Moduli of sheaves supported on curves of low genus contained in a quadric surface

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- M. Maican. *On two moduli spaces of sheaves supported on quadric surfaces.* Osaka Journal of Mathematics **54** (2017), 323–333.
- M. Maican. *Moduli of sheaves supported on curves of genus two in a quadric surface.* Geometriae Dedicata **199** (2019), 307–334.
- M. Maican. *Moduli of stable sheaves supported on curves of genus three contained in a quadric surface.* Advances in Geometry, to appear.
- M. Maican. *On the geometry of the moduli space of sheaves supported on curves of genus four contained in a quadric surface.* arXiv:1704.07011

$\mathbb{P}^1$ complex projective line

$\mathbb{P}^1 \times \mathbb{P}^1$ quadric surface endowed with the polarization $\mathcal{O}(1,1)$

$\mathcal{F}$ coherent algebraic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with support of dimension 1

Recall: $\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} \mathrm{H}^i(\mathcal{F}) = \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{F}) - \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{F})$ is the *Euler characteristic* of $\mathcal{F}$.

There are $r, s, t \in \mathbb{Z}$, $r, s \geq 0$, such that, for all $m, n \in \mathbb{Z}$,

$$\chi(\mathcal{F} \otimes \mathcal{O}(m,n)) = rm + sn + t$$

$t = \chi(\mathcal{F})$. The *multiplicity* of $\mathcal{F}$ is $r + s$.

$P_{\mathcal{F}}(m,n) = rm + sn + t$ *Hilbert polynomial* of $\mathcal{F}$

$\mathrm{p}(\mathcal{F}) = \dfrac{t}{r+s}$ *slope* of $\mathcal{F}$ (with respect to the fixed polarization)

Definition (Gieseker and Maruyama)

$\mathcal{F}$ is *semi-stable* relative to $\mathcal{O}(1,1)$ if

1. $\mathcal{F}$ is pure, i.e. there are no subsheaves supported on points;
2. for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $\mathrm{p}(\mathcal{F}') \le \mathrm{p}(\mathcal{F})$.

Theorem (Simpson)

There exists a coarse moduli space $\mathrm{M}(rm + sn + t)$ *of sheaves* $\mathcal{F}$ *on* $\mathbb{P}^1 \times \mathbb{P}^1$ *with* $P_{\mathcal{F}} = rm + sn + t$ *that are semi-stable relative to* $\mathcal{O}(1,1)$*.* $\mathrm{M}(rm + sn + t)$ *is a projective variety. If* $\gcd(r + s, t) = 1$*, then it is smooth.*

Proposition (Le Potier)

$\mathrm{M}(rm + sn + t)$ *is irreducible, of dimension* $2rs + 1$ *if* $r > 0$ *and* $s > 0$*.*

Theorem (Maican)

Assume that $\mathcal{F} \in \mathrm{M}(rm + sn + t)$. *Then*

$$\mathrm{H}^0(\mathcal{F}(i,j)) = 0 \quad \textit{if} \quad i, j < 1 - \frac{rs + t}{r + s}$$
$$\mathrm{H}^1(\mathcal{F}(i,j)) = 0 \quad \textit{if} \quad i, j > -1 + \frac{rs - t}{r + s}$$

Easy examples (Genus zero case)

1. $\mathrm{M}(rm + r) = \{\mathcal{O}_C | \ \deg(C) = (0, r)\} = |\mathcal{O}(0, r)| \simeq \mathbb{P}^r$.
2. $\mathrm{M}(rm + t) = \emptyset$ if $0 < t < r$. Assume that $\mathcal{F} \in \mathrm{M}(rm + t)$.

$$\left.\begin{array}{l} 0 < 1 - \dfrac{r \cdot 0 + t}{r + 0} \Longrightarrow \mathrm{H}^0(\mathcal{F}) = 0 \\ 0 > -1 + \dfrac{r \cdot 0 - t}{r + 0} \Longrightarrow \mathrm{H}^1(\mathcal{F}) = 0 \end{array}\right\} \Longrightarrow t = \chi(\mathcal{F}) = 0. \text{ Absurd!}$$

3. $\mathrm{M}(rm + n + t) = \{\mathcal{O}_C(0, t-1) | \ \deg(C) = (1, r)\} = |\mathcal{O}(1, r)| \simeq \mathbb{P}^{2r+1}$ for $0 \leq t \leq r$.

For a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ and a point $p \in C$ we denote by $\mathcal{O}_C(p)$ the unique non-split extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(p) \longrightarrow \mathbb{C}_p \longrightarrow 0$$

Proposition (Ballico and Huh)

$\mathrm{M}(2m + 2n + 1)$ *is isomorphic to the universal curve of degree* $(2, 2)$*, so it is a bundle with base* $\mathbb{P}^1 \times \mathbb{P}^1$ *and fiber* $\mathbb{P}^7$*. More precisely,*

$$\mathrm{M}(2m + 2n + 1) = \{\mathcal{O}_C(p) | \ \deg(C) = (2, 2), \ p \in C\}$$

$\mathrm{M}(2m+2n+2)$ first non-trivial example

$\mathrm{N}(4;2,2) = \{\varphi \in \mathrm{M}_{2,2}(\mathbb{C}^4) |\ \varphi$ has linearly independent rows and linearly independent columns$\}/\!\!/ \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) =$ Kronecker moduli space of 2×2-matrices with entries in $\mathbb{C}^4$

Proposition (Chung and Moon, Maican)

$\mathrm{M}(2m+2n+2)$ *is the blow-up of* $\mathrm{N}(4;2,2)$ *at two (regular) points.*

Proof.

$\mathbb{W} = \mathrm{Hom}(2\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1), \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \oplus 2\mathcal{O})$

The following group acts by left and right multiplication: $\mathbb{G} = \mathrm{Aut}(2\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \times \mathrm{Aut}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \oplus 2\mathcal{O})$

$\mathbb{W}_0 = \{\psi \in \mathbb{W} |\ \psi$ injective, $\mathcal{C}oker(\psi)$ semi-stable$\}$

$\mathrm{M}(2m+2n+2) \simeq \mathbb{W}_0 /\!\!/ \mathbb{G}$

$\square$

Chung and Moon use Fourier-Mukai transforms and also the isomorphism $\mathrm{N}(4;2,2) \simeq \mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ due to Le Potier

$\mathbb{P} = (\mathrm{S}^2\,\mathbb{C}^4 \oplus \bigwedge^4 \mathbb{C}^4)/\mathbb{C}^*$ where the action is given by $a.(f,e) = (af, a^2e)$

$\mathbb{P} = \mathbb{P}(\underbrace{1,\ldots,1}_{\text{10 times}},2)$

Proposition (Maican)

We have an isomorphism

$$\mathrm{N}(4;2,2) \longrightarrow \{(f,e)|\ \mathrm{res}(f) = e^2\} \subset \mathbb{P}$$

$$[\varphi] \longmapsto (\det(\varphi), \varphi_{11} \wedge \varphi_{12} \wedge \varphi_{21} \wedge \varphi_{22})$$

Aim: compute the Betti numbers $\mathrm{b}_i = \dim_{\mathbb{Q}} \mathrm{H}^i(\mathrm{M}(rm + sn + t), \mathbb{Q})$

We will do this only for the following moduli spaces:

$\mathrm{M}(3m + 2n + 1)$. Here the supports are curves of genus 2.

$\mathrm{M}(4m + 2n + 1)$. Here the supports are curves of genus 3.

$\mathrm{M}(3m + 3n + 1)$. Here the supports are curves of genus 4.

Recall: the *Poincaré polynomial* of a smooth projective variety M is

$$\mathrm{P}(M) = \sum_{i \geq 0} \mathrm{b}_i(M) x^i$$

$\chi(M) = \sum_{i \geq 0} (-1)^i \, \mathrm{b}_i(M)$ is the *topological Euler characteristic* of M.

If M is quasiprojective, $\mathrm{P}(M)$ is defined using the virtual Betti numbers.

Recall: For any quasiprojective variety X and $i \in \mathbb{Z}_{\geq 0}$ there exists a unique integer $\mathrm{b}_i^{\mathrm{vir}}(X)$ such that the collection of all such integers satisfies the following properties:

1. $\mathrm{b}_i^{\mathrm{vir}}(X) = \mathrm{b}_i(X)$ if X is smooth and projective;
2. $\mathrm{b}_i^{\mathrm{vir}}(X \setminus Y) = \mathrm{b}_i^{\mathrm{vir}}(X) - \mathrm{b}_i^{\mathrm{vir}}(Y)$ if $Y \subset X$ is a closed subvariety.

The virtual Betti numbers are algebraic invariants depending only on the reduced structure. In general, they can be negative. Their existence can be proved using the Weak Factorization Theorem. The (virtual) Poincaré polynomial satisfies the following motivic properties:

1. $\mathrm{P}(X \setminus Y) = \mathrm{P}(X) - \mathrm{P}(Y)$ if $Y \subset X$ is a closed subvariety.
2. $\mathrm{P}(X) = \mathrm{P}(Y)\,\mathrm{P}(F)$ if $X \to Y$ is a bundle (i.e. Zariski locally trivial fibration) with fiber F.

$X =$ Calabi-Yau (i.e. ω_X is trivial) threefold

$\beta \in \mathrm{H}_2(X, \mathbb{Z})$

$\mathbf{M} =$ moduli space of D-branes supported on curves of class β

The mathematical definition of a D-brane is not clear. It is thought that for a smooth curve $C \subset X$, the D-branes supported on C are the line bundles on C of fixed degree (viewed as sheaves on X).

Gopakumar and Vafa define invariants $\mathrm{n}_\beta^g \in \mathbb{Z}$ "counting the number of curves of genus g and class β in X."

Consider the support map $\mathbf{M} \to B$. Assume that B admits a decomposition $B = \amalg B_i$, where B_i are connected and the support map is a product over B_i. Thus, $\mathbf{M} = \amalg B_i \times F_i$. The Gopakumar-Vafa invariants are defined by the formula

$$\sum_i (-1)^{\dim B_i} \chi(B_i) \frac{\mathrm{P}(F_i)}{x^{\dim F_i}} = \sum_{g \geq 0} \mathrm{n}_\beta^g \frac{(x+1)^{2g}}{x^g}$$

Let N^g_β be the Gromov-Witten numbers of genus g curves of class β in X.

Conjecture (Gopakumar and Vafa)

$$\sum_{g\geq 0,\, \beta\neq 0} \mathrm{N}^g_\beta\, x^{2g-2} y^\beta = \sum_{g\geq 0,\, \beta\neq 0,\, k>0} \frac{\mathrm{n}^g_\beta}{k}\left(2\sin\left(\frac{kx}{2}\right)\right)^{2g-2} y^{k\beta}$$

In particular,

$$\mathrm{N}^0_\beta = \sum_{k|\beta,\, k>0} \frac{\mathrm{n}^0_{\beta/k}}{k^3}$$

For mathematicians, **M** is the moduli space $\mathrm{M}_X(\beta)$ of sheaves $\mathcal{F}$ on X, that are semi-stable with respect to a certain polarization, that are supported on curves of class β, and such that $\chi(\mathcal{F}) = 1$.

Definition (Katz)

The genus-zero Gopakumar-Vafa invariants of X are

$$\mathrm{n}_\beta(X) = \deg[\mathrm{M}_X(\beta)]^{\mathrm{vir}}$$

$\operatorname{vdim} \mathrm{M}_X(\beta) = \dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) - \dim \operatorname{Ext}^2(\mathcal{F}, \mathcal{F})$ vanishes by Serre duality $\implies \mathrm{n}_\beta(X) \in \mathrm{H}_0(\mathrm{M}_X^\beta(X)) = \mathbb{Z}$.

Conjecture (Katz)

$$\mathrm{N}_\beta^0(X) = \sum_{k|\beta,\, k>0} \frac{\mathrm{n}_{\beta/k}(X)}{k^3}$$

Assume now that $\mathbf{X}$ is the total space of $\omega_{\mathbb{P}^1 \times \mathbb{P}^1}$.

The polarization is the pull-back of $\mathcal{O}(1,1)$.

$\mathrm{H}_2(\mathbf{X}) = \mathrm{H}_2(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z}$

Given $\beta = (r, s)$, $r, s > 0$, Choi observed that a semi-stable sheaf on $\mathbf{X}$ supported on a curve of class β, is, in fact, supported on the zero-section $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbf{X}$.

$\Longrightarrow \mathrm{M}_{\mathbf{X}}(\beta) = \mathrm{M}_{\mathbb{P}^1 \times \mathbb{P}^1}(rm + sn + 1)$, denoted $\mathbf{M}$.

Since $\mathrm{M}_{\mathbf{X}}(\beta)$ is smooth, $[\mathrm{M}_{\mathbf{X}}(\beta)]^{\mathrm{vir}}$ is the top Chern class of its obstruction bundle. By Serre duality the obstruction bundle is isomorphic to the cotangent bundle: pointwise we have

$$\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) \simeq (\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}))^*.$$

Gauss-Bonnet formula $\Longrightarrow$

$$\mathrm{n}_{(r,s)}(\mathbf{X}) = (-1)^{\dim \mathbf{M}} \chi(\mathbf{M}) = -\chi(\mathbf{M})$$

Definition

A *coherent system* $(\Gamma, \mathcal{F})$ on $\mathbb{P}^1 \times \mathbb{P}^1$ consists of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and a vector subspace $\Gamma \subset \mathrm{H}^0(\mathcal{F})$. Consider $\alpha \in (0, \infty)$. The α*-slope* of $(\Gamma, \mathcal{F})$ is

$$\mathrm{p}_\alpha(\Gamma, \mathcal{F}) = \frac{\alpha \dim_{\mathbb{C}} \Gamma + t}{r + s}$$

We say that $(\Gamma, \mathcal{F})$ is α*-semi-stable* if $\mathcal{F}$ is pure and for any coherent subsystem $(\Gamma', \mathcal{F}') \subset (\Gamma, \mathcal{F})$ we have $\mathrm{p}_\alpha(\Gamma', \mathcal{F}') \leq \mathrm{p}_\alpha(\Gamma, \mathcal{F})$.

Theorem (Le Potier)

There exists a coarse moduli space $\mathrm{Syst}^\alpha(rm + sn + t)$ *parametrizing* α*-semi-stable coherent systems* $(\Gamma, \mathcal{F})$ *on* $\mathbb{P}^1 \times \mathbb{P}^1$ *for which* $P_{\mathcal{F}} = rm + sn + t$. *Moreover, this moduli space is a projetive variety.*

$\mathrm{Syst}^\alpha(rm + sn + t)$ decomposes into disconnected components according to $\dim \Gamma$. The component corresponding to $\Gamma = 0$ is $\mathrm{M}(rm + sn + t)$. The component corresponding to $\dim \Gamma = 1$ will be denoted $\mathrm{M}^\alpha(rm + sn + t)$.

There exist finitely many *singular values* of α (also called *walls*) $\alpha_1 < \ldots < \alpha_k$ $(k \geq 0)$ such that $\mathrm{M}^{\alpha}(rm+sn+t)$ remains unchanged if α varies in one of the intervals $(0, \alpha_1)$, (α_1, α_2), ..., (α_k, ∞). When α transits a singular value we have a wall-crossing diagram as below, which we expect to be a flipping diagram. Given $(\Gamma, \mathcal{F}) \in \mathrm{M}^{\alpha_i \pm \epsilon}(rm+sn+t)$ then $(\Gamma, \mathcal{F}) \in \mathrm{M}^{\alpha_i}(rm+sn+t)$. We obtain canonical morphisms

$$\begin{tikzcd}
\mathrm{M}^{\alpha_i-\epsilon}(rm+sn+t) \arrow[dr, "\rho_-"'] & & \mathrm{M}^{\alpha_i+\epsilon}(rm+sn+t) \arrow[dl, "\rho_+"] \\
& \mathrm{M}^{\alpha_i}(rm+sn+t) &
\end{tikzcd}$$

Moreover, ρ_- and ρ_+ are isomorphisms over the set of stable points $\mathrm{M}^{\alpha_i}(rm+sn+t)^{\mathrm{s}}$.

$\mathrm{M}^{0+}(rm+sn+t) = \mathrm{M}^{\alpha}(rm+sn+t)$ for $\alpha \in (0, \alpha_1)$

$\mathrm{M}^{\infty}(rm+sn+t) = \mathrm{M}^{\alpha}(rm+sn+t)$ for $\alpha \in (\alpha_k, \infty)$

If $(\Gamma, \mathcal{F}) \in \mathrm{M}^{0+}(rm+sn+t)$ then $\mathcal{F} \in \mathrm{M}(rm+sn+t)$. Indeed, for any subsheaf $\mathcal{F}' \subset \mathcal{F}$, we have the subsystem $(0, \mathcal{F}') \subset (\Gamma, \mathcal{F}) \Longrightarrow$

$$\mathrm{p}(\mathcal{F}') = \mathrm{p}_{\alpha}(0, \mathcal{F}') \leq \mathrm{p}_{\alpha}(\Gamma, \mathcal{F}) = \frac{\alpha}{r+s} + \mathrm{p}(\mathcal{F})$$

for all $\alpha \in (0, \alpha_1)$. Taking limit at $\alpha \to 0$ yields $\mathrm{p}(\mathcal{F}') \leq \mathrm{p}(\mathcal{F})$.
We have the *forgetful morphism*

$$\Phi \colon \mathrm{M}^{0+}(rm+sn+t) \longrightarrow \mathrm{M}(rm+sn+t), \qquad \Phi(\Gamma, \mathcal{F}) = [\mathcal{F}].$$

At the other extreme we have the following:

Proposition (Pandharipande and Thomas)

For $\alpha \gg 0$, a coherent system $(\Gamma, \mathcal{F})$ with $\dim \Gamma = 1$ is α-semi-stable if and only if $\mathcal{F}$ is pure and $\mathcal{F}/\mathcal{O}_C$ has dimension zero or is zero, where $\mathcal{O}_C$ is the subsheaf of $\mathcal{F}$ generated by Γ. In this case there is a unique subscheme $Z \subset C$ of dimension zero and length $t - r - s + rs$ such that $\mathcal{F}/\mathcal{O}_C \simeq \mathcal{E}xt^2_{\mathcal{O}}(\mathcal{O}_Z, \mathcal{O})$. We have an isomorphism

$$\mathrm{M}^{\infty}(rm + sn + t) \longrightarrow \mathrm{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(t - r - s + rs, rm + sn + r + s - rs)$$

$$(\Gamma, \mathcal{F}) \longmapsto (Z, C)$$

The r.h.s. denotes the flag Hilbert scheme of zero-dimensional schemes of length $t - r - s + rs$ contained in curves of degree (s, r).

If $t = r + s - rs$, then Z is empty, so we have the isomorphism

$$\mathrm{M}^{\infty}(rm + sn + r + s - rs) \simeq |\mathcal{O}(s, r)| \simeq \mathbb{P}^{r+s+rs}.$$

Proposition

For $P(m,n) = 3m + 2n + 1$ *there is only one singular value* $\alpha = 4$.

Proof.

If α is singular $\Longrightarrow$ there is $(\Gamma, \mathcal{F}) \in \mathrm{M}^{\alpha}(3m + 2n + 1)$ that is not stable $\Longrightarrow$ there exists $(\Gamma', \mathcal{F}') \in \mathrm{M}^{\alpha}(rm + sn + t)$, which is either a subsystem or a quotient system of $(\Gamma, \mathcal{F})$, and such that $\mathrm{p}_{\alpha}(\Gamma', \mathcal{F}') = \mathrm{p}_{\alpha}(\Gamma, \mathcal{F})$. We need to solve the equation

$$\frac{\alpha + t}{r + s} = \frac{\alpha + 1}{5}$$

where $0 \leq r \leq 3$, $0 \leq s \leq 2$, $1 \leq r + s \leq 4$, $\alpha \in (0, \infty)$.
$\mathrm{M}^{\alpha}(rm + sn + t) \neq \emptyset \Longrightarrow t \geq r + s - rs$.
If $r \leq 1$ or $s \leq 1$, then $t \geq 1$ and there are no positive solutions α.
If $r = 2$ and $s = 2$, the only positive solution is $\alpha = 4$ for $t = 0$. □

$\mathbf{M} = \mathrm{M}(3m + 2n + 1)$ smooth of dimension 13

$\mathbf{M}^\alpha = \mathrm{M}^\alpha(3m + 2n + 1)$

$\mathbf{M}^\infty \simeq \mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 3m + 2n - 1) =$ bundle with base $\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(2)$ and fiber $\mathbb{P}^9$

$$\begin{tikzcd}
\mathbf{M}^\infty \arrow[dr, "\rho_\infty"'] & & \mathbf{M}^{0+} \arrow[dl, "\rho_0"] \arrow[dr, "\Phi"] & \\
& \mathbf{M}^4 & & \mathbf{M}
\end{tikzcd}$$

Corollary (of the proof)

If $(\Gamma, \mathcal{F}) \in \mathbf{M}^4$ *but is not stable (we say that* $(\Gamma, \mathcal{F})$ *is* strictly semi-stable) then we have one of the following exact sequences:

$$0 \longrightarrow (\Gamma'', \mathcal{F}'') \longrightarrow (\Gamma, \mathcal{F}) \longrightarrow (\Gamma', \mathcal{F}') \longrightarrow 0,$$

$$0 \longrightarrow (\Gamma', \mathcal{F}') \longrightarrow (\Gamma, \mathcal{F}) \longrightarrow (\Gamma'', \mathcal{F}'') \longrightarrow 0,$$

where $(\Gamma', \mathcal{F}') \in \mathrm{M}^4(2m + 2n)$ and $(\Gamma'', \mathcal{F}'') \in \mathrm{M}(m + 1)$.

There are no singular values of α for $P(m, n) = 2m + 2n \Longrightarrow$

$$\mathrm{M}^4(2m + 2n) = \mathrm{M}^\infty(2m + 2n) = |\mathcal{O}(2, 2)| \simeq \mathbb{P}^8$$

and $(\Gamma', \mathcal{F}') = (\mathrm{H}^0(\mathcal{O}_Q), \mathcal{O}_Q)$ for a curve $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\deg(Q) = (2, 2)$. Also, $(\Gamma'', \mathcal{F}'') = (0, \mathcal{O}_L)$ for a line $L \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\deg(L) = (0, 1) \Longrightarrow$ $\mathrm{M}(m + 1) \simeq \mathbb{P}^1$. Conclusion:

$$(\mathbf{M}^4)^{\mathrm{sss}} \simeq \mathrm{M}^4(2m + 2n) \times \mathrm{M}(m + 1) \simeq \mathbb{P}^8 \times \mathbb{P}^1$$

$$
\begin{tikzcd}
\mathbf{M}^{\infty} \arrow[dr, "\rho_\infty"'] & & \mathbf{M}^{0+} \arrow[dl, "\rho_0"] \\
& \mathbf{M}^4 &
\end{tikzcd}
$$

ρ_∞ and ρ_0 are isomorphisms over $(\mathbf{M}^4)^{\mathrm{s}}$

$\rho_\infty^{-1}((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}'')) \simeq \mathbb{P}(\mathrm{Ext}^1((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}'')))$

$\mathrm{Ext}^1((\mathrm{H}^0(\mathcal{O}_Q), \mathcal{O}_Q), (0, \mathcal{O}_L)) \simeq \mathbb{C}^3$

$F^\infty = \rho_\infty^{-1}((\mathbf{M}^4)^{\mathrm{sss}})$ bundle with base $(\mathbf{M}^4)^{\mathrm{sss}}$ and fiber $\mathbb{P}^2$

$\rho_0^{-1}((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}'')) \simeq \mathbb{P}(\mathrm{Ext}^1((\Gamma'', \mathcal{F}''), (\Gamma', \mathcal{F}')))$

$\mathrm{Ext}^1((0, \mathcal{O}_L), (\mathrm{H}^0(\mathcal{O}_Q), \mathcal{O}_Q)) \simeq \mathbb{C}^2$

$F^0 = \rho_0^{-1}((\mathbf{M}^4)^{\mathrm{sss}})$ bundle with base $(\mathbf{M}^4)^{\mathrm{sss}}$ and fiber $\mathbb{P}^1$

$\Phi^{-1}([\mathcal{F}]) \simeq \mathbb{P}(\mathrm{H}^0(\mathcal{F})) \Longrightarrow$
$\Phi\colon \mathbf{M}^{0+} \longrightarrow \mathbf{M}$ is an isomorphism over $\{[\mathcal{F}] \in \mathbf{M} |\ \mathrm{H}^0(\mathcal{F}) \simeq \mathbb{C}\}$

Proposition

For $[\mathcal{F}] \in \mathbf{M}$, $\dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{F}) > 1$ *if and only if* $\mathcal{F} \simeq \mathcal{O}_C(0,1)$ *for a curve* $C \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\deg(C) = (2,3)$. *The Brill-Noether locus*

$$\mathbf{M}_2 = \{[\mathcal{F}] \in \mathbf{M} |\ \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{F}) > 1\}$$

is closed and smooth of codimension 2*. In fact,* $\mathbf{M}_2 = |\mathcal{O}(2,3)| \simeq \mathbb{P}^{11}$.

$\mathrm{H}^0(\mathcal{O}_C(0,1)) \simeq \mathbb{C}^2 \Longrightarrow \Phi^{-1}(\mathbf{M}_2)$ is a bundle with base $\mathbf{M}_2$ and fiber $\mathbb{P}^1$

Theorem (Maican)

$\mathrm{H}^i(\mathbf{M}, \mathbb{Z})$ *have no torsion. The Poincaré polynomial of* $\mathbf{M}$ *is*

$$x^{26} + 3x^{24} + 8x^{22} + 10x^{20} + 11x^{18} + 11x^{16} + 11x^{14} + 11x^{12} + 11x^{10} + 11x^{8} + 10x^{6} + 8x^{4} + 3x^{2} + 1.$$

The Euler characteristic of $\mathbf{M}$ *is* 110.

Katz' conjecture takes the form: $\mathrm{N}^0_{(3,2)}(\mathbf{X}) = -110$.

Proof.

Recall the diagram $\mathbf{M}^{\infty} \xrightarrow{\rho_\infty} \mathbf{M}^4 \xleftarrow{\rho_0} \mathbf{M}^{0+} \xrightarrow{\Phi} \mathbf{M}$.

$$\begin{aligned}
\mathrm{P}(\mathbf{M}) &= \mathrm{P}(\mathbf{M} \setminus \mathbf{M}_2) + \mathrm{P}(\mathbf{M}_2) = \mathrm{P}(\mathbf{M} \setminus \mathbf{M}_2) + \mathrm{P}(\mathbf{M}_2)\,\mathrm{P}(\mathbb{P}^1) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\mathbf{M} \setminus \mathbf{M}_2) + \mathrm{P}(\Phi^{-1}(\mathbf{M}_2)) - \mathrm{P}(\mathbf{M}_2)x^2 = \mathrm{P}(\mathbf{M}^{0+}) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\rho_0^{-1}(\mathbf{M}^{\mathrm{s}})) + \mathrm{P}(F^0) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\mathbf{M}^{\mathrm{s}}) + \mathrm{P}(F^0) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\rho_\infty^{-1}(\mathbf{M}^{\mathrm{s}})) + \mathrm{P}(F^0) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\mathbf{M}^{\infty}) - \mathrm{P}(F^{\infty}) + \mathrm{P}(F^0) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 3m+2n-1)) \\
&\quad - \mathrm{P}((\mathbf{M}^4)^{\mathrm{sss}})\,\mathrm{P}(\mathbb{P}^2) + \mathrm{P}((\mathbf{M}^4)^{\mathrm{sss}})\,\mathrm{P}(\mathbb{P}^1) - \mathrm{P}(\mathbf{M}_2)x^2 \\
&= \mathrm{P}(\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(2))\,\mathrm{P}(\mathbb{P}^9) \\
&\quad - \mathrm{P}(\mathbb{P}^8 \times \mathbb{P}^1)\,\mathrm{P}(\mathbb{P}^2) + \mathrm{P}(\mathbb{P}^8 \times \mathbb{P}^1)\,\mathrm{P}(\mathbb{P}^1) - \mathrm{P}(\mathbb{P}^{11})x^2
\end{aligned}$$

$\square$

Theorem

We have a flipping diagram

$$\begin{tikzcd}
 & \widetilde{\mathbf{M}} \arrow[dl, "\beta_\infty"'] \arrow[dr, "\beta_0"] & \\
\mathbf{M}^\infty \arrow[dr, "\rho_\infty"'] & & \mathbf{M}^{0+} \arrow[dl, "\rho_0"] \\
 & \mathbf{M}^4 &
\end{tikzcd}$$

β_∞ *is the blow-up with center* F^∞ *and* β_0 *is the blow-up with center* F^0.

Proof.

The argument is due to Choi and Chung. The normal space of F^∞ at a point $(\Gamma, \mathcal{F}) \in \rho_\infty^{-1}((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}''))$ is

$$\mathrm{Ext}^1((\Gamma'', \mathcal{F}''), (\Gamma', \mathcal{F}')).$$

Here the smoothness of $\mathbf{M}^\infty$ is used. Analogously, the normal space of F^0 at a point $(\Gamma, \mathcal{F}) \in \rho_0^{-1}((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}''))$ is

$$\mathrm{Ext}^1((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}'')).$$

Thus, $\beta_\infty^{-1}\rho_\infty^{-1}((\mathbf{M}^4)^{\mathrm{sss}})$ is a fibre bundle with base $(\mathbf{M}^4)^{\mathrm{sss}}$ and fiber

$$\mathbb{P}(\mathrm{Ext}^1((\Gamma', \mathcal{F}'), (\Gamma'', \mathcal{F}''))) \times \mathbb{P}(\mathrm{Ext}^1((\Gamma'', \mathcal{F}''), (\Gamma', \mathcal{F}')))$$

and the same is true for $\beta_0^{-1}\rho_0^{-1}((\mathbf{M}^4)^{\mathrm{sss}})$. $\square$

Proposition

$\Phi\colon \mathbf{M}^{0+} \longrightarrow \mathbf{M}$ *is a blow-up with center* $\mathbf{M}_2$.

Proof.

$\mathbf{M}^{0+} \setminus F^0 \simeq (\mathbf{M}^4)^s$ is smooth because it can be identified with an open subset of $\mathbf{M}^\infty$, which is smooth. For $(\Gamma, \mathcal{F}) \in F^0$ we can check that $\mathrm{Ext}^2((\Gamma,\mathcal{F}),(\Gamma,\mathcal{F})) = 0 \Longrightarrow \mathbf{M}^{0+}$ is also smooth along F^0.

$$\left.\begin{array}{l} \mathbf{M}^{0+},\ \mathbf{M},\ \text{and } \mathbf{M}_2 \text{ are smooth} \\ \operatorname{codim} \mathbf{M}_2 = 2 \\ \Phi \text{ is an isomorphism away from } \mathbf{M}_2 \\ \Phi^{-1}(x) \simeq \mathbb{P}^1 \text{ for any } x \in \mathbf{M}_2 \end{array}\right\} \Longrightarrow \Phi \text{ is a blow-up along } \mathbf{M}_2.$$

□

Global description of the geometry of $\mathrm{M}(3m+2n+1)$:

$$\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 3m+2n-1) \overset{\text{flip}}{\dashrightarrow} \mathbf{M}^{0+} \xrightarrow{\text{blow-up}} \mathrm{M}(3m+2n+1)$$

If we want to study $\mathbf{M} = \mathrm{M}(3m+2n+2)$, we cannot use the wall-crossing method because we do not have a good description of $\mathrm{M}^{\infty}(3m+2n+2)$. Using the Beilinson spectral sequence we can classify the sheaves in $\mathbf{M}$. We obtain a stratification $\mathbf{M} = \mathbf{M}_0 \cup \mathbf{M}_1 \cup \mathbf{M}_2$, where

$\mathbf{M}_0 \subset \mathbf{M}$ open consisting of sheaves $\mathcal{F}$ having resolution

$$0 \longrightarrow \mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,-1) \xrightarrow{\varphi} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}, \quad \varphi_{12},\ \varphi_{22} \text{ are linearly independent } (1,1)\text{-forms.}$$

$\mathbf{M}_1 = \{\mathcal{O}_C(p)(0,1) \mid \deg(C) = (2,3),\ p \in C\} \subset \mathbf{M}$ closed of codimension 1, isomorphic to the universal curve of degree $(2,3)$.
$\mathbf{M}_1 =$ bundle with base $\mathbb{P}^1 \times \mathbb{P}^1$ and fiber $\mathbb{P}^{10}$.

$\mathbf{M}_2 = \{\mathcal{O}_C \otimes \mathcal{O}(1,0) \mid \deg(C) = (2,3)\} \subset \mathbf{M}$ closed of codimension 2.
$\mathbf{M}_2 \simeq \mathbb{P}^9$.

$\mathbb{W} = \mathrm{Hom}(\mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,-1), 2\mathcal{O})$

$\mathbb{G} = \mathrm{Aut}(\mathcal{O}(-1,-2) \oplus \mathcal{O}(-1,-1)) \times \mathrm{Aut}(2\mathcal{O})$ acts by left and right multiplication

$\mathbb{W}_0 \subset \mathbb{W}$ open subset given by the conditions that $\varphi_{12}, \varphi_{22}$ be linearly independent and the first column be not a multiple of the second column

$\mathbb{W}_0/\mathbb{G}$ is a bundle with base $\mathrm{Grass}(2, \mathbb{C}^4)$ and fiber $\mathbb{P}^9$;
$[\varphi] \mapsto \mathrm{span}\{\varphi_{12}, \varphi_{22}\}$ is the map to the base

$\mathbf{M}_0 \subset \mathbb{W}_0/\mathbb{G}$ open subset; its complement has two disjoint components isomorphic to $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$

$$
\begin{aligned}
\mathrm{P}(\mathrm{M}(3m+2n+2)) &= \mathrm{P}(\mathbf{M}_0) + \mathrm{P}(\mathbf{M}_1) + \mathrm{P}(\mathbf{M}_2) \\
&= \mathrm{P}(\mathbb{W}_0/\mathbb{G}) - \mathrm{P}(\mathbb{P}^1) - \mathrm{P}(\mathbb{P}^1 \times \mathbb{P}^1) + \mathrm{P}(\mathbf{M}_1) + \mathrm{P}(\mathbf{M}_2) \\
&= \mathrm{P}(\mathrm{Grass}(2, \mathbb{C}^4))\,\mathrm{P}(\mathbb{P}^9) - \mathrm{P}(\mathbb{P}^1) - \mathrm{P}(\mathbb{P}^1 \times \mathbb{P}^1) \\
&\qquad + \mathrm{P}(\mathbb{P}^1 \times \mathbb{P}^1)\,\mathrm{P}(\mathbb{P}^{10}) + \mathrm{P}(\mathbb{P}^9) \\
&= \mathrm{P}(\mathrm{M}(3m+2n+1))
\end{aligned}
$$

It is thought that $\mathrm{M}(rm + sn + t)$ is birational to $\mathrm{M}(rm + sn + 1)$.

Chung and Moon have constructed a birational map from $\mathrm{M}(3m + 2n + 2)$ to $\mathrm{M}(3m + 2n + 1)$.

By analogy with the fact that birational Calabi-Yau threefolds have the same Betti numbers, it is believed that $\mathrm{M}(rm + sn + t)$ has the same Betti numbers as $\mathrm{M}(rm + sn + 1)$ if $\gcd(r + s, t) = 1$.

$\mathbf{M} = \mathrm{M}(4m+2n+1)$ smooth of dimension 17, $\mathbf{M}^{\alpha} = \mathrm{M}^{\alpha}(4m+2n+1)$

$\mathbf{M}^{\infty} \simeq \mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(3, 4m+2n-2) =$ bundle with base $\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(3)$ and fiber $\mathbb{P}^{11}$

Two flipping squares

$$\begin{tikzcd}
 & \widetilde{\mathbf{M}}^{\infty} \arrow[dl, "\beta_\infty"'] \arrow[dr, "\beta_{11}"] & & \widetilde{\mathbf{M}}^{5+\epsilon} \arrow[dl, "\beta_5"'] \arrow[dr, "\beta_0"] & & \\
\mathbf{M}^{\infty} \arrow[dr, "\rho_\infty"'] & & \mathbf{M}^{5+\epsilon} \arrow[dl, "\rho_{11}"] \arrow[dr, "\rho_5"'] & & \mathbf{M}^{0+} \arrow[dl, "\rho_0"] \arrow[dr, "\Phi"] & \\
 & \mathbf{M}^{11} & & \mathbf{M}^{5} & & \mathbf{M}
\end{tikzcd}$$

Φ is a blow-up along the Brill-Noether locus

$$\begin{aligned}
\mathbf{M}_2 &= \{[\mathcal{F}] \in \mathbf{M} |\ \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{F}) > 1\} \\
&= \{[\mathcal{O}_C(p)(0,1) |\ \deg(C) = (2,4),\ p \in C\} \\
&= \text{universal curve of degree } (2,4).
\end{aligned}$$

Theorem (Maican)

$\mathrm{H}^i(\mathbf{M}, \mathbb{Z})$ *have no torsion. The Poincaré polynomial of* $\mathbf{M}$ *is*

$$x^{34} + 3x^{32} + 8x^{30} + 16x^{28} + 21x^{26} + 23x^{24} + 24x^{22} + 24x^{20} + 24x^{18} + 24x^{16} + 24x^{14} + 24x^{12} + 23x^{10} + 21x^{8} + 16x^{6} + 8x^{4} + 3x^{2} + 1.$$

The Euler characteristic of $\mathbf{M}$ *is* 264.

Katz' conjecture takes the form

$$\begin{aligned}
\mathrm{N}^0_{(4,2)}(\mathbf{X}) =& (-1)^{\dim \mathrm{M}(4m+2n+1)}\chi(\mathrm{M}(4m+2n+1)) \\
&+ \frac{1}{8}(-1)^{\dim \mathrm{M}(2m+n+1)}\chi(\mathrm{M}(2m+n+1)) \\
=& (-1)^{\dim \mathbf{M}}\chi(\mathbf{M}) + \frac{1}{8}(-1)^{\dim \mathbb{P}^5}\chi(\mathbb{P}^5) \\
=& (-1)^{17}264 + \frac{1}{8}(-1)^{5}6 = -264.75
\end{aligned}$$

$\mathbf{M} = \mathrm{M}(3m + 3n + 1)$ smooth of dimension 19
$\mathbf{M}^\alpha = \mathrm{M}^\alpha(3m + 3n + 1)$

$\mathbf{M}^\infty \simeq \mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(4, 3m + 3n - 3)$ = bundle with base $\mathrm{Hilb}_{\mathbb{P}^1\times\mathbb{P}^1}(4)$ and fiber $\mathbb{P}^{11}$

Three flipping squares

$$\begin{tikzcd}
 & \widetilde{\mathbf{M}}^\infty \arrow[dl, "\beta_\infty"'] \arrow[dr, "\beta_{11}"] & & \widetilde{\mathbf{M}}^{5+\epsilon} \arrow[dl, "\beta_{5+}"'] \arrow[dr, "\beta_{5-}"] & & \widetilde{\mathbf{M}}^{2+\epsilon} \arrow[dl, "\beta_2"'] \arrow[dr, "\beta_0"] & \\
\mathbf{M}^\infty \arrow[dr, "\rho_\infty"'] & & \mathbf{M}^{5+\epsilon} \arrow[dl, "\rho_{11}"] \arrow[dr, "\rho_{5+}"'] & & \mathbf{M}^{2+\epsilon} \arrow[dl, "\rho_{5-}"] \arrow[dr, "\rho_2"'] & & \mathbf{M}^{0+} \arrow[dl, "\rho_0"] \\
 & \mathbf{M}^{11} & & \mathbf{M}^5 & & \mathbf{M}^2 &
\end{tikzcd}$$

Proposition

Assume that $[\mathcal{F}] \in \mathbf{M}$ *and* $\mathrm{H}^1(\mathcal{F}) \neq 0$. *Then* $\mathrm{H}^1(\mathcal{F}) \simeq \mathbb{C}$. *Consequently, we have an isomorphism*

$$\{[\mathcal{F}] \in \mathbf{M} | \ \dim_{\mathbb{C}} \mathrm{H}^0(\mathcal{F}) > 1\} = \mathbf{M}_2 \longrightarrow \mathrm{M}^{0+}(3m+3n-1)$$

$$[\mathcal{F}] \longmapsto ((\mathrm{H}^1(\mathcal{F}))^*, \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{F}, \omega))$$

The smoothness and Betti numbers of $\mathrm{M}^{0+}(3m+3n-1)$ can be determined from the wall-crossing diagram

$$\begin{tikzcd} & \widetilde{\mathrm{M}}(3m+3n-1) \arrow[dl, "\beta_\infty"'] \arrow[dr, "\beta_0"] & \\ \mathrm{M}^{\infty}(3m+3n-1) \arrow[dr, "\rho_\infty"'] & & \mathrm{M}^{0+}(3m+3n-1) \arrow[dl, "\rho_0"] \\ & \mathrm{M}^1(3m+3n-1) & \end{tikzcd}$$

Theorem (Maican)

$\mathrm{H}^i(\mathbf{M},\mathbb{Z})$ *have no torsion. The Poincaré polynomial of* $\mathbf{M}$ *is*

$$x^{38}+3x^{36}+10x^{34}+22x^{32}+41x^{30}+53x^{28}+60x^{26}+62x^{24}+63x^{22}+63x^{20}$$
$$+63x^{18}+63x^{16}+62x^{14}+60x^{12}+53x^{10}+41x^{8}+22x^{6}+10x^{4}+3x^{2}+1.$$

The Euler characteristic of $\mathbf{M}$ *is* 756.

Katz' conjecture takes the form

$$\begin{aligned}\mathrm{N}^0_{(3,3)}(\mathbf{X}) =&(-1)^{\dim \mathrm{M}(3m+3n+1)}\chi(\mathrm{M}(3m+3n+1)) \\ &+\frac{1}{27}(-1)^{\dim \mathrm{M}(m+n+1)}\chi(\mathrm{M}(m+n+1)) \\ =&(-1)^{\dim \mathbf{M}}\chi(\mathbf{M})+\frac{1}{27}(-1)^{\dim \mathbb{P}^3}\chi(\mathbb{P}^3) \\ =&(-1)^{19}756+\frac{1}{27}(-1)^{3}4=-756-\frac{4}{27}\end{aligned}$$