Cosmological Two-field α-attractor Models

(Hidden Symmetries and Exact Solutions)

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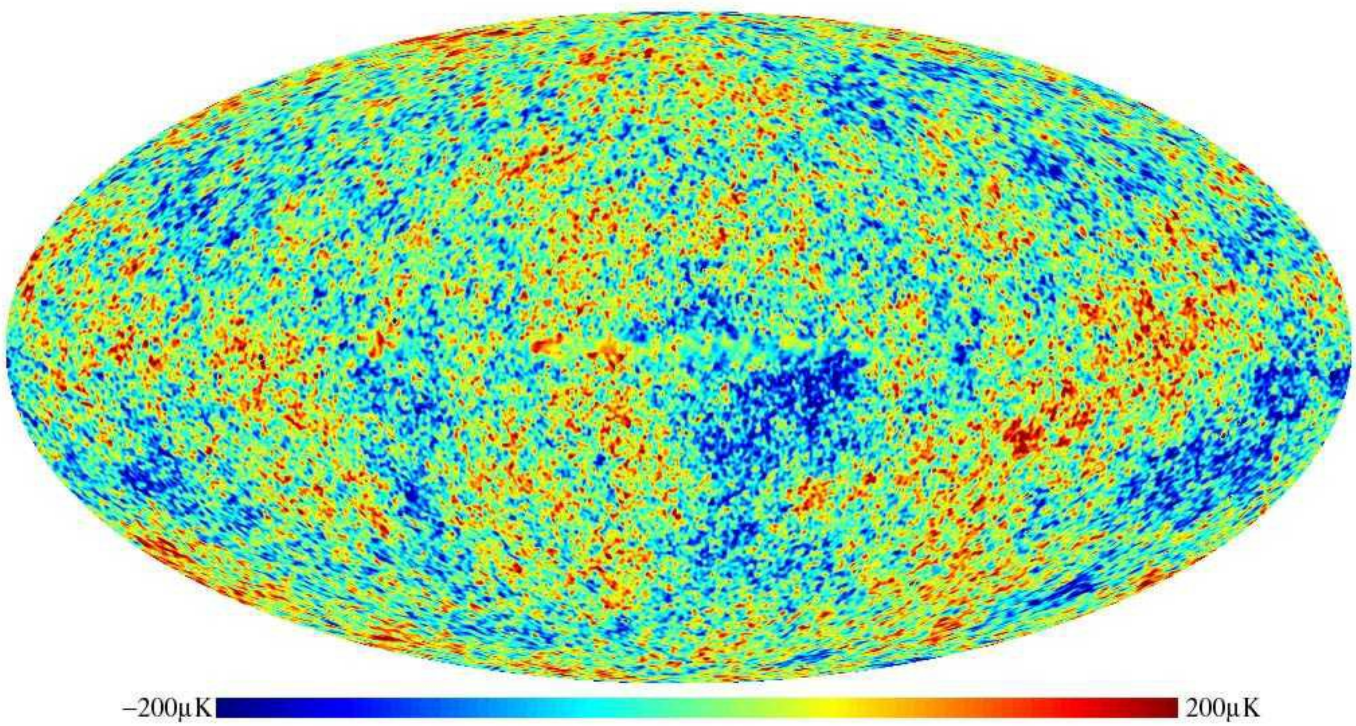
Motivation

Cosmic Microwave Background (CMB) radiation:

WMAP (2003-2012) and Planck (2013) satellites:

Detailed map of CMB temperature fluctuations on the sky



$\overline{T} = 2.7\text{K}$

According to CMB data:

Temperature fluctuations $\dfrac{\delta T(\theta,\varphi)}{\overline{T}}$, (θ,φ) coord. on S^2 ,

measured with great precision:

- On large scales:

  Universe is homogeneous and isotropic

- In Early Universe:

  Small perturbations that seed structure formation

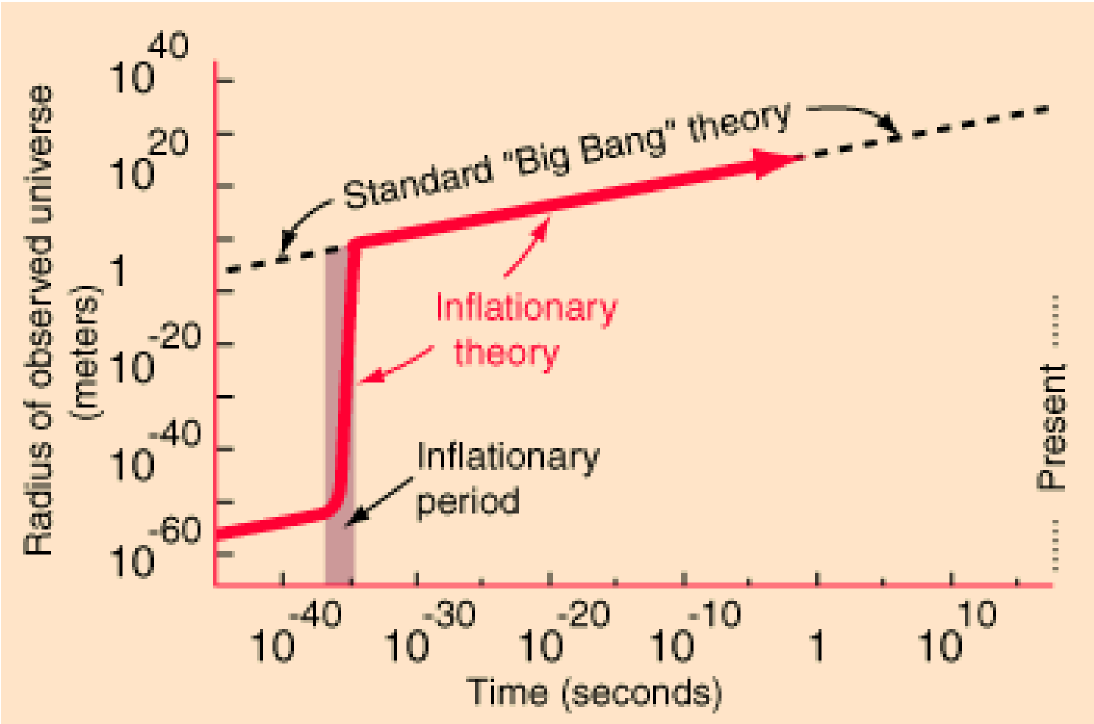
  [(Clusters of) Galaxies]

Cosmological Inflation:

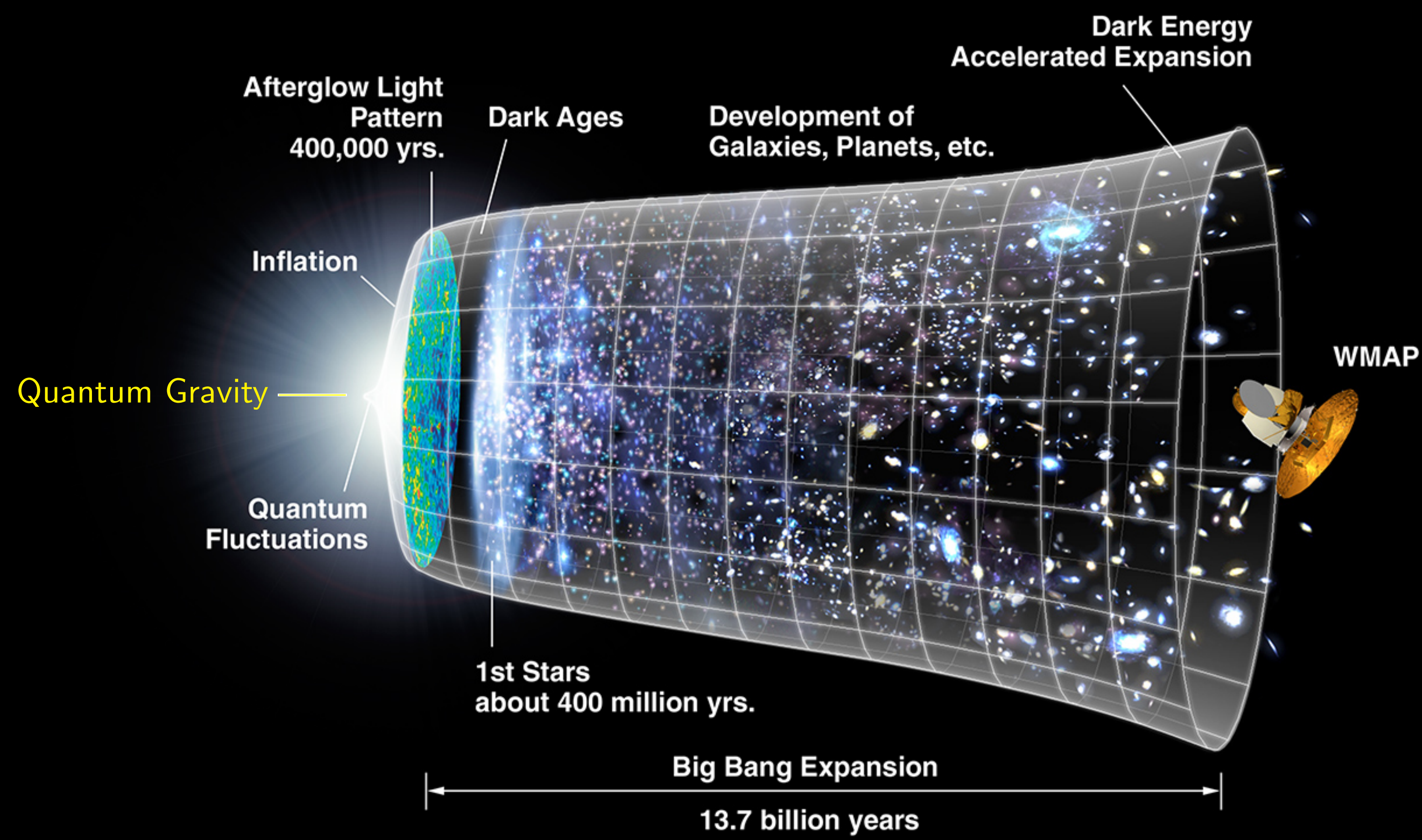
Period of very fast expansion of space in the Early Universe

(faster than speed of light)

$\Rightarrow$ homogeneity and isotropy observed today



Inflation: Traces of Quantum Gravity ?



(Shortly after) Big Bang: Origin of all structure we see today!

NASA/WMAP Science Team

Cosmological Inflation:

Standard description:

- expansion driven by the potential energy of a single scalar field φ called inflaton

- weakly coupled Lagrangian for the inflaton within QFT framework:

$$S = \int d^4x \sqrt{-\det g} \left[\frac{R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi) \right]$$

- slow roll approximation:

$$\epsilon_{\mathrm{v}} \stackrel{\mathrm{def.}}{=} \tfrac{1}{2} \left[\frac{V'(\varphi)}{V(\varphi)} \right]^2 \ll 1 \quad , \quad \eta_{\mathrm{v}} \stackrel{\mathrm{def.}}{=} \frac{V''(\varphi)}{V(\varphi)} \ll 1$$

BUT: Many reasons to consider non-standard models

- Embedding in a fundamental theory:
  - In string compactifications 4d scalars arise in pairs (chiral superfields)
  - Compatibility with quantum gravity ('swampland' conjectures, in particular, constraints on $V(\varphi)$; very restrictive for a single scalar)
- Richer phenomenology:
  - Decoupling the generation of curvature perturbations (curvaton) from the inflaton
  - Non-Gaussianity of primordial fluctuations

Two-field α-attractor Models

Action:

$$S = \int d^4x \, \sqrt{-\det g} \left[\frac{R}{2} - \frac{1}{2} G_{ij}(\varphi) \, g^{\mu\nu} \, \partial_\mu \varphi^i \, \partial_\nu \varphi^j - V(\varphi) \right] ,$$

$g_{\mu\nu}(x)$ - spacetime metric ,

Ansatz: $ds_g^2 = -dt^2 + a(t)^2 d\vec{x}^2$, $\varphi^i = \varphi^i(t)$,

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \quad - \quad \text{Hubble parameter} \; ,$$

$G_{ij}(\varphi)$ - target space metric: $i, j = 1, 2$

Gaussian curvature of G_{ij} - constant and negative

Two-field α-attractors:

Kallosh, Linde et al. (arXiv:1311.0472 [hep-th], arXiv:1405.3646 [hep-th], arXiv:1503.06785 [hep-th], arXiv:1504.05557 [hep-th])

Two-dim. manifold $\mathcal{M}$ with metric $ds_G^2 = G_{ij} d\varphi^i d\varphi^j$ and Gaussian curvature $K_G = const < 0$: hyperbolic surface

$\rightarrow$ simplest example: Poincaré disk

Initial studies: radial trajectories on the Poincaré disk

Generalization to any hyperbolic surface:

Lazaroiu and Shahbazi (arXiv:1702.06484 [hep-th])

Two-field α-attractors:

Note:

In single-field models potential $V(\varphi)$ plays key role:

Always: field redefinition $\rightarrow$ canonical kinetic term

$(G_{ij}\partial\varphi^i\partial\varphi^j \rightarrow \delta_{ij}\partial\hat{\varphi}^i\partial\hat{\varphi}^j \Rightarrow$ Can transfer complexity to the potential)

In multi-field models:

Cannot redefine away the curvature of G_{ij} !

$\Rightarrow$ kinetic term becomes important

In particular: Can have genuine two (or multi-) field trajectories, $\{\varphi^i(t)\}$, even when $\partial_{\varphi^i}V = 0$!

Action:

Substituting ansatz $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$, $\varphi^i = \varphi^i(t)$:

$$L = -3a\dot{a}^2 + a^3 \left[\frac{1}{2} G_{ij} \dot{\varphi}^i \dot{\varphi}^j - V(\varphi) \right]$$

$\rightarrow$ classical mechanical action for $\{a, \varphi^i\}$ ds.o.f.

Euler-L. eqs of $L \equiv$ original EoMs, when imposing constraint:

$$E_L \equiv \dot{a} \frac{\partial L}{\partial \dot{a}} + \dot{\varphi}^i \frac{\partial L}{\partial \dot{\varphi}^i} - L = 0$$

Note: $E_L = const$ on solutions of EL eqs., so Hamiltonian constraint $\rightarrow$ relation between integration constants

Noether Symmetry

Will impose condition that L has Noether symmetry

Motivation:

- can restrict:
  - form of potential V (expected)
  - value of Gaussian curvature K_G (unexpected!)

(hence: may help for embedding in fundamental theory)

- can facilitate finding exact solutions of EoMs

(as opposed to numerical ones)

- conserved quantity may play important role

Noether symmetry:

Recall: $L = -3a\dot{a}^2 + a^3\left[\frac{1}{2}G_{ij}\dot{\varphi}^i\dot{\varphi}^j - V(\varphi)\right]$

Denote $q^I \equiv \{a, \varphi^i\}$ - generalized coordinates on $\widetilde{\mathcal{M}}$

Consider transformation $q^I \to Q^I(q)$:

– generated by: $X = X^a(a,\varphi)\partial_a + X^i(a,\varphi)\partial_{\varphi^i}$

– induces transf. on tangent bundle $T\widetilde{\mathcal{M}}$, generated by :

(with coord. $\{q^I, \dot{q}^I\}$)

$$\hat{X} = X + \dot{X}^a(a,\varphi,\dot{a},\dot{\varphi})\partial_{\dot{a}} + \dot{X}^i(a,\varphi,\dot{a},\dot{\varphi})\partial_{\dot{\varphi}^i}$$

Symmetry condition: $\mathcal{L}_{\hat{X}}(L) = 0$

Noether symmetry:

(arXiv:1905.01611 [hep-th])

$\mathcal{L}_{\hat{X}}(L) = 0 \quad \Rightarrow$ coupled system of equations:

$$\begin{aligned} X^a + 2a\partial_a X^a &= 0 \\ -6\partial_i X^a + a^2 G_{ij}\partial_a X^j &= 0 \\ 3G_{ij}X^a + a\left(\nabla_i X_j + \nabla_j X_i\right) &= 0 \\ 3VX^a + aX^i\partial_i V &= 0 \quad , \end{aligned}$$

∇_i - covariant derivative on $\mathcal{M}$ (with coord. $\{\varphi^i\}$)

Look for functions $X^a(a,\varphi)$, $X^i(a,\varphi)$ satisfying this system identically

(I.e., look for global symmetries, independent of t !)

Noether symmetry:

(arXiv:1905.01611 [hep-th])

Have shown: The solutions of $\mathcal{L}_{\hat{X}}(L) = 0$ have the form:

$$X^a = \frac{\Lambda(\varphi)}{\sqrt{a}} \quad , \quad X^i = Y^i(\varphi) - \frac{4}{a^{3/2}} G^{ij} \partial_j \Lambda \quad ,$$

where Λ and Y^i satisfy:

- $$\nabla_i Y_j + \nabla_j Y_i = 0 \quad , \quad Y^i \partial_i V = 0$$

  $\rightarrow$ Y^i - Killing vector on $\mathcal{M}$, preserving $V(\varphi)$

- $$\nabla_i \nabla_j \Lambda = \frac{3}{8} G_{ij} \Lambda \quad , \quad G^{ij} \partial_i V \partial_j \Lambda = \frac{3}{4} V \Lambda$$

  $\rightarrow$ Λ - Hessian symmetry (hidden symmetry)

Hidden symmetry:

Convenient to rescale $\hat{G} \stackrel{\text{def.}}{=} \frac{3}{8}G$

Then the Λ-conditions become:

$$\nabla \mathrm{d}\Lambda = \hat{G}\Lambda \quad ,$$
$$\langle \mathrm{d}V, \mathrm{d}\Lambda \rangle_{\hat{G}} = 2\, V\, \Lambda$$

Note: These eqs. are invariant under the natural action of the isometry group of $\mathcal{M}$, i.e. under

$$(\Lambda, V) \to (\Lambda \circ \psi^{-1}, V \circ \psi^{-1}) \quad , \ \forall \psi \in \mathrm{Iso}(\mathcal{M}, \hat{G})$$

$\to$ Very useful for finding general solutions!

Hidden symmetry:

Remark on scalar potential:

Consider $\gamma(s)$ - gradient flow curve of Λ with gradient flow parameter s:

$$\frac{\mathrm{d}\gamma(s)}{\mathrm{d}s} = -(\mathrm{grad}_{\hat{G}}\Lambda)(\gamma(s))$$

Then equation $\langle \mathrm{d}V, \mathrm{d}\Lambda\rangle_{\hat{G}} = 2\,V\Lambda$ implies:

$$V(\gamma(s)) = V(\gamma(s_0))\,\exp\left(-2\int_{s_0}^{s}\Lambda(\gamma(s'))\,\mathrm{d}s'\right)$$

$\rightarrow$ Can find V in full generality, once we know Λ

Rotationally-invariant 2-field models

Consider rot.-invariant metric G_{ij} on $\mathcal{M}$ with $i, j = 1, 2$:

$$ds_G^2 = dr^2 + f(r)d\theta^2 \quad , \quad \{\varphi^i\} = \{r, \theta\}$$

Then the hidden symmetry conditions become:

- Hessian equation $\nabla_i \nabla_j \Lambda = \frac{3}{8} G_{ij} \Lambda$:

$$\partial_r^2 \Lambda = \frac{3}{8}\Lambda \quad , \quad \partial_r \partial_\theta \Lambda - \frac{f'}{2f}\partial_\theta \Lambda = 0$$

$$\partial_\theta^2 \Lambda + \frac{f'}{2}\partial_r \Lambda = \frac{3}{8} f \Lambda$$

- Λ-V equation: $\quad \partial_r V \partial_r \Lambda + \frac{1}{f}\partial_\theta V \partial_\theta \Lambda = \frac{3}{4} V \Lambda$

Rotationally-invariant G_{ij}: (recall: $i, j = 1, 2$)

Showed that Hessian equation implies:

$$K_G = -\frac{3}{8} \qquad (K_G \text{ - Gaussian curvature of } \mathcal{M})$$

$\rightarrow$ Λ-symmetry requires hyperbolic $\mathcal{M}$!

Rotationally-invariant hyperbolic surfaces:

$$\left(z \in \mathbb{C} \ , \ \rho \stackrel{\text{def.}}{=} |z| \ , \ \theta \stackrel{\text{def.}}{=} \arg(z)\right)$$

- Poincaré disk $\mathbb{D}$

$$\left(\rho < 1 \quad , \quad ds^2_{\mathbb{D}} = \tfrac{4}{(1-\rho^2)^2} \left(d\rho^2 + \rho^2 d\theta^2\right) \ , \ K_{\mathbb{D}} = -1 \right)$$

- hyperbolic punctured disk $\mathbb{D}^*$

$$\left(0 < \rho < 1 \quad , \quad ds^2_{\mathbb{D}^*} = \tfrac{1}{(\rho \log \rho)^2} \left(d\rho^2 + \rho^2 d\theta^2\right) \ , \ K_{\mathbb{D}^*} = -1 \right)$$

- hyperbolic annuli $\mathbb{A}$

$$\left(\tfrac{1}{R} < \rho < R \quad , \quad ds^2_{\mathbb{A}} = \left(\tfrac{\pi}{2 \log R}\right)^2 \frac{\left(d\rho^2 + \rho^2 d\theta^2\right)}{\left[\rho \cos\left(\frac{\pi \log \rho}{2 \log R}\right)\right]^2} \ , \ K_{\mathbb{A}} = -1 \right)$$

Poincaré disk case:

Metric G_{ij}: $ds_G^2 = \frac{4}{\beta^2(1-\rho^2)^2}\left(d\rho^2 + \rho^2 d\theta^2\right) \quad , \quad \rho < 1$

$$r = \frac{2}{\beta}\,\mathrm{arctanh}(\rho) \in (0,\infty) \quad \to \quad ds_G^2 = dr^2 + f(r)d\theta^2$$

Showed that general solution for Λ is:

$$\Lambda = B_0 \cosh(\beta r) + (B_1 \cos\theta + B_2 \sin\theta)\sinh(\beta r)\,,$$

where $\beta \equiv \sqrt{\frac{3}{8}}$ and $B_{0,1,2} = const$

Finding V complicated! To simplify Λ-V equation, note:

Can write $\Lambda = B_\mu \Xi^\mu \;, \quad (\mu = 0,1,2)$

where $(\Xi^0)^2 - (\Xi^1)^2 - (\Xi^2)^2 = 1 \quad \text{and} \quad \Xi^0 > 0$

Poincaré disk case:

Ξ^μ - Weierstrass coordinates for the Poincaré disk D

Weierstrass map: $\quad \Xi \,:\, \mathrm{D} \to S^+$,

where S^+ - future sheet of the unit hyperboloid in 3d Minkowski space $\mathbb{R}^{1,2}$

Can identify orientation-preserving isometries of $\mathbb{D}$ with proper and orthochronous Lorentz transf. in 3d

$\rightarrow$ Solve Λ-V equation in 3 simple canonical cases

(B_μ: timelike, spacelike, lightlike)

$\Rightarrow$ Find general solution for V (in each case) by Lorentz transf.

Orientation-preserving isometries of $\mathbb{D}$:

$\mathrm{Iso}_o(\mathbb{D})$ - orientation preserving isometries of $\mathbb{D}$

$\mathrm{SO}_o(1,2)$ - connected component of Lorentz group in 3d

Can identify the two groups by using $\mathrm{PSU}(1,1)$:

1) Consider morphism of groups:

$$\psi \,:\, \mathrm{SU}(1,1) \to \mathrm{Diff}(\mathrm{D})\,,$$

where $$\psi_U(z) = \frac{\eta z + \sigma}{\bar{\sigma} z + \bar{\eta}} \quad , \quad z \in \mathrm{D} \quad , \quad \psi_U \stackrel{\text{def.}}{=} \psi(U)$$

and $$\eta, \sigma \in \mathbb{C} \quad , \quad U(\eta,\sigma) \stackrel{\text{def.}}{=} \begin{bmatrix} \eta & \sigma \\ \bar{\sigma} & \bar{\eta} \end{bmatrix} \in \mathrm{SU}(1,1)$$

$$\rightarrow \quad \psi(\mathrm{PSU}(1,1)) = \mathrm{Iso}_o(\mathbb{D})$$

($\mathrm{PSU}(1,1) \stackrel{\text{def.}}{=} \mathrm{SU}(1,1)/\{-I_2, I_2\}$: for effective action)

Orientation-preserving isometries of $\mathbb{D}$:

2) Identify Lie algebra $\mathrm{su}(1,1)$ with 3d Minkowski space $\mathbb{R}^{1,2}$:

$$Z = Z(X) \stackrel{\text{def.}}{=} \begin{bmatrix} X^0 & X^1 + \mathbf{i}X^2 \\ X^1 - \mathbf{i}X^2 & X^0 \end{bmatrix} ,$$

$X \stackrel{\text{def.}}{=} (X^0, X^1, X^2) \in \mathbb{R}^3$, $Z = \frac{\mathrm{i}}{\sqrt{8}} AJ$, $A \in \mathrm{su}(1,1)$, $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

AND

adjoint representation $\mathrm{Ad} : \mathrm{SU}(1,1) \to \mathrm{Aut}_{\mathbb{R}}(\mathrm{su}(1,1))$

$$\Big(\to \quad \mathrm{Ad}(U)(Z) = UZU^\dagger \quad , \quad \forall U \in \mathrm{SU}(1,1) \Big)$$

with $\mathrm{SO}_o(1,2)$ Lorentz transformations

($\mathrm{su}(1,1)$ Killing form $\to$ pairing $(X,Y) = X^0Y^0 - X^1Y^1 - X^2Y^2$)

Poincaré disk case:

Exact solutions in a special case:

(arising from separation-of-variables Ansatz)

$$V = V_0 \cosh^2(\beta r) \coth^m(\beta r) \left(C_1 \cos\theta - C_2 \sin\theta \right)^{-m} \ ,$$

where $m, V_0, C_1, C_2 = const$

To solve EL equations, transform to generalized coord., adapted to the symmetry: $(a, r, \theta) \to (u, v, w)$, $\frac{\partial L}{\partial w} = 0$

[see arXiv:1809.10563 [hep-th] for the explicit expressions for:

$a = a(u,v,w)$, $r = r(u,v,w)$, $\theta = \theta(u,v,w)$]

$\to$ easily solve EL eq. for cyclic variable : $w = w(t)$

Poincaré disk case:

Exact solutions:

- $m = 0$:

$$u(t) = C_1^u \sinh(\kappa\, t) + C_2^u \cosh(\kappa\, t) \quad , \quad \kappa = \tfrac{1}{2}\sqrt{3V_0}$$

$$v(t) = C_1^v\, t + C_2^v$$

- $m = -2$:

$$u(t) = C_1^u\, t + C_2^u$$

$$v(t) = C_1^v \sin(\omega t) + C_2^v \cos(\omega t) \quad , \quad \omega = \tfrac{1}{2}\sqrt{3V_0(C_1^2 + C_2^2)}$$

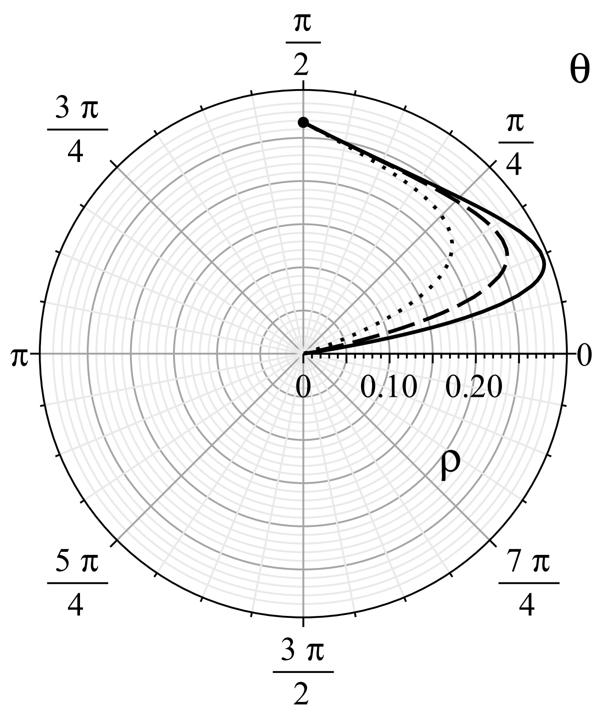
- $m = -1$:

$$v = [C_1^v \cosh(\hat{\kappa} t) + C_2^v \sinh(\hat{\kappa} t)] \cos(\hat{\kappa} t)$$
$$+ [C_3^v \cosh(\hat{\kappa} t) + C_4^v \sinh(\hat{\kappa} t)] \sin(\hat{\kappa} t) \quad , \quad u = const \times \ddot{v}$$

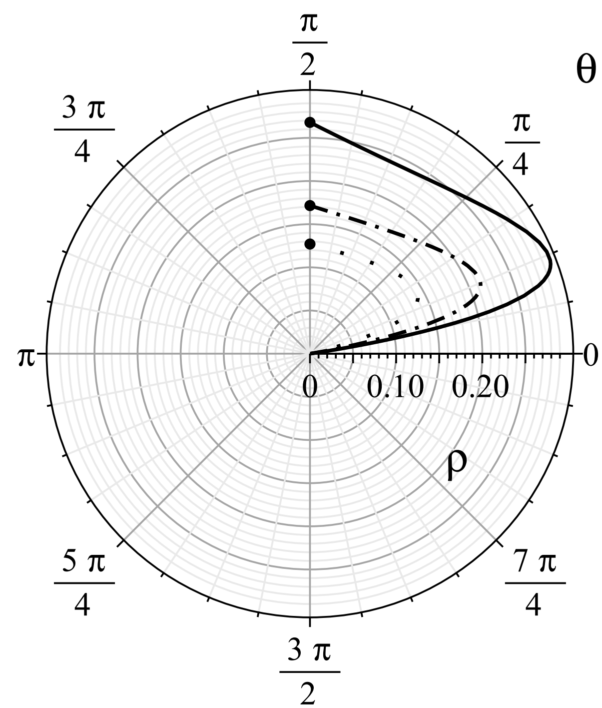
Note:

For $m = 0$: V is θ-independent, but still there are genuine 2-field trajectories $(\rho(t), \theta(t))$!

Illustration: (all constants fixed, except one)



C_1^u - varies



C_2^u - varies

Summary

Found so far:

- Most general hidden symmetries of cosmological two-field α-attractor models with rot.-invariant scalar manifold metric

  [In particular: Gaussian curvature - fixed !]

- Form of scalar potential compatible with hidden symmetry
- Exact solutions in special case [separation-of-variables Ansatz]

Open issues:

- Exact solutions in general case ?...
- Embedding in string theory (points of enhanced symmetry) ?...
- Perturbations, cosmological observables ?...

Thank you!