associated G_σ-equivariant structure on $\mathcal{G}_p$ **descendable**.

The many faces of a G_σ-equivariant structure:

- extension of the $(p+1)$-cocycle in $\mathbb{H}^{p+1}(M, \mathcal{D}(p+1)^\bullet)$ for $\mathcal{G}_p$ to a $(p+1)$-cocycle in an extension of the Čech–de Rham bicomplex in the direction of G_σ-cohomology;
- extension of the 0-cell $\mathcal{G}_p$ to a *descent* $(p+2)$-tuple $(\mathcal{G}_p, \Upsilon_p, \Upsilon_{p-1}, \ldots, \Upsilon_0)$ over $\mathsf{N}^\bullet(\mathrm{G}_\sigma \ltimes M)$;
- [rrS '12] geometric data for the topological gauge-symmetry defect of the σ-model over Ω_p (based on [Runkel & rrS '09]).

Applications:

- geometrisation and cohomological classification of obstructions against gauging and of inequivalent gaugings, and hence
- natural mapping of the moduli space of σ-models, with beautiful connections to TFT
- reconstruction of T-duality outside the topological context. . .

Part II

Putting a spin on it
or
The Brave New Superworld

The goal:

A rigorous definition of a super-σ-model of 'mappings' $[\Omega_p, \mathcal{M}]$ for $\mathcal{M}$ a superm_fold endowed with an action of a **supersymmetry** Lie supergroup G.

Problems:

- What is the meaning of (all this, and in particular of) $[\Omega_p, \mathcal{M}]$?
- Inherent non-compactness of $\mathrm{G} \implies H^\bullet_{\mathrm{dR}}(\mathcal{M})^{\mathrm{G}} \not\equiv H^\bullet_{\mathrm{dR}}(\mathcal{M})$.
- …

Physical motivation:

Understanding the (super)geometric structure (*sensu largissimo*) of superstring theory-inspired & -related FTs, with view to elucidation of the deep nature of the tremendously robust yet notoriously elusive

AdS/CFT correspondence.

Def$^{\underline{\text{n}}}$: A **superm_fold** of superdimension $(m|n)$ is a ringed space $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ composed of

- $|\mathcal{M}| \in \mathrm{Ob}\,\mathbf{TopMan}$ of $\dim |\mathcal{M}| = m$ (the **body**) and
- $\mathcal{O}_{\mathcal{M}} \,:\, \mathscr{T}(|\mathcal{M}|)^{\mathrm{op}} \longrightarrow \mathbf{sAlg}_{\mathbf{scomm}}$ (the **structure sheaf**),

locally modelled on $(\mathbb{R}^m, C^\infty(\cdot,\mathbb{R}) \otimes \wedge^\bullet \mathbb{R}^n) \equiv \mathbb{R}^{m|n}$, *i.e.*,

$$\exists_{\{\mathcal{U}_i\}_{i\in I} \in \mathrm{Cov}(|\mathcal{M}|) \subset \mathscr{T}(|\mathcal{M}|)} \forall_{j\in I} \exists_{\mathcal{V}_j \in \mathscr{T}(\mathbb{R}^m),\ \mathcal{V}_j \cong \mathcal{U}_j} \;:\; \mathcal{O}_{\mathcal{M}}(\mathcal{U}_j) \cong C^\infty(\mathcal{V}_i, \mathbb{R}) \otimes \wedge^\bullet \mathbb{R}^n \,.$$

A **superm_fold morphism**

$$\varphi \equiv (|\varphi|, \varphi^*) \;:\; (|\mathcal{M}_1|, \mathcal{O}_{\mathcal{M}_1}) \longrightarrow (|\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_2})$$

consists of

- $|\varphi| \in \mathrm{Hom}_{\mathbf{TopMan}}(|\mathcal{M}_1|, |\mathcal{M}_2|)$ and
- $\varphi^* \,:\, \mathcal{O}_{\mathcal{M}_2} \Longrightarrow |\varphi|_* \mathcal{O}_{\mathcal{M}_1}$, *i.e.*,
  a family $\varphi^*_{\mathcal{U}} \in \mathrm{Hom}_{\mathbf{sAlg}_{\mathbf{scomm}}}(\mathcal{O}_{\mathcal{M}_2}(\mathcal{U}), \mathcal{O}_{\mathcal{M}_1}(|\varphi|^{-1}(\mathcal{U})))$
  indexed by $\mathcal{U} \in \mathscr{T}(|\mathcal{M}_2|)$.

Together, these form the **category of superm_folds** $\mathbf{sMan}$, with product $\mathcal{M}_1 \times \mathcal{M}_2 = (|\mathcal{M}_1| \times |\mathcal{M}_2|, \mathcal{O}_{\mathcal{M}_1} \widehat{\otimes} \mathcal{O}_{\mathcal{M}_2})$.

E.g.,

- $(M, C^\infty(\cdot, \mathbb{R}))$ is a superm_fold of superdimension $(\dim M|0)$;
- $(M, \Omega^\bullet(\cdot))$ is a superm_fold of superdimension $(\dim M|\dim M)$;
- super-Minkowski spacetime, super-AdS spacetime, Lie supergroups, their homogeneous spaces. . .

The geometric perspective: By the Yoneda Lemma,

$$\mathrm{Yon.} \ : \ \mathbf{sMan} \hookrightarrow \mathbf{Presh}(\mathbf{sMan}) \qquad \text{faithfully,}$$

and so we get, for any $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}}) \in \mathrm{Ob}\,\mathbf{sMan}$,

$$\mathrm{Yon}_{\mathcal{M}}(\cdot) \equiv \mathrm{Hom}_{\mathbf{sMan}}(\cdot, \mathcal{M}) \ : \ \mathbf{sMan}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,,$$

with $\mathrm{Yon.}(\mathcal{S}) \equiv \mathrm{Hom}_{\mathbf{sMan}}(\mathcal{S}, \mathcal{M})$ the **set of $\mathcal{S}$-points** in $\mathcal{M}$.

In particular, $\mathrm{Hom}_{\mathbf{sMan}}(\mathbb{R}^{0|0}, \mathcal{M}) \equiv |\mathcal{M}|$ is the set of **topological points** in $\mathcal{M}$ (also, $\mathcal{S} \dashrightarrow \mathbb{R}^{0|0} \longrightarrow \mathcal{M}$).

In the $\mathcal{S}$-point picture, we obtain **local coordinates**

$$(x^a, \theta^\alpha)\,, \qquad (a, \alpha) \in \overline{1, m} \times \overline{1, n}$$

of the respective Graßmann parities $|x^a| = 0 = |\theta^\alpha| - 1$. Sections take the form

$$f(x, \theta) = \sum_{k=0}^{n} F_{\alpha_1 \alpha_2 \ldots \alpha_k}(x) \otimes \theta^{\alpha_1}\,\theta^{\alpha_2}\,\ldots\,\theta^{\alpha_k}\,.$$

Furthermore, we have the locally free $\mathbb{Z}/2\mathbb{Z}$-graded **tangent sheaf**

$$\mathcal{T}\mathcal{M} = \mathrm{sDer}\,\mathcal{O}_{\mathcal{M}}\,,$$

with (homogeneous) sections $\mathcal{X}$ subject to the super-Leibniz rule, written for homogeneous sections f, g of $\mathcal{O}_{\mathcal{M}}$,

$$\mathcal{X}(f \cdot g) = \mathcal{X}(f) \cdot g + (-1)^{|\mathcal{X}|\cdot|f|}\, f \cdot \mathcal{X}(g)\,,$$

and the dual (locally free $\mathbb{Z}/2\mathbb{Z}$-graded) **cotangent sheaf**

$$\mathcal{T}^*\mathcal{M} = \mathrm{Hom}_{\mathbf{Sh}}(\mathcal{T}\mathcal{M}, \mathcal{O}_{\mathcal{M}})\,.$$

In the local coörds (x^a, θ^α), we obtain the local bases:

$$\Big\{\partial_a \equiv \tfrac{\partial}{\partial x^a}, \vec{\partial}_\alpha \equiv \tfrac{\vec{\partial}}{\partial \theta^\alpha}\Big\} \quad \text{of} \quad \mathcal{T}\mathcal{M} \qquad \text{and} \qquad \big\{\mathrm{d}x^a, \mathrm{d}\theta^\alpha\big\} \quad \text{of} \quad \bigwedge{}^{\bullet}\mathcal{T}^*\mathcal{M}\,,$$

and the corresponding presentations of sections:

$$\underset{(p)}{\omega}(x,\theta) = \sum_{r+s=p} \omega_{a_1 a_2 \ldots a_r \alpha_1 \alpha_2 \ldots \alpha_s}(x,\theta)\, \mathrm{d}x^{a_1} \wedge \mathrm{d}x^{a_2} \wedge \cdots \wedge \mathrm{d}x^{a_r} \wedge \mathrm{d}\theta^{\alpha_1} \wedge \mathrm{d}\theta^{\alpha_2} \wedge \cdots \wedge \mathrm{d}\theta^{\alpha_s}\,,$$

$$\mathcal{X}(x,\theta) = \mathcal{X}^a(x,\theta)\,\partial_a + \mathcal{X}^\alpha(x,\theta)\,\vec{\partial}_\alpha\,.$$

Among Ob **sMan**, we find

Def$^{\underline{\mathrm{n}}}$: A **Lie supergroup** is a group object

$$\big(\mathrm{G} = (|\mathrm{G}|, \mathcal{O}_{\mathrm{G}}), \mu \ :\ \mathrm{G} \times \mathrm{G} \longrightarrow \mathrm{G}, \mathrm{Inv} \ :\ \mathrm{G} \circlearrowleft, \varepsilon \ :\ \mathbb{R}^{0|0} \longrightarrow \mathrm{G}\big)$$

in **sMan**, with body $|\mathrm{G}| \in \mathrm{Ob}\,\mathbf{LieGrp}$. Together with (obvious) morphisms, they form the **category of Lie supergroups** **sLieGrp**.

<u>**NB:**</u> $\forall_{\mathcal{S} \in \mathrm{Ob}\,\mathbf{sMan}} \ :\ \mathrm{Yon.}(\mathcal{S}) \ :\ \mathbf{sLieGrp} \longrightarrow \mathbf{TopGrp}$.

On $\mathrm{G} \in \mathrm{Ob}\,\mathbf{sLieGrp}$, we have **LI vector fields** $L \in \mathcal{T}\mathrm{G}(|\mathrm{G}|)$ determined by the condition

$$\begin{tikzcd} \mathcal{O}_{\mathrm{G}} \widehat{\otimes} \mathcal{O}_{\mathrm{G}} \arrow[d, "\mathrm{id}_{\mathcal{O}_{\mathrm{G}}} \otimes L"'] & \mathcal{O}_{\mathrm{G}} \arrow[l, "\mu^*"'] \arrow[d, "L"] \\ \mathcal{O}_{\mathrm{G}} \widehat{\otimes} \mathcal{O}_{\mathrm{G}} & \mathcal{O}_{\mathrm{G}} \arrow[l, "\mu^*"] \end{tikzcd} ,$$

and their duals – the **LI 1-forms**.

The RI objects are defined analogously

The supercommutator closes on the LI vector fields, giving rise to the **tangent Lie superalgebra** of G,

$$\left(\mathrm{sLie}\,(\mathrm{G}) \cong \mathsf{T}_e\mathrm{G}, [\cdot,\cdot\}\right), \qquad \mathfrak{g} \ni X \longmapsto L_X \equiv \left(\mathrm{id}_{\mathcal{O}_\mathrm{G}} \otimes X\right) \circ \mu^* \in \mathcal{T}\mathrm{G}(|\mathrm{G}|)_\mathrm{L}\,,$$

an example of

Def$^{\underline{\mathrm{n}}}$: A **Lie superalgebra** is $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \in \mathrm{Ob}\,\mathrm{sVect}$ with the **Lie superbracket**

$$[\cdot,\cdot\} \;:\; \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \qquad \text{s.t.}$$

(sLB1) $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}\} \subset \mathfrak{g}^{(i+_2 j)}$;

(sLB2) $\forall_{X,Y\in\mathfrak{g}\ (\mathrm{hom.})} \;:\; [Y,X\} = (-1)^{|X|\cdot|Y|}\,[X,Y\}$;

(sLB3) the super-Jacobi identity holds true.

Together with the obvious morphisms, they form the **category of Lie superalgebras** $\mathbf{sLieAlg}$.

Th$^{\underline{\text{m}}}$[Kostant '77]: $\exists\,\mathscr{K}\ :\ \textbf{sLieGrp}\xrightarrow{\cong}\textbf{sHCp}$

Objects and morphisms of the latter category are given in

Def$^{\underline{\text{n}}}$: A **super-Harish-Chandra pair** is a. . . triple $\mathrm{G}\equiv(|\mathrm{G}|,\mathfrak{g},\rho)$ composed of

- $|\mathrm{G}|\in\mathrm{Ob}\,\textbf{LieGrp}$,
- $\mathfrak{g}=\mathfrak{g}^{(0)}\oplus\mathfrak{g}^{(1)}\in\mathrm{Ob}\,\textbf{sLieAlg}$ with $\mathfrak{g}^{(0)}\equiv\mathrm{Lie}\,(|\mathrm{G}|)$, and
- a realisation $\rho\ :\ |\mathrm{G}|\longrightarrow\mathrm{End}_{\textbf{sLieAlg}}(\mathfrak{g})$ s.t., for any $g\in|\mathrm{G}|$, $\rho(g)\restriction_{\mathfrak{g}^{(0)}}\equiv\mathsf{T}_e\mathrm{Ad}_g$.

A **sHCp morphism** $(\Phi,\phi)\ :\ (|\mathrm{G}_1|,\mathfrak{g}_1,\rho_1)\longrightarrow(|\mathrm{G}_2|,\mathfrak{g}_2,\rho_2)$ is composed of

- $\Phi\in\mathrm{Hom}_{\textbf{LieGrp}}\,(|\mathrm{G}_1|,|\mathrm{G}_2|)$,
- $\phi\in\mathrm{Hom}_{\textbf{sLieAlg}}\,(\mathfrak{g}_1,\mathfrak{g}_2)$ s.t. $\phi\restriction_{\mathfrak{g}^{(0)}}=\mathsf{T}_e\Phi$ and, for any $g\in|\mathrm{G}|$, $\rho_2\circ\Phi(g)\circ\phi=\phi\circ\rho_1(g)$.

Remark: $\mathscr{K}$ uses the Hopf-superalgebra structure on $\mathfrak{U}(\mathfrak{g})$.
It yields $\mathrm{sLie}\big(\mathscr{K}(|\mathrm{G}|,\mathfrak{g},\rho)\big)=\mathfrak{g}$.

Examples of Lie supergroups:

- $\mathrm{sMink}(d,1\,|\,D_{d,1})$ as an abstract Lie supergroup is

$$\mathrm{sMink}(d,1\,|\,D_{d,1}) = \big(\mathbb{R}^{d+1}, C^\infty(\cdot,\mathbb{R})\otimes \bigwedge{}^\bullet\mathbb{R}^{D_{d,1}}\big)\,, \qquad D_{d,1} = \dim S_{d,1}\,,$$

with $S_{d,1}$ a distinguished Majorana-spinor $\mathrm{Cliff}\,(\mathbb{R}^{d,1})$-module.
It admits global coörds $\{x^a,\theta^\alpha\}^{(a,\alpha)\in\overline{0,d}\times\overline{1,D_{d,1}}}$ and

$$\mu^*\ :\ \big(x^a,\theta^\alpha\big)\longmapsto\big(x^a\otimes\mathbf{1}+\mathbf{1}\otimes x^a-\tfrac{1}{2}\,\theta^\alpha\otimes\big(C\,\Gamma^a\big)_{\alpha\beta}\,\theta^\beta,\theta^\alpha\otimes\mathbf{1}+\mathbf{1}\otimes\theta^\alpha\big)\,,$$

$$\mathrm{Inv}^*\ :\ \big(x^a,\theta^\alpha\big)\longmapsto\big(-x^a,-\theta^\alpha\big)\,,$$

or, equivalently, in the $\mathcal{S}$-point picture,

$$\big(x_1^a,\theta_1^\alpha\big)\cdot\big(x_2^b,\theta_2^\beta\big)=\big(x_1^a+x_2^a-\tfrac{1}{2}\,\theta_1\,\overline{\Gamma}^a\,\theta_2,\theta_1^\alpha+\theta_2^\alpha\big)\,,\qquad \big(x^a,\theta^\alpha\big)^{-1}=\big(-x^a,-\theta^\alpha\big)\,.$$

As a sHCp,

$$\mathrm{sMink}(d,1\,|\,D_{d,1})=\big(\mathrm{Mink}(d,1),\mathfrak{smink}(d,1\,|\,D_{d,1})=\bigoplus_{a=0}^{d}\langle P_a\rangle\oplus\bigoplus_{\alpha=1}^{D_{d,1}}\langle Q_\alpha\rangle\,,0\big)\,,$$

$$\{Q_\alpha,Q_\beta\}=\big(C\,\Gamma^a\big)_{\alpha\beta}\,P_a\,,\qquad [P_a,P_b]=0=[Q_\alpha,P_a]\,.$$

- $\mathrm{SU}(2,2\,|\,4)$ as a sHCp with the body Lie group

$$|\mathrm{SU}(2,2\,|\,4)| = \mathrm{SO}(4,2)\times\mathrm{SO}(6)\,,$$

the Lie superalgebra

$$\mathfrak{su}(2,2\,|\,4) = \Big(\big(\bigoplus_{a=0}^{4}\langle P_a\rangle \oplus \bigoplus_{a'=5}^{9}\langle P_{a'}\rangle\big) \oplus \bigoplus_{(\alpha,\alpha',I)\in\overline{1,4}\times\overline{1,4}\times\{1,2\}}\langle Q_{\alpha\alpha' I}\rangle\Big)$$
$$\oplus\Big(\bigoplus_{a,b=0}^{4}\langle J_{ab}=-J_{ba}\rangle \oplus \bigoplus_{a',b'=5}^{9}\langle J_{a'b'}=-J_{b'a'}\rangle\Big)$$

$$\{Q_{\alpha\alpha' I}, Q_{\beta\beta' J}\} = \mathrm{i}\,\big(-2(\widehat{C}\,\widehat{\Gamma}^{\widehat{a}}\otimes\mathbf{1})_{\alpha\alpha' I\beta\beta' J}\,P_{\widehat{a}} + (\widehat{C}\,\widehat{\Gamma}^{\widehat{a}\widehat{b}}\otimes\sigma_2)_{\alpha\alpha' I\beta\beta' J}\,J_{\widehat{a}\widehat{b}}\big)\,,$$

$$[Q_{\alpha\alpha' I}, P_{\widehat{a}}] = -\tfrac{1}{2}\,(\widehat{\Gamma}_{\widehat{a}}\otimes\sigma_2)^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}}, P_{\widehat{b}}] = \varepsilon_{\widehat{a}\widehat{b}}\,J_{\widehat{a}\widehat{b}}\,,\quad \varepsilon_{\widehat{a}\widehat{b}} = \begin{cases} +1 & \text{if } \widehat{a},\widehat{b}\in\overline{0,4} \\ -1 & \text{if } \widehat{a},\widehat{b}\in\overline{5,9} \\ 0 & \text{otherwise}\end{cases}\,,$$

$$[J_{\widehat{a}\widehat{b}}, J_{\widehat{c}\widehat{d}}] = \eta_{\widehat{a}\widehat{d}}\,J_{\widehat{b}\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\,J_{\widehat{b}\widehat{d}} + \eta_{\widehat{b}\widehat{c}}\,J_{\widehat{a}\widehat{d}} - \eta_{\widehat{b}\widehat{d}}\,J_{\widehat{a}\widehat{c}}\,,$$

$$[Q_{\alpha\alpha' I}, J_{\widehat{a}\widehat{b}}] = -\tfrac{1}{2}\,\varepsilon_{\widehat{a}\widehat{b}}\,(\widehat{\Gamma}_{\widehat{a}\widehat{b}}\otimes\mathbf{1})^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}}, J_{\widehat{b}\widehat{c}}] = \eta_{\widehat{a}\widehat{b}}\,P_{\widehat{c}} - \eta_{\widehat{a}\widehat{c}}\,P_{\widehat{b}}\,.$$

and the standard spinor realisation of the former

on the Graßmann-odd component of the latter.

Towards homogeneous spaces of Lie supergroups...

Def$^{\underline{n}}$: A **superm_fold with action** of a Lie supergroup G is a pair $(\mathcal{M}, \lambda_\cdot)$ composed of

- $\mathcal{M} \in \mathrm{Ob}\,\mathbf{sMan}$ and
- $\lambda_\cdot \in \mathrm{Hom}_{\mathbf{sMan}}(\mathrm{G} \times \mathcal{M}, \mathcal{M})$ s.t.

$$\begin{tikzcd} \mathbb{R}^{0|0} \times \mathcal{M} \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "\varepsilon \times \mathrm{id}_{\mathcal{M}}"'] & & \mathcal{M} \\ & \mathrm{G} \times \mathcal{M} \arrow[ur, "\lambda_\cdot"'] & \end{tikzcd} , \qquad \begin{tikzcd} \mathrm{G} \times \mathrm{G} \times \mathcal{M} \arrow[r, "\mathrm{id}_{\mathrm{G}} \times \lambda_\cdot"] \arrow[d, "\mu \times \mathrm{id}_{\mathcal{M}}"'] & \mathrm{G} \times \mathcal{M} \arrow[d, "\lambda_\cdot"] \\ \mathrm{G} \times \mathcal{M} \arrow[r, "\lambda_\cdot"'] & \mathcal{M} \end{tikzcd} .$$

Together with the (obvious) morphisms (equivariant supermanifold morphisms), these form the **category of G-superm_folds** $\mathrm{G}-\mathbf{sMan}$.

Among $\mathrm{Ob}\,\mathrm{G}-\mathbf{sMan}$, we have G with the L action $\ell_\cdot$ and the R action $\wp_\cdot$ induced from μ. These restrict to actions of $|\mathrm{G}| \ni g$ by superdiffeo_s *as per*

$$|l|_g \equiv \mu \circ (\widehat{g} \times \mathrm{id}_{\mathrm{G}}) \; : \; \mathbb{R}^{0|0} \times \mathrm{G} \cong \mathrm{G} \longrightarrow \mathrm{G} \,,$$

$$|r|_g \equiv \mu \circ (\mathrm{id}_{\mathrm{G}} \times \widehat{g}) \; : \; \mathrm{G} \times \mathbb{R}^{0|0} \cong \mathrm{G} \longrightarrow \mathrm{G} \,.$$

Th$^{\underline{\text{m}}}$[Kostant '77, Koszul '82, Fioresi *et al.* '07]: Given $\mathrm{G} \in \mathrm{Ob}\,\mathbf{sLieGrp}$ and its closed subsupergroup H with $\mathrm{sLie}\,\mathrm{H} \equiv \mathfrak{h}$, $\exists$ ess. unique superm_fold structure on the **homogeneous space**

$$\mathrm{G}/\mathrm{H} = \left(|\mathrm{G}|/|\mathrm{H}|, \mathcal{O}_{\mathrm{G}/\mathrm{H}}\right) \qquad \text{s.t.}$$

$$\mathcal{O}_{\mathrm{G}/\mathrm{H}} = \left\{\, f \in \mathcal{O}_{\mathrm{G}}\restriction_{|\mathrm{G}|/|\mathrm{H}|} \quad \middle| \quad \forall_{(J,h)\in\mathfrak{h}\times|\mathrm{H}|} \; : \; L_J(f) = 0 \;\wedge\; |r|_h^*(f) = f \right\}$$

(sHS1) $\pi_{\mathrm{G}/\mathrm{H}} = \left(\pi_{|\mathrm{G}|/|\mathrm{H}|}, \iota_{\mathcal{O}_{\mathrm{G}/\mathrm{H}}} : \mathcal{O}_{\mathrm{G}/\mathrm{H}} \hookrightarrow \mathcal{O}_{\mathrm{G}}\right)$ is a submersion with

$$\begin{tikzcd} \mathrm{H} \arrow[r] & \mathrm{G} \arrow[r, dashed] \arrow[d, "\pi_{\mathrm{G}/\mathrm{H}}"'] & {|\mathrm{G}|} \arrow[d, "\pi_{|\mathrm{G}|/\mathrm{H}}"] \\ & \mathrm{G}/\mathrm{H} \arrow[r, dashed] & {|\mathrm{G}|/\mathrm{H}} \end{tikzcd} \;;$$

(sHS2) $\ell.$ descends to G/H *as per*

$$\begin{tikzcd} \mathrm{G}\times\mathrm{G} \arrow[r, "\ell."] \arrow[d, "\mathrm{id}_{\mathrm{G}}\times\pi_{\mathrm{G}/\mathrm{H}}"'] & \mathrm{G} \arrow[d, "\pi_{\mathrm{G}/\mathrm{H}}"] \\ \mathrm{G}\times\mathrm{G}/\mathrm{H} \arrow[r, "{[\ell].}"] & \mathrm{G}/\mathrm{H} \end{tikzcd} \;.$$

We have **local trivialisations** over (distinguished) $\mathcal{U} \in \mathscr{T}(|\mathrm{G}|/|\mathrm{H}|)$,

$$\tau^{-1} \, : \, \widehat{\mathcal{U}} \times \mathrm{H} \equiv \left(\mathcal{U}, \mathcal{O}_{\mathrm{G}/\mathrm{H}}\restriction_{\mathcal{U}}\right) \times \mathrm{H} \longrightarrow \left(\pi^{-1}_{|\mathrm{G}|/|\mathrm{H}|}(\mathcal{U}), \mathcal{O}_{\mathrm{G}}\restriction_{\pi^{-1}_{|\mathrm{G}|/|\mathrm{H}|}(\mathcal{U})}\right),$$

and so also local sections

$$\sigma_{\widehat{\mathcal{U}}} = \tau^{-1} \circ \left(\mathrm{id}_{\widehat{\mathcal{U}}} \times \widehat{e}\right) \, : \, \widehat{\mathcal{U}} \times \mathbb{R}^{0|0} \cong \widehat{\mathcal{U}} \longrightarrow \mathrm{G}\,.$$

With the help of the $[|l|]_{g_i}$, for some $g_i \in |\mathrm{G}|$, $i \in I$, induced from $|l|_\cdot$ *via* $[\ell]_\cdot$, we obtain a trivialising cover from $\widehat{\mathcal{U}}_0 \ni \mathrm{H} \equiv |l|_e(\mathrm{H})$,

$$\left\{[|l|]_{g_i}(\widehat{\mathcal{U}}_0) \equiv \widehat{\mathcal{U}}_i\right\}_{i \in I},$$

with the corresponding local trivialising sections

$$\sigma_i = |l|_{g_i} \circ \sigma_{\widehat{\mathcal{U}}_0} \circ [l]_{g_i^{-1}}\,.$$

We may be quite explicit for a class of pairs $(\mathfrak{g}, \mathfrak{h})$…

Reductive homogeneous spaces

Let $(\mathrm{G}, \mathrm{H} \subset |\mathrm{G}|)$ with

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}\,, \qquad \mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)} \equiv \bigoplus_{a=0}^{d_0} \langle P_a \rangle \oplus \bigoplus_{\alpha=1}^{d_1} \langle Q_\alpha \rangle\,, \qquad \mathfrak{h} = \bigoplus_{\kappa=1}^{d_S} \langle J_\kappa \rangle$$

be **reductive**, *i.e.*,

$$[\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}\,.$$

We may then take $\mathcal{U}_0 \equiv |\widehat{\mathcal{U}}_0|$ sufficiently small to have

$$\mathcal{U}_0 = \{\, \exp\left(x^a \otimes P_a\right) \quad | \quad x^a \,:\, \mathcal{V}_0 \longrightarrow \mathbb{R} \quad \text{a local chart near } 0 \in \mathbb{R}^{d_0+1} \,\}\,,$$

and then also, for $\{\theta^\alpha\}^{\alpha \in \overline{1, d_1}}$ a global chart on $\mathbb{R}^{0|d_1}$,

$$\sigma_0(x^a, \theta^\alpha) = \exp\left(\theta^\alpha \otimes Q_\alpha\right) \circ \exp\left(x^a \otimes P_a\right).$$

In the $\mathcal{S}$-point picture, at any $\xi \equiv (x, \theta) \in \widehat{\mathcal{U}}_{ij}$, we find

$$\sigma_j(\xi) = \sigma_j(\xi) \cdot h_{ij}(\xi)\,, \qquad h_{ij}(\xi) \in \mathrm{H}\,.$$

Furthermore, components of the LI $\mathfrak{g}$-valued Maurer–Cartan 1-form

$$\theta_{\rm L} = \theta_{\rm L}^A \otimes t_A = \theta_{\rm L}^\mu \otimes T_\mu + \theta_{\rm L}^\kappa \otimes J_\kappa\,, \qquad \bigoplus_{\mu=0}^{d_0+d_1} \langle T_\mu \rangle \equiv \mathfrak{t}$$

- transform as H-tensors for $A = \mu \in \overline{0, d_0 + d_1}$,

$$|r|_\cdot^* \theta_{\rm L}^\mu(g,h) = \rho(h)^\mu{}_\nu\, \theta_{\rm L}^\nu(g)\,, \qquad (g,h) \in {\rm Yon}_{\rm G} \times {\rm Yon}_{\rm H}$$

- compose a principal H-connection on ${\rm G} \longrightarrow {\rm G/H}$ as

$$\Theta = \theta_{\rm L}^\kappa \otimes J_\kappa\,.$$

Upshot:

- the notion of horizontality ($\in \ker \Theta$);
- $T = \tau_{\mu_1\mu_2\ldots\mu_n}\, \theta_{\rm L}^{\mu_1} \otimes \theta_{\rm L}^{\mu_2} \otimes \cdots \otimes \theta_{\rm L}^{\mu_n}$ with const. coefficients s.t.

$$\tau_{\mu_1\mu_2\ldots\mu_n} = \tau_{\nu_1\nu_2\ldots\nu_n}\, \rho(h)^{\nu_1}{}_{\mu_1}\, \rho(h)^{\nu_2}{}_{\mu_2} \cdots \rho(h)^{\nu_n}{}_{\mu_n}\,, \qquad h \in {\rm H}$$

descend to ${\rm G/H}$ along the σ_i, $i \in I$.

Examples of reductive homogeneous spaces of Lie supergroups:

- $\mathrm{sMink}(d,1\,|\,D_{d,1}) \equiv \mathrm{sISO}(d,1\,|\,D_{d,1})/\mathrm{SO}(d,1)$ for $\mathrm{sISO}(d,1\,|\,D_{d,1}) = \mathrm{sMink}(d,1\,|\,D_{d,1}) \rtimes_{L_{d,1}\oplus S_{d,1}} \mathrm{SO}(d,1)$, with

$$\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^{a} \otimes \theta_{\mathrm{L}}^{b}\,,$$

$$\underset{(p+2)}{\mathrm{H}} = \begin{cases} \theta_{\mathrm{L}}^{\alpha} \wedge (C\,\Gamma_{11})_{\alpha\beta}\,\theta_{\mathrm{L}}^{\beta} & (p=0) \\ \theta_{\mathrm{L}}^{\alpha} \wedge (C\,\Gamma_{a_1 a_2 \ldots a_p})_{\alpha\beta}\,\theta_{\mathrm{L}}^{\beta} \wedge \theta_{\mathrm{L}}^{a_1} \wedge \theta_{\mathrm{L}}^{a_2} \wedge \cdots \wedge \theta_{\mathrm{L}}^{a_p} & (1<p<8) \end{cases}$$

the admissible (d,p,N) filling up the ‘old brane scan’

- $\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5) \equiv \mathrm{SU}(2,2|4)/(\mathrm{SO}(4,1)\times\mathrm{SO}(5))$, with

$$\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^{a} \otimes \theta_{\mathrm{L}}^{b} + \delta_{a'b'}\,\theta_{\mathrm{L}}^{a'} \otimes \theta_{\mathrm{L}}^{b'}\,,$$

$$\underset{(3)}{\mathrm{H}} = \theta_{\mathrm{L}}^{\alpha\alpha' I} \wedge \big(\widehat{C}\,\widehat{\Gamma}_{\hat a} \otimes \sigma_3\big)_{\alpha\alpha' I\,\beta\beta' J}\,\theta_{\mathrm{L}}^{\beta} \wedge \theta_{\mathrm{L}}^{\hat a}$$

Def$^{\underline{\mathrm{n}}}$: Given a *closed* m_fold Ω_p of $\dim \Omega_p = p + 1$, a Lie supergroup G and its closed Lie subgroup $\mathrm{H} \subset |\mathrm{G}|$ with $(\mathfrak{g}, \mathfrak{h})$ reductive, as described above, fix a tesellation $\triangle(\Omega_p)$ of Ω_p subordinate, for a given $\xi \in [\Omega_p, \mathrm{G}/\mathrm{H}]$, to a trivialising cover $\{\widehat{\mathcal{U}}_i\}_{i\in I}$ of G/H, described earlier. Assume given H-basic LI tensors on G:

$$\mathrm{g} = \mathrm{g}_{(ab)}\, \theta^a_{\mathrm{L}} \otimes \theta^b_{\mathrm{L}}\,, \qquad\qquad \underset{(p+2)}{\chi} \equiv \pi^*_{\mathrm{G}/\mathrm{H}}\underset{(p+2)}{\mathrm{H}} \in Z^{p+2}_{\mathrm{dR}}(\mathrm{G})^{\mathrm{G}}\,.$$

The **Green–Schwarz super-σ-model in the Nambu–Goto formulation** is a theory of mappings $\xi \in [\Omega_p, \mathrm{G}/\mathrm{H}]$ determined by the PLA for the DF amplitudes defined in terms of ($\xi_\tau \equiv \xi{\restriction}_\tau$)

$$S^{(\mathrm{NG})}_{\mathrm{GS},p}[\xi] = \sum_{\tau\in\mathfrak{T}_{p+1}} \int_\tau \sqrt{\det\left((\sigma_{i_\tau} \circ \xi_\tau)^*\mathrm{g}\right)} + \int_{\Omega_p} \xi^*\mathrm{d}^{-1} \underset{(p+2)}{\mathrm{H}}$$

The meaning of the super-σ-model [Freed '95]: The mappings from

$$[\Omega_p, \mathcal{M}] \equiv \underline{\mathrm{Hom}}_{\mathbf{sMan}}(\Omega_p, \mathcal{M}) := \mathrm{Hom}_{\mathbf{sMan}}(\cdot \times \Omega_p, \mathcal{M})$$

$$\in \mathrm{Ob}\,\mathbf{Fun}(\mathbf{sMan}^{\mathrm{op}}, \mathbf{Set})$$

are to be evaluated on (*cp* Sorokin's superembedding formalism!)

$$\mathbb{R}^{0\,|\,N}\,, \qquad N \in \mathbb{N}^\times$$

with the resp. structure sheaves $\mathbb{R}[\eta^1, \eta^2, \ldots, \eta^N]$, whereupon

$$\xi^a = \xi^a_0 + \xi^a_{i_1 i_2}\,\eta^{i_1}\,\eta^{i_2} + \cdots + \xi^a_{i_1 i_2 \ldots 2[\frac{N}{2}]}\,\eta^{i_1}\,\eta^{i_2}\,\cdots\,\eta^{i_{2[\frac{N}{2}]}}\,,$$

$$\xi^\alpha = \xi^\alpha_{i_1}\,\eta^{i_1} + \xi^\alpha_{i_1 i_2 i_3}\,\eta^{i_1}\,\eta^{i_2}\,\eta^{i_3} + \cdots + \xi^\alpha_{i_1 i_2 \ldots 2[\frac{N-1}{2}]+1}\,\eta^{i_1}\,\eta^{i_2}\,\cdots\,\eta^{i_{2[\frac{N-1}{2}]+1}}\,,$$

& the $\xi^a_{i_1 i_2 \ldots i_k}, \xi^\alpha_{i_1 i_2 \ldots i_k}$ become the fields of the super-σ-model.

Physically relevant models:

(i) the original Green-Schwarz-… super-p-branes on
$\mathrm{sMink}(d,1\,|\,ND_{d,1}) \equiv \mathrm{sISO}(d,1\,|\,ND_{d,1})/\mathrm{SO}(d,1),\ N \in \mathbb{N}^\times$;

(ii) the Metsaev-Tseytlin superstring and super-p-branes on
$\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5) \equiv \mathrm{SU}(2,2\,|\,4)/(\mathrm{SO}(4,1) \times \mathrm{SO}(5))$;

(iii) the Zhou super-0-brane and superstring on
$\mathrm{s}(\mathrm{AdS}_2 \times \mathbb{S}^2) \equiv \mathrm{SU}(1,1\,|\,2)_2/(\mathrm{SO}(1,1) \times \mathrm{SO}(2))$;

(iv) super-p-branes on
$\mathrm{s}(\mathrm{AdS}_3 \times \mathbb{S}^3), \mathrm{s}(\mathrm{AdS}_4 \times \mathbb{S}^7), \mathrm{s}(\mathrm{AdS}_7 \times \mathbb{S}^4), \ldots$

Empirical facts:

(H) (i) & (iii) have $[\underset{(p+2)}{\chi}]_{\mathrm{dR}} = 0$, but $[\underset{(p+2)}{\chi}]^{\mathrm{G}}_{\mathrm{dR}} \in \mathrm{CaE}^{p+2}(\mathrm{G}) \setminus \{0\}$.

(İW) (ii) has $[\underset{(3)}{\chi}]^{\mathrm{SU}(2,2\,|\,4)}_{\mathrm{dR}} = 0 \in \mathrm{CaE}^3(\mathrm{SU}(2,2\,|\,4))$, but
the supersymmetric primitive does NOT İnönü-Wigner–contract to the one of (i).

What are the **PROBLEMS** with the empirical facts?

Ad ($\dot{\mathrm{I}}$W) Signals apparent ill-definedness of the MT super-σ-model whose construction was based upon the asymptotic correspondence with the GS super-σ-model for $\mathrm{sMink}(9,1\,|\,32)$. [rrS '18]

Ad (H) The choice of the cohomology critical for the meaning of the topological term in $\mathcal{A}_{\rm DF}$.

AND

Physics (SUSY) favours the Cartan–Eilenberg cohomology

$$\mathrm{CaE}^\bullet(\mathrm{G}) \equiv H^\bullet_{\rm dR}(\mathrm{G})^{\mathrm{G}}\,,$$

BUT

(How) Does $\mathrm{CaE}^\bullet(\mathrm{G}) \setminus H^\bullet(\mathrm{G})$ topologise?

The Rabin-Crane-type argument/hypothesis:

Secretly, the GS super-σ-model for $[\Omega_p, \mathrm{G}/\mathrm{H} \equiv \mathcal{M}]$ is a theory of mappings from $[\Omega_p, \mathcal{M}/\Gamma_{\mathrm{KR}}]$ for $\Gamma_{\mathrm{KR}} \subset \mathrm{G}$ s.t.

$$\mathcal{M}/\Gamma_{\mathrm{KR}} \cong_{\mathrm{loc.}} \mathcal{M} \qquad \wedge \qquad H^\bullet_{\mathrm{dR}}(\mathcal{M})^{\mathrm{G}} \cong H^\bullet_{\mathrm{dR}}(\mathcal{M}/\Gamma_{\mathrm{KR}}) .$$

A working model:

In the case $\mathcal{M} = \mathrm{sMink}(d, 1|D_{d,1})$, the subgroup was identified in [Crane & Rabin '85] as the discrete Kostelecký-Rabin supergroup generated by *integer* supertranslations

$$(x^a, \theta^\alpha) \longmapsto (y^b, \varepsilon^\beta) \cdot (x^a, \theta^\alpha)$$

with $y^b_{i_1 i_2 \ldots i_k}, \varepsilon^\alpha_{i_1 i_2 \ldots i_k} \in \mathbb{Z}$ (in the $\mathcal{S}$-point picture).

Field-theoretic consequences: We ought to take into account the Γ_{KR}-twisted sector in $[\Omega_p, \mathrm{G/H}]$, but then the Poisson-Lie superalgebra of the Noether charges of supersymmetry of the GS super-σ-model,

$$\{h_A, h_B\} = f_{AB}{}^C\, h_C + \mathcal{A}_{AB}\,,$$

exhibits a (classical!) **wrapping anomaly**.

Conclusion: Need to consider extensions of the supersymmetry algebra $\mathfrak{g}$.

The latter is merely an (exact) intuition with a rigorous cohomology story behind it. . .

Towards geometrisation of supersymmetric de Rham cocycles…

Th$\underline{\text{m}}$: $\exists$ isomorphism

$$[\gamma]\ :\ H^\bullet(\mathfrak{g},\mathbb{R}) \equiv \mathrm{CE}^\bullet(\mathfrak{g},\mathbb{R}) \xrightarrow{\cong} \mathrm{CaE}^\bullet(\mathrm{G}) \cong H^\bullet_{\mathrm{dR}}(\mathrm{G})^{\mathrm{G}}\,.$$

Th$\underline{\text{m}}$: $\exists$ a correspondence

$$\mathrm{CE}^2(\mathfrak{g},\mathbb{R}) \xleftrightarrow{1:1} \{\text{ equivalence classes of supercentral extensions of } \mathfrak{g}\ \}\,,$$

where

$$\widetilde{\mathfrak{g}}_2 \sim \widetilde{\mathfrak{g}}_1 \quad \Longleftrightarrow \quad \begin{tikzcd} & & \widetilde{\mathfrak{g}}_1 \arrow[dd, "\cong"] \arrow[dr] & & \\ \mathbf{0} \arrow[r] & \mathfrak{a} \arrow[ur] \arrow[dr] & & \mathfrak{g} \arrow[r] & \mathbf{0}\,. \\ & & \widetilde{\mathfrak{g}}_2 \arrow[ur] & & \end{tikzcd}$$

Th$\underline{\text{m}}$: $\mathbf{0} \longrightarrow \mathbb{R} \longrightarrow \widetilde{\mathfrak{g}}_{[\omega]} \longrightarrow \mathfrak{g} \longrightarrow \mathbf{0}$ determined by $[\omega] \in \mathrm{CaE}^2(\mathrm{G})$ integrates to $\mathbf{1} \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{\mathrm{G}} \longrightarrow \mathrm{G} \longrightarrow \mathbf{1}$ iff $\mathrm{Per}(\omega) \subset 2\pi\mathbb{Z}$ and $\ell_\cdot\ :\ \mathrm{G}\times(\mathrm{G},\omega) \longrightarrow (\mathrm{G},\omega)$ has a momentum map.

Idea of geometrisation:

(**Inspiration:** extended superspacetimes of [de Azcárraga *et al.*])

1. Look for G-invariant 2-cocycles among the

$$\omega_{\mu_1\mu_2\ldots\mu_p} := \iota_{L_{\mu_1}}\iota_{L_{\mu_2}}\cdots\iota_{L_{\mu_p}}\underset{(p+2)}{\chi}\,.$$

2. Associate a supercentral extension $\mathbb{C}^\times \longrightarrow \widetilde{\mathrm{G}} \xrightarrow{\widetilde{\pi}} \mathrm{G}$ to it and partially reduce $\widetilde{\pi}^*\underset{(p+2)}{\chi}$ in $\mathrm{CaE}^\bullet(\widetilde{\mathrm{G}})$.

3. Repeat 1.-2. until complete reduction of $\widehat{\pi}^*\underset{(p+2)}{\chi}$ is obtained over an extension $\widehat{\mathrm{G}} \xrightarrow{\widehat{\pi}} \mathrm{G}$ in the corresponding $\mathrm{CaE}^\bullet(\widehat{\mathrm{G}})$, *i.e.*,

$$\exists\ \underset{(p+1)}{\beta} \in \Omega^{p+1}(\widehat{\mathrm{G}})^{\widehat{\mathrm{G}}}\ :\ \mathrm{d}\underset{(p+1)}{\beta} = \widehat{\pi}^*\underset{(p+2)}{\chi}\,.$$

4. Descend $\underset{(p+1)}{\beta}$ to $\widehat{\mathrm{G}}/\mathrm{H}$.

5. Use $\widehat{\mathrm{G}}/\mathrm{H}$ as THE surjective submersion of $\mathcal{G}_p$ & continue à la [Murray & Stevenson *et al.*].

Constructive results:

Theorem I [rrS '17('12)] Consecutive resolution, through central extensions, of the various CaE super-2-cocycles encountered in the analysis of the GS super-$(p+2)$-cocycles on $\mathrm{sMink}(d,1\,|\,(N\cdot)D_{d,1})$, induces a hierarchy of surjective submersions necessary for the geometrisation of the latter, leading to the emergence of the corresponding **Green–Schwarz super-p-gerbes** (explicited for $p \in \{0,1,2\}$).

Abstraction: The super-Minkowskian construction yields **Cartan–Eilenberg super-p-gerbes** that are, *morally*, p-gerbe objects in $\mathbf{sLieGrp}$. ($\mathrm{sMink}(d,1\,|\,(N\cdot)D_{d,1})$ is a Lie supergroup!)

More generally, we obtain p-gerbes with a lift of $[\ell]_\cdot$ to the constitutive surjective submersions that preserves the entire connective structure.

Constructive results – ct^{d}:

- The success of the super-Minkowskian geometrisation (of Theorem I) is repeated in [rrS '18] in the setting of Zhou's super-σ-model of [Zhou '99] for the superparticle in $\mathrm{s}(\mathrm{AdS}_2 \times \mathbb{S}^2)$.
- The celebrated Metsaev–Tseytlin super-σ-model of [Metsaev & Tseytlin '98] for the superstring in $\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5)$, on the other hand, is problematic. There exists an İnönü–Wigner-*non*contractible trivial super-1-gerbe, and a collection of no-go theorems.

Constructive results – ct^{d}:

Theorem II [rrS ’19(’17)] The GS super-p-gerbes of Theorem I with $p \in \{0, 1\}$ are endowed with a canonical **supersymmetric** Ad.**-equivariant structure**.

NB: This conforms with the purely even (WZW) story.

But there is even more verifiable Physics in the construction. . .

A self-consistent dscription of the supersymmetric vacuum

The translational component of the global supersymmetry G of $\mathcal{A}_{\mathrm{DF}}^{\mathrm{GS},p(\mathrm{NG})}$ is broken spontaneously by the (class.) vacuum Ψ_{VAC}.
$\Longrightarrow$ Also its Graßmann-odd component has to be reduced on Ψ_{VAC}.

Q: How to remove the spurious (Goldstone) Graßmann-odd fields?

A: A ***κ*-symmetry** extension of $\mathfrak{h}$ on Ψ_{VAC}, spanning the latter as its (super)diffeo_s. [de Azcárraga & Lukierski '82, Siegel '83]

Problems: Ω_p-locality of κ-symmetry & the mixing of the metric and topological DOFs prevent geometrisation in the NG picture

BUT

Here come Hughes & Polchinski…

Consider $\mathfrak{t}^{(0)}_{\rm VAC} \subset \mathfrak{t}^{(0)}_{\rm VAC} \oplus \mathfrak{e}^{(0)} \subset \mathfrak{t}^{(0)} (\subset \mathfrak{t} \subset \mathfrak{t} \oplus \mathfrak{h} \equiv \mathfrak{g})$ representing $\Psi_{\rm VAC}$, & its maximal ad-isotropy algebra $\mathfrak{h}_{\rm VAC} \subset \mathfrak{h}$ over $\mathrm{H}_{\rm VAC} \subset \mathrm{H}$. Write

$$\mathfrak{t} \oplus \mathfrak{h} = \mathfrak{g} = \left(\mathfrak{t} \oplus \mathfrak{d}\right) \oplus \mathfrak{h}_{\rm VAC} \equiv \mathfrak{f} \oplus \mathfrak{h}_{\rm VAC}\,, \qquad \mathfrak{d} = \bigoplus_{\widehat{S}=1}^{T} \left\langle J_{\widehat{S}} \right\rangle\,,$$

& assume reductivity of the latter, $[\mathfrak{h}_{\rm VAC}, \mathfrak{f}] \subset \mathfrak{f}$, and

$$[\mathfrak{h}_{\rm VAC}, \mathfrak{e}^{(0)}] \subset \mathfrak{e}^{(0)}\,, \qquad [\mathfrak{d}, \mathfrak{t}^{(0)}_{\rm VAC}] \subset \mathfrak{e}^{(0)}\,, \qquad [\mathfrak{d}, \mathfrak{e}^{(0)}] \subset \mathfrak{t}^{(0)}_{\rm VAC}\,,$$

as well as unimodularity of ρ on $\Psi_{\rm VAC}$ (volume preservation)

$$\forall\, h \in \mathrm{H}_{\rm VAC} \;:\; \det \rho(h)\restriction_{\mathfrak{t}^{(0)}_{\rm VAC}} \equiv \det \mathrm{T}_e \mathrm{Ad}_h \restriction_{\mathfrak{t}^{(0)}_{\rm VAC}} \stackrel{!}{=} 1\,.$$

Replace the former local sections of $\mathrm{G} \longrightarrow \mathrm{G}/\mathrm{H}$ by those of $\mathrm{G} \longrightarrow \mathrm{G}/\mathrm{H}_{\rm VAC}$ given by top. translates of

$$\begin{aligned} \sigma_0^{\rm VAC}\left(x^a, \theta^\alpha, \phi^{\widehat{S}}\right) &= \exp\left(\theta^\alpha \otimes Q_\alpha\right) \cdot \exp\left(x^a \otimes P_a\right) \cdot \exp\left(\phi^{\widehat{S}} \otimes J_{\widehat{S}}\right) \\ &\equiv \sigma_0\left(x^a, \theta^\alpha\right) \cdot \exp\left(\phi^{\widehat{S}} \otimes J_{\widehat{S}}\right) \end{aligned}$$

Th$^{\underline{\mathrm{m}}}$[rrS '19('17)]: If $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_{\mathrm{VAC}}, \mathfrak{t}^{(0)}_{\mathrm{VAC}})$ and ρ are constrained as above & there exists a $\mathsf{T}_e\mathrm{Ad}_{\mathrm{H}}$-invariant scalar product g on $\mathfrak{t}^{(0)}$ s.t.

$$\mathfrak{t}^{(0)}_{\mathrm{VAC}} \perp_{\mathrm{g}} \mathfrak{e}^{(0)}$$

then $\mathcal{A}_{\mathrm{DF}}^{\mathrm{GS},p(\mathrm{NG})}$ in the gauge σ_0 is (classically) equivalent to the **Green–Schwarz super-σ-model in the Hughes–Polchinski formulation** for $[\Omega_p, \mathrm{G}/\mathrm{H}_{\mathrm{VAC}}] \ni \widehat{\xi}$ in the gauge σ_0^{VAC} with

$$S^{(\mathrm{HP})}_{\mathrm{GS},p}[\widehat{\xi}] = \sum_{\tau \in \mathfrak{T}_{p+1}} \int_\tau \big(\sigma^{\mathrm{VAC}}_{i_\tau} \circ \widehat{\xi}\big)^* \Big(\underset{(p+1)}{\beta^{(\mathrm{HP})}} + \mathsf{d}^{-1}\, \pi^*_{\mathrm{G/H}} \underset{(p+2)}{\mathrm{H}} \Big)\,,$$

$$\underset{(p+1)}{\beta^{(\mathrm{HP})}} = \frac{\lambda_p}{(p+1)!}\, \epsilon_{\underline{a}_0 \underline{a}_1 \ldots \underline{a}_p}\, \theta^{\underline{a}_0}_{\mathrm{L}} \wedge \theta^{\underline{a}_1}_{\mathrm{L}} \wedge \cdots \wedge \theta^{\underline{a}_p}_{\mathrm{L}}\,, \qquad \lambda_p \in \mathbb{R}^\times\,,$$

partially reduced through imposition the **Inverse Higgs Constraints**

$$\big(\sigma^{\mathrm{VAC}}_{i_\tau} \circ \widehat{\xi}\big)^* \mathsf{P}^{\mathfrak{g}}_{\mathfrak{e}^{(0)}} \circ \theta_{\mathrm{L}} \overset{!}{=} 0\,,$$

$\Longleftrightarrow$ the EL eq$^{\underline{\mathrm{n}}}$s for the Goldstone modes $\phi^{\widehat{S}}$ are imposed.

Inspiration: [Hughes & Polchinski '86; Gauntlett *et al.* '90].

For the reduction of the Graßmann-odd DOFs, assume

$$\exists\, \mathsf{P}^{(1)}_{\rm VAC} = \mathsf{P}^{(1)}_{\rm VAC} \circ \mathsf{P}^{(1)}_{\rm VAC} \text{ w/ im}\, \mathsf{P}^{(1)}_{\rm VAC} \equiv \mathfrak{t}^{(1)}_{\rm VAC} \subset \mathfrak{t}^{(1)}_{\rm VAC} \oplus \mathfrak{e}^{(1)} \equiv \mathfrak{t}^{(1)} \subset \mathfrak{t} \quad \text{s.t.}$$

$$[\mathfrak{h}_{\rm VAC}, \mathfrak{t}^{(1)}_{\rm VAC}] \subset \mathfrak{t}^{(1)}_{\rm VAC}\,, \quad [\mathfrak{h}_{\rm VAC}, \mathfrak{e}^{(1)}] \subset \mathfrak{e}^{(1)} \quad \wedge \quad \{\mathfrak{t}^{(1)}_{\rm VAC}, \mathfrak{t}^{(1)}_{\rm VAC}\} \subset \mathfrak{t}^{(0)}_{\rm VAC} \oplus \mathfrak{h}$$

& the gauge invariance

$$\forall_{\kappa \in [\Omega_p, \mathfrak{t}^{(1)}_{\rm VAC}]} \,:\, \tfrac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0} \mathcal{A}^{{\rm GS},p({\rm HP})}_{\rm DF}\big[\Phi_{L_\kappa}(\widehat{\xi}(\cdot), t)\big] = 0$$

realised *exactly* as $\iota_{L_\kappa}(\underset{(p+1)}{\mathsf{d}\ \,\beta^{({\rm HP})}} + \pi^*_{{\rm G}/{\rm H}}\underset{(p+2)}{\rm H}) = 0$. Then*, the **vacuum slice** $\Sigma_{\rm VAC}$ within

$$\Sigma = \bigsqcup_{i \in I} \sigma_i^{\rm VAC}(\widehat{\mathcal{U}}_i^{\rm VAC})\,, \qquad \{[|l|]^{\rm VAC}_{g_i}(\widehat{\mathcal{U}}_0^{\rm VAC}) \equiv \widehat{\mathcal{U}}_i^{\rm VAC}\}_{i \in I} \in {\rm cov}({\rm G}/{\rm H}_{\rm VAC})$$

with the normal determined by the EL eq$^{\underline{\rm n}}$s of $\mathcal{A}^{{\rm GS},p({\rm HP})}_{\rm DF}$:

$$\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}^{(0)} \oplus \mathfrak{e}^{(1)} \oplus \mathfrak{d}} \circ \theta_{\rm L}\restriction_{\mathcal{T}\Sigma_{\rm VAC}} = 0$$

is covered by local flows of **linearised local κ-symmetries**:

$$\Big\langle \mathcal{K}^{\rm R}_{\delta\xi} \equiv \delta\xi^\mu \otimes \big(L_{t_\mu} + \Delta^{\underline{S}}_{i\,\mu}\, L_{J_{\underline{S}}}\big) \Big\rangle_{\delta\xi^\mu \in \Gamma_{\rm cord}(\mathcal{O}_{\mathfrak{g}_{\rm VAC} = \mathfrak{t}_{\rm VAC} \oplus \mathfrak{h}_{\rm VAC}})}\,.$$

Problems with the symmetries of the vacuum:

- they do not span an involutive distribution $\Longleftarrow$ their Lie superbracket closes (over $\Sigma_{\rm VAC}$!) only up to 'invisible' gauge transformations from $\mathfrak{h}$;
- they do not descend to $G/H_{\rm VAC}$.

Way out: κ-symmetry as a **gauged supervector-space symmetry of the vacuum slice** $\Sigma_{\rm VAC}$.

Reminder: Gauge symmetries of the (super-)σ-model geometrise as equivariant structures on the corresponding (super-)p-gerbes, of which the flat ones descend the models and the gerbes to the orbispace.

Upshot: Need to check the existence of a **supersymmetric linearised descendable κ-equivariant structure** on…

Remarkable upshot: The **extended Hughes–Polchinski p-gerbe**

$$\widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}} := \pi^*_{\mathrm{G/H}}\mathcal{G}^{(p)}_{\mathrm{GS}} \otimes \mathcal{I}^p_{\underset{(p+1)}{\beta^{(\mathrm{HP})}}}\,,$$

written in terms of $\pi_{\mathrm{G/H}} \ :\ \mathrm{G} \longrightarrow \mathrm{G/H}$, to be restricted to Σ_{VAC}.

NB: The metric DOFs of the NG formulation have been topologised/gerbified!

Theorem III [rrS '19, in writing] $\widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}}$ carries a *canonical* linearised (descendable) κ-equivariant structure

$$\Upsilon^\kappa_p \ :\ \mathscr{L}_{\mathcal{K}^{\wp}\cdot}\widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}} \cong \mathcal{I}^p_0 \qquad \text{over} \quad \mathfrak{g}_{\mathrm{VAC}} \times \Sigma_{\mathrm{VAC}}\,.$$

Q: What about compatibility with the global supersymmetry?

Problem: Large supersymmetry transformations $g \in \mathrm{G}$ map between sections in Σ_{VAC}, & κ-symmetries do NOT glue...

Way out: Consider linearised supersymmetry,

$$\Phi_A \ : \ \mathscr{L}_{\mathcal{K}^{\ell.}_A} \widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}} \cong \mathcal{I}^p_0 \,, \qquad A \in \overline{1, \dim \mathfrak{g}} \qquad \text{over} \quad \Sigma_{\mathrm{VAC}} \,,$$

implying – in the light of Th$^{\underline{\mathrm{m}}}$ III –

$$\mathscr{L}_{\widetilde{\mathcal{K}}^{\ell.}_A} \Upsilon_p \ : \ \mathscr{L}_{\widetilde{\mathcal{K}}^{\ell.}_A} \mathscr{L}_{\mathcal{K}^{\wp}_{\cdot}} \widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}} \cong \mathcal{I}^p_0 \,, \qquad A \in \overline{1, \dim \mathfrak{g}} \qquad \text{over} \quad \mathfrak{g}_{\mathrm{VAC}} \times \Sigma_{\mathrm{VAC}}$$

for natural lifts $\widetilde{\mathcal{K}}^{\ell.}_A$ of the $\mathcal{K}^{\ell.}_A$. We demand the existence of

$$\begin{tikzcd}
\mathscr{L}_{\widetilde{\mathcal{K}}^{\ell.}_A} \mathscr{L}_{\mathcal{K}^{\wp}_{\cdot}} \widehat{\mathcal{G}}^{(p)}_{\mathrm{HP}} \arrow[rr, "\mathscr{L}_{\widetilde{\mathcal{K}}^{\ell.}_A} \Upsilon^\kappa_p"] \arrow[d, "\Phi^\kappa_A"'] & & \mathscr{L}_{\widetilde{\mathcal{K}}^{\ell.}_A} \mathcal{I}^p_0 \equiv \mathcal{I}^p_0 \arrow[d, equal] \arrow[dll, Rightarrow, "\gamma^\kappa_A"'] \\
\mathcal{I}^p_0 & & \mathcal{I}^p_0 \arrow[ll, equal]
\end{tikzcd} \;.$$

Theorem IV [rrS '19, in writing] The $\{\widetilde{\mathcal{K}}^{\ell.}_A\}_{A \in \overline{1, \dim \mathfrak{g}}}$ exist, and so Υ^κ_p is *canonically* supersymmetric (the γ^κ_A exist).

Conclusions:

1. The physically relevant CaE super-$(p+2)$-cocycles on (supersymmetry) Lie supergroups should be geometrised and do geometrise in a large class of supergeometric settings as the GS super-p-gerbes of [rrS '17].
2. So do the supersymmetries, global and local. [rrS '19, in writing]
3. The super-p-gerbes are endowed with (the expected and) natural equivariant structures with respect to the supersymmetries of the relevant super-σ-models, in conformity with the underlying physics and the bosonic intuition. [rrS '19]
4. The construction generalises to physically relevant curved homogeneous spaces of supersymmetry Lie supergroups, and sometimes suggests corrections to the existing field-theory results. [rrS '18]

Outlook:

- Uniqueness of the construction and its relation to the approach of Schreiber *et al.*. Reconstruction of the full-fledged (weak) $(p+1)$-categories of super-p-gerbes.
- The relevance of the İnönü–Wigner-contractibility & the ultimate fate of the Metsaev–Tseytlin background.
- The higher supergeometry and superalgebra of supersymmetric defects (including boundary states) & state fusion.
- Relation to the worldvolume supersymmetry, possibly *via* Sorokin's Superembedding Formalism.
- Relation to the $\mathfrak{String}$-structure.
- The bosonisation/fermionisation defect.
- T-duality *via* the Hughes–Polchinski formulation, also in the bosonic setting.
- The gauging of the Ad.-supersymmetry and the ensuing Chern–Simons-type sTFT.
- ...

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Part III

Xtras

Intrusion: The Lie supergroup of the Metsaev-Tseytlin super-σ-model:

$\mathrm{SU}(2,2\,|\,4)$ with the body

$$|\mathrm{SU}(2,2\,|\,4)| = \mathrm{SO}(4,2)\times\mathrm{SO}(6)$$

and the Lie superalgebra (R-rescaled, for $R\in\mathbb{R}$)

$$\mathfrak{su}(2,2\,|\,4)^{(R)} = \Big(\big(\bigoplus_{a=0}^{4}\langle P_a\rangle\oplus\bigoplus_{a'=5}^{9}\langle P_{a'}\rangle\big)\oplus\bigoplus_{(\alpha,\alpha',I)\in\overline{1,4}\times\overline{1,4}\times\{1,2\}}\langle Q_{\alpha\alpha' I}\rangle\Big)$$
$$\oplus\Big(\bigoplus_{a,b=0}^{4}\langle J_{ab}=-J_{ba}\rangle\oplus\bigoplus_{a',b'=5}^{9}\langle J_{a'b'}=-J_{b'a'}\rangle\Big)$$

$$\{Q_{\alpha\alpha' I},Q_{\beta\beta' J}\} = \mathrm{i}\,\big(-2(\widehat{C}\,\widehat{\Gamma}^{\widehat{a}}\otimes\mathbf{1})_{\alpha\alpha' I\beta\beta' J}\,P_{\widehat{a}}+\tfrac{1}{R^2}\,(\widehat{C}\,\widehat{\Gamma}^{\widehat{ab}}\otimes\sigma_2)_{\alpha\alpha' I\beta\beta' J}\,J_{\widehat{ab}}\big)\,,$$

$$[Q_{\alpha\alpha' I},P_{\widehat{a}}] = -\tfrac{1}{2R}\,(\widehat{\Gamma}_{\widehat{a}}\otimes\sigma_2)^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}},P_{\widehat{b}}] = \tfrac{1}{R^2}\,\varepsilon_{\widehat{ab}}\,J_{\widehat{ab}}\,,\quad \varepsilon_{\widehat{ab}} = \begin{cases} +1 & \text{if } \widehat{a},\widehat{b}\in\overline{0,4} \\ -1 & \text{if } \widehat{a},\widehat{b}\in\overline{5,9} \\ 0 & \text{otherwise}\end{cases}\,,$$

$$[J_{\widehat{ab}},J_{\widehat{cd}}] = \eta_{\widehat{ad}}\,J_{\widehat{bc}}-\eta_{\widehat{ac}}\,J_{\widehat{bd}}+\eta_{\widehat{bc}}\,J_{\widehat{ad}}-\eta_{\widehat{bd}}\,J_{\widehat{ac}}\,,$$

$$[Q_{\alpha\alpha' I},J_{\widehat{ab}}] = -\tfrac{1}{2}\,\varepsilon_{\widehat{ab}}\,(\widehat{\Gamma}_{\widehat{ab}}\otimes\mathbf{1})^{\beta\beta' J}{}_{\alpha\alpha' I}\,Q_{\beta\beta' J}\,,\qquad [P_{\widehat{a}},J_{\widehat{bc}}] = \eta_{\widehat{ab}}\,P_{\widehat{c}}-\eta_{\widehat{ac}}\,P_{\widehat{b}}\,.$$

with the İnönü-Wigner asymptote $\mathfrak{su}(2,2\,|\,4)^{(R)}\xrightarrow{R\to\infty}\mathfrak{smink}(9,1\,|\,32)$.

Some Lie-superalgebra cohomology…

Def$^{\underline{n}}$: A (**left**) $\widehat{\mathfrak{g}}$**-module** of an LSA $\widehat{\mathfrak{g}}$ is a pair $(\widehat{V}, \ell_\cdot)$ composed of a $\mathbb{K}$-linear superspace $\widehat{V} = \widehat{V}^{(0)} \oplus \widehat{V}^{(1)}$ and a left $\widehat{\mathfrak{g}}$-action

$$\ell_\cdot \; : \; \widehat{\mathfrak{g}} \times \widehat{V} \longrightarrow \widehat{V} \; : \; (X, v) \longmapsto X \triangleright v$$

consistent with the $\mathbb{Z}/2\mathbb{Z}$-gradings, $\widetilde{X \triangleright v} = \widetilde{X} + \widetilde{v}$, and such that for any two homogeneous elements $X_1, X_2 \in \mathfrak{g}$ and $v \in \widehat{V}$,

$$[X_1, X_2\} \triangleright v = X_1 \triangleright (X_2 \triangleright v) - (-1)^{\widetilde{X_1} \cdot \widetilde{X_2}} X_2 \triangleright (X_1 \triangleright v) \,.$$

and the fundamental…

Def$^{\underline{n}}$: Let $(\widehat{\mathfrak{g}}, [\cdot,\cdot\})$ be an LSA over field $\mathbb{K}$ and let $(\widehat{V}, \ell.)$ be a $\widehat{\mathfrak{g}}$-module. A **p-cochain on $\widehat{\mathfrak{g}}$ with values in $\widehat{V}$** is a p-linear map $\underset{(p)}{\varphi} : \widehat{\mathfrak{g}}^{\times p} \longrightarrow \widehat{V}$ that is totally super-skewsymmetric,

$$\begin{aligned} & \underset{(p)}{\varphi}(X_1, X_2, \ldots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_p) \\ = \quad & -(-1)^{\widetilde{X_i}\widetilde{X_{i+1}}} \underset{(p)}{\varphi}(X_1, X_2, \ldots, X_p) . \end{aligned}$$

They form a $\mathbb{Z}_2$-graded **group of p-cochains on $\widehat{\mathfrak{g}}$ valued in $\widehat{V}$**,

$$C^p(\widehat{\mathfrak{g}}, \widehat{V}) = C^p_0(\widehat{\mathfrak{g}}, \widehat{V}) \oplus C^p_1(\widehat{\mathfrak{g}}, \widehat{V}) ,$$

with $\underset{(p)}{\varphi}(X_1, X_2, \ldots, X_p) \in \widehat{V}_{\sum_{i=1}^p \widetilde{X_i}+n}$ for $\underset{(p)}{\varphi} \in C^p_n(\widehat{\mathfrak{g}}, \widehat{V})$, composed of **even** ($n = 0$) and **odd** ($n = 1$) p-cochains.

These groups form a semi-bounded complex

$$C^\bullet(\widehat{\mathfrak{g}}, \widehat{V}) \ : \ C^0(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(0)}_{\widehat{\mathfrak{g}}}} C^1(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(1)}_{\widehat{\mathfrak{g}}}} \cdots \xrightarrow{\delta^{(p-1)}_{\widehat{\mathfrak{g}}}} C^p(\widehat{\mathfrak{g}}, \widehat{V}) \xrightarrow{\delta^{(p)}_{\widehat{\mathfrak{g}}}} \cdots$$

The coboundary operators

$$\delta_{\mathfrak{g}}^{(p)} \ : \ C_n^p(\mathfrak{g}, V) \longrightarrow C_n^{p+1}(\mathfrak{g}, V)$$

evaluate on the homogeneous $X_i \in \mathfrak{g}, \ i \in \overline{0, p+1}, \ \underset{(p)}{\varphi} \in C^p(\mathfrak{g}, V)$

as

$$(\delta_{\mathfrak{g}}^{(0)} \underset{(0)}{\varphi})(X) := (-1)^{|X_0| \cdot |\underset{(0)}{\varphi}|} X_0 \triangleright \underset{(0)}{\varphi},$$

$$(\delta_{\mathfrak{g}}^{(p)} \underset{(p)}{\varphi})(X_1, X_2, \ldots, X_{p+1}) := \sum_{j=1}^{p+1} (-1)^{|X_j| \cdot |\underset{(p)}{\varphi}| + S(X_j)} X_j \triangleright \underset{(p)}{\varphi}(X_1, X_2, \underset{\widehat{j}}{\ldots}, X_{p+1})$$

$$+ \sum_{1 \leq j < k \leq p+1} (-1)^{S(X_j) + S(X_k) + |X_j| \cdot |X_k|} \underset{(p)}{\varphi}([X_j, X_k\}, X_1, X_2, \underset{\widehat{j,k}}{\ldots}, X_{p+1}),$$

$$S(X_i) := |X_i| \cdot \sum_{j=1}^{i-1} |X_j| + i - 1 .$$

The $\mathbb{Z}/2\mathbb{Z}$-graded **V-valued cohomology groups of $\mathfrak{g}$** are

$$H^p(\mathfrak{g}, V) := H_0^p(\mathfrak{g}, V) \oplus H_1^p(\mathfrak{g}, V), \quad H_n^p(\mathfrak{g}, V) := \frac{\ker \delta_{\mathfrak{g}}^{(p)} \restriction_{C_n^p(\mathfrak{g}, V)}}{\operatorname{im} \delta_{\mathfrak{g}}^{(p-1)} \restriction_{C_n^{p-1}(\mathfrak{g}, V)}} .$$

Def$^{\underline{\mathrm{n}}}$: Let $(\widehat{\mathfrak{g}}, [\cdot,\cdot\}_{\widehat{\mathfrak{g}}})$ and $(\widehat{\mathfrak{a}}, [\cdot,\cdot\}_{\widehat{\mathfrak{a}}})$ be LSAs over field $\mathbb{K}$. A **supercentral extension of $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{a}}$** is an LSA $(\widetilde{\mathfrak{g}}, [\cdot,\cdot\}_{\widetilde{\mathfrak{g}}})$ over $\mathbb{K}$ that determines a short exact sequence of LSAs

$$\mathbf{0} \longrightarrow \mathfrak{a} \xrightarrow{\jmath_{\widehat{\mathfrak{a}}}} \widetilde{\mathfrak{g}} \xrightarrow{\pi_{\widehat{\mathfrak{g}}}} \mathfrak{g} \longrightarrow \mathbf{0}\,,$$

written in terms of an LSA mono $\jmath_{\widehat{\mathfrak{a}}}$ and of an LSA epi $\pi_{\widehat{\mathfrak{g}}}$, and s.t. $\jmath_{\widehat{\mathfrak{a}}}(\widehat{\mathfrak{a}}) \subset \mathfrak{z}(\widetilde{\mathfrak{g}})$ (the supercentre of $\widetilde{\mathfrak{g}}$). Whenever $\pi_{\mathfrak{g}}$ admits an LSA **section**, *i.e.*, there exists

$$\sigma \in \mathrm{Hom}_{\mathrm{sLie}}(\widehat{\mathfrak{g}}, \widetilde{\mathfrak{g}})\,, \qquad \pi_{\widehat{\mathfrak{g}}} \circ \sigma = \mathrm{id}_{\widehat{\mathfrak{g}}}\,,$$

the supercentral extension is said to **split**.

An **equivalence of supercentral extensions** $\widetilde{\mathfrak{g}}_\alpha, \alpha \in \{1,2\}$ of $\widehat{\mathfrak{g}}$ by $\widehat{\mathfrak{a}}$ is represented by a commutative diagram of LSAs

$$\begin{tikzcd}
 & & \widetilde{\mathfrak{g}}_1 \arrow[dd, "\cong"] \arrow[dr] & & \\
\mathbf{0} \arrow[r] & \widehat{\mathfrak{a}} \arrow[ur] \arrow[dr] & & \widehat{\mathfrak{g}} \arrow[r] & \mathbf{0}\,. \\
 & & \widetilde{\mathfrak{g}}_2 \arrow[ur] & &
\end{tikzcd}$$

The relevant one-way ticket:

Given an LSA $(\mathfrak{g}, [\cdot,\cdot\}_{\mathfrak{g}})$ and its supercommutative module $\mathfrak{a}$, as well as a representative $\Theta \in Z_0^2(\mathfrak{g},\mathfrak{a})$ of a class in $H_0^2(\mathfrak{g},\mathfrak{a})$, we define

$$\widetilde{\mathfrak{g}} := \mathfrak{a} \oplus \mathfrak{g}$$

and put on it the Lie superbracket

$$\begin{aligned} [\cdot,\cdot\}_\Theta \quad &: \quad \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}} \\ &: \quad \big((A_1, X_1), (A_2, X_2)\big) \longmapsto \big(\Theta(X_1, X_2), [X_1, X_2\}_{\mathfrak{g}}\big) . \end{aligned}$$