TWISTED JACOBI VERSUS JACOBI WITH BACKGROUND STRUCTURES

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Outline

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid
- Jacobi-like line bundles
- Jacobi-like line bundles encompass Jacobi-like pairs

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From Jacobi pairs to twisted Jacobi pairs

Let M be a smooth manifold. By definition, a Jacobi pair (Π, E) consists in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M)$$

that enjoy the properties

$$[\Pi, \Pi] + 2\Pi \wedge E = 0, \quad [\Pi, E] = 0, \tag{1}$$

with $[\bullet, \bullet]$ the Schouten-Nijenhuis bracket in the Gerstenhaber algebra of multi-vector fields

$$\mathfrak{X}^\bullet(M) \equiv \mathcal{F}(M) \oplus \mathfrak{X}^1(M) \oplus \cdots \oplus \mathfrak{X}^{\dim M}(M)$$

coming from the Lie algebra of smooth vector fields

$$\left(\mathfrak{X}(M) \equiv \mathfrak{X}^1(M), [\bullet, \bullet]\right).$$

From Jacobi pairs to twisted Jacobi pairs

A Jacobi pair, naturally structures the vector space $\mathcal{F}(M)$ as a Lie algebra, but not a Poisson one with respect to

$$\begin{aligned} &\{\bullet,\bullet\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M), \\ &\{f,g\} \equiv i_{\Pi}\mathrm{d}f \wedge \mathrm{d}g + i_{E}(f\mathrm{d}g - g\mathrm{d}f). \end{aligned} \tag{2}$$

The bracket in the above display the 'Hamiltonian' morphism of Lie algebras

$$\begin{aligned} &\mathcal{H} : \mathcal{F}(M) \to \mathfrak{X}^{1}(M), \\ &\mathcal{H}(f) \equiv X_{f} = \Pi^{\sharp}\mathrm{d}f + fE, \end{aligned} \tag{3}$$

with

$$\Pi^{\sharp} : T^{*}M \to TM, \quad \Pi^{\sharp}\alpha \equiv -j_{\alpha}\Pi. \tag{4}$$

The Hamiltonian vector fields enjoy the properties

$$[X_{f}, X_{g}] = X_{\{f,g\}}, \quad [X_{f}, E] = -X_{\mathcal{L}_{E}f} \tag{5}$$

From Jacobi pairs to twisted Jacobi pairs

A Jacobi pair is said to be transitive if the 'Hamiltonian' distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi^\sharp_x, E_x\rangle = T_xM, \quad x \in M. \tag{6}$$

Example

A locally conformal symplectic structure on an even-dimensional smooth manifold M consists in a pair (Ω, α) with Ω non-degenerate, α closed and

$$\mathrm{d}\Omega + \alpha \wedge \Omega = 0.$$

This results in a Jacobi pair (Π, E) with

$$\langle \rho \wedge \lambda, \Pi\rangle \equiv \langle \Omega, \Omega^\sharp\rho \wedge \Omega^\sharp\lambda\rangle, \quad E \equiv \Omega^\sharp\alpha.$$

By $\Omega^\sharp$ we denoted the inverse of the isomorphism

$$\Omega^\flat : \mathfrak{X}^1(M) \to \Omega^1(M), \quad \Omega^\flat X \equiv -i_X\Omega.$$

From Jacobi pairs to twisted Jacobi pairs

Example

A coorientable contact structure on an odd-dimensional smooth manifold M is given by a 1-form θ such that

$$\mu \equiv \theta \wedge (\mathrm{d}\theta)^m$$

is a volume form, i.e.,

$$\mu^\flat : \mathfrak{X}^1(M) \to \Omega^{2m}(M), \quad \mu^\flat X \equiv -i_X \mu$$

is an isomorphism. The pair (Π, E) is a Jacobi one where E is the Reeb vector field, i.e., the unique solution to

$$i_E \theta = 1, \quad i_E \mathrm{d}\theta = 0$$

and

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \equiv \langle \mathrm{d}\theta, X_f \wedge X_g \rangle.$$

From Jacobi pairs to twisted Jacobi pairs

Example

Previously, by X_f we meant the Hamiltonian vector field associated with the smooth function $f \in \mathcal{F}(M)$ given by the considered coorientable contact structure, i.e., the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}\mathrm{d}\theta = i_E\left(\mathrm{d}f \wedge \theta\right).$$

From Jacobi pairs to twisted Jacobi pairs

Theorem

If a Jacobi pair (Π, E) on a smooth manifold M is transitive then M is either a locally conformal symplectic manifold or a coorientable contact one.

Theorem

The characteristic distribution of a Jacobi pair is completely integrable with the characteristic leaves either locally conformal symplectic manifolds or coorientable contact ones.

From Jacobi pairs to twisted Jacobi pairs

Let $(M, \mathcal{A}_M)$ be a smooth manifold. By definition, a twisted Jacobi pair $((\Pi, E), \omega)$ consists in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M), \quad \omega \in \Omega^2(M)$$

that enjoy the properties

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \mathrm{d}\omega + \Pi^\sharp \omega \wedge E \tag{7}$$

$$[E, \Pi] = -\left(\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E \omega \wedge E\right). \tag{8}$$

From Jacobi pairs to twisted Jacobi pairs

A twisted Jacobi pair endows the vector space $\mathcal{F}(M)$ with the $\mathbb{R}$-linear and skew-symmetric bracket

$$\begin{aligned} &\{\bullet,\bullet\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)\,, \\ &\{f,g\} \equiv i_{\Pi}\mathrm{d}f \wedge \mathrm{d}g + i_E\left(f\mathrm{d}g - g\mathrm{d}f\right), \end{aligned} \tag{9}$$

which verifies

$$\{f,gh\} - g\{f,h\} - h\{f,g\} = gh\mathcal{L}_E f, \tag{10}$$

and

$$\begin{aligned} \mathrm{Jac}\{f,g,h\} = {} & i_{\Pi^\sharp \mathrm{d}\omega + \Pi^\sharp \omega \wedge E}\left(\mathrm{d}f \wedge \mathrm{d}g \wedge \mathrm{d}h\right) \\ & - i_{\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E \omega \wedge E}\left(f\mathrm{d}g \wedge \mathrm{d}h + g\mathrm{d}h \wedge \mathrm{d}f + h\mathrm{d}f \wedge \mathrm{d}g\right), \end{aligned} \tag{11}$$

From Jacobi pairs to twisted Jacobi pairs

The bracket in the above allows display the introduction of Hamiltonian vector fields

$$\begin{aligned}
&\mathcal{H} : \mathcal{F}(M) \to \mathfrak{X}^1(M), \\
&\mathcal{H}(f) \equiv X_f = \Pi^\sharp \mathrm{d}f + fE,
\end{aligned} \tag{12}$$

which verify the relations

$$\begin{aligned}
[X_f, X_g] - X_{\{f,g\}} = {}& \Pi^\sharp i_{X_f \wedge X_g} \mathrm{d}\omega - (\mathcal{L}_E f)\, \Pi^\sharp i_{X_g}\omega \\
& + (\mathcal{L}_E g)\, \Pi^\sharp i_{X_f}\omega + \left(i_{X_f \wedge X_g}\omega\right) E.
\end{aligned} \tag{13}$$

$$[X_f, E] + X_{\mathcal{L}_E f} = \Pi^\sharp \left(i_{X_f \wedge E}\mathrm{d}\omega - (\mathcal{L}_E f)\, i_E\omega\right) + \left(i_{X_f \wedge E}\omega\right) E. \tag{14}$$

From Jacobi pairs to twisted Jacobi pairs

A twisted Jacobi pair $((\Pi, E), \omega)$ is said to be transitive if the characteristic distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi_x^\sharp, E_x \rangle = T_x M, \quad x \in M. \tag{15}$$

Example

The pair (Ω, α), with Ω non-degenerate and α closed, is said to be a locally conformal symplectic structure twisted by $\omega \in \Omega^2(M)$ if

$$\mathrm{d}(\Omega - \omega) + \alpha \wedge (\Omega - \omega) = 0.$$

This results in a twisted Jacobi pair $((\Pi, E), \omega)$ with

$$\langle \rho \wedge \lambda, \Pi \rangle \equiv \langle \Omega, \Omega^\sharp \rho \wedge \Omega^\sharp \lambda \rangle, \quad E \equiv \Omega^\sharp \alpha.$$

By $\Omega^\sharp$ we denoted the inverse of the isomorphism

$$\Omega^\flat : \mathfrak{X}^1(M) \to \Omega^1(M), \quad \Omega^\flat X \equiv -i_X \Omega.$$

From Jacobi pairs to twisted Jacobi pairs

Example

The contact structure θ is said to be twisted by the 2-form ω if

$$\mu \equiv \theta \wedge (\mathrm{d}\theta + \omega)^m$$

is a volume form, i.e.,

$$\mu^\flat : \mathfrak{X}^1(M) \to \Omega^{2m}(M), \quad \mu^\flat X \equiv -i_X \mu$$

is an isomorphism. The structure $((\Pi, E), \omega)$ is a twisted Jacobi pair where E is the twisted Reeb vector field, i.e., the unique solution to

$$i_E \theta = 1, \quad i_E (\mathrm{d}\theta + \omega) = 0$$

and

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \equiv \langle \mathrm{d}\theta, X_f \wedge X_g \rangle.$$

Example

Previously, by X_f we meant the twisted Hamiltonian vector field associated with the smooth function $f \in \mathcal{F}(M)$ given by the considered twisted coorientable contact structure, i.e., the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}\left(\mathrm{d}\theta + \omega\right) = i_E\left(\mathrm{d}f \wedge \theta\right).$$

Theorem

If a twisted Jacobi pair $((\Pi, E), \omega)$ *on a smooth manifold* M *is transitive then* M *is either a twisted locally conformal symplectic manifold or a twisted coorientable contact one.*

Theorem

The characteristic distribution of a twisted Jacobi pair is completely integrable with the characteristic leaves either twisted locally conformal symplectic manifolds or twisted coorientable contact ones.

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Relaxing twisted Jacobi pairs: Jacobi pair with background

Definition

A pair $((\Pi, E), (\phi, \omega))$ consisting in

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M), \quad \phi \in \Omega^3(M), \quad \omega \in \Omega^2(M)$$

which enjoys the properties

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \phi + \Pi^\sharp \omega \wedge E \tag{16}$$

$$[E, \Pi] = -\left(\Pi^\sharp i_E \phi + \Pi^\sharp i_E \omega \wedge E\right) \tag{17}$$

is called Jacobi pair (Π, E) with background (ϕ, ω).

It is immediate that if in the above we take

$$\phi \equiv \mathrm{d}\omega \tag{18}$$

then we recover the twisted Jacobi pair.

Example

Let's consider the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = \left(x^1, x^2, x^3, x^4\right)$ and the real smooth functions $f, e \in \mathcal{C}^\infty\left(\mathbb{R}^4\right)$ among which f is nowhere vanishing and

$$e = e\left(x^1, x^2\right).$$

The geometric objects

$$\Pi = \frac{1}{f}\left(\partial_1 \wedge \partial_4 + \partial_2 \wedge \partial_3\right),$$

$$E = -\frac{1}{f}\left(\left(\partial_1 e\right)\partial_4 + \left(\partial_2 e\right)\partial_3\right) = -\Pi^\sharp \mathrm{d}e, \quad \omega = 0,$$

$$\phi = \mathrm{d}\left(f \ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f \ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right) - f\mathrm{d}\left(e \ \mathrm{d}x^2 \wedge \mathrm{d}x^3 + e \ \mathrm{d}x^1 \wedge \mathrm{d}x^4\right),$$

organize $\mathbb{R}^4$ as a Jacobi pair with background whose 3-form is non-closed and twisting 2-form ω vanishes.

Example

Let's consider the same four-dimensional smooth manifold $\mathbb{R}^4$ and take the smooth functions a, b with a nowhere vanishing. We introduce the objects

$$\begin{aligned}
&\Omega = a\left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4\right) \quad \omega = a\ \mathrm{d}x^1 \wedge \mathrm{d}x^2,\\
&\phi = \mathrm{d}\omega + (\mathrm{d}a + a\ \mathrm{d}b) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4,\\
&\Pi = -\frac{1}{a}\left(\partial_1 \wedge \partial_2 + \partial_3 \wedge \partial_4\right), \quad E = \Omega^\sharp \mathrm{d}b.
\end{aligned}$$

With these tools at hand $((\Pi, E), (\phi, \omega))$ is nothing but a Jacobi pair with background defined by 3-form ϕ and non-trivial twisting 2-form ω. The background 3-form is closed if and only if

$$(\mathrm{d}a + a\mathrm{d}b) \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 = 0.$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

A Jacobi pair with background endows the vector space $\mathcal{F}(M)$ with the $\mathbb{R}$-linear and skew-symmetric bracket

$$\begin{aligned} &\{\bullet,\bullet\}:\mathcal{F}(M)\times\mathcal{F}(M)\to\mathcal{F}(M)\,,\\ &\{f,g\}\equiv i_{\Pi}\mathrm{d}f\wedge\mathrm{d}g+i_{E}\left(f\mathrm{d}g-g\mathrm{d}f\right), \end{aligned} \tag{19}$$

which verifies

$$\{f,gh\}-g\{f,h\}-h\{f,g\}=gh\mathcal{L}_{E}f, \tag{20}$$

and

$$\begin{aligned} \mathrm{Jac}\{f,g,h\} &= i_{\Pi^{\sharp}\phi+\Pi^{\sharp}\omega\wedge E}\left(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h\right)\\ &\quad - i_{\Pi^{\sharp}i_{E}\phi+\Pi^{\sharp}i_{E}\omega\wedge E}\left(f\mathrm{d}g\wedge\mathrm{d}h+g\mathrm{d}h\wedge\mathrm{d}f+h\mathrm{d}f\wedge\mathrm{d}g\right) \end{aligned} \tag{21}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

The bracket in the above allows display the introduction of Hamiltonian vector fields

$$\mathcal{H}:\mathcal{F}(M)\to\mathfrak{X}^{1}(M),$$
$$\mathcal{H}(f)\equiv X_{f}=\Pi^{\sharp}\mathrm{d}f+fE, \tag{22}$$

which verify the relations

$$[X_{f},X_{g}]-X_{\{f,g\}}=\Pi^{\sharp}i_{X_{f}\wedge X_{g}}\phi-(\mathcal{L}_{E}f)\,\Pi^{\sharp}i_{X_{g}}\omega + (\mathcal{L}_{E}g)\,\Pi^{\sharp}i_{X_{f}}\omega+\left(i_{X_{f}\wedge X_{g}}\omega\right)E. \tag{23}$$
$$[X_{f},E]+X_{\mathcal{L}_{E}f}=\Pi^{\sharp}\left(i_{X_{f}\wedge E}\phi-(\mathcal{L}_{E}f)\,i_{E}\omega\right)+\left(i_{X_{f}\wedge E}\omega\right)E. \tag{24}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

A Jacobi pair with background $((\Pi, E), (\phi, \omega))$ is said to be transitive if its characteristic distribution coincides with the tangent one i.e iff

$$\langle \mathrm{Im}\Pi^{\sharp}_x, E_x \rangle = T_x M, \quad x \in M. \tag{25}$$

Example

A locally conformal symplectic structure (Ω, α), with Ω non-degenerate and α closed, is said to be with background (ϕ, ω) if

$$\phi = \mathrm{d}\Omega + \alpha \wedge (\Omega - \omega).$$

It generates a transitive Jacobi pair with background $((\Pi, E), (\phi, \omega))$ where

$$\langle \rho \wedge \lambda, \Pi \rangle = \left\langle \Omega, \Omega^{\sharp}\rho \wedge \Omega^{\sharp}\lambda \right\rangle, \quad E = \Omega^{\sharp}\alpha.$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

Let M be a smooth manifold and $((\Pi, E), (\phi_1, \omega_1))$ and $((\Pi, E), (\phi_2, \omega_2))$ be two Jacobi pairs with background on M. If both structures are transitive then the following alternative cases hold:

1. $\dim M$ *is even: there exists a* 2*-form,* $\omega \in \Omega^2(M)$*, such that*
$$\omega_1 = \omega_2 + \omega, \quad \phi_1 = \phi_2 - \omega \wedge \Pi^\flat E; \tag{26}$$
2. $\dim M$ *is odd:*
$$\omega_1 = \omega_2, \quad \phi_1 = \phi_2. \tag{27}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

If a Jacobi pair with background $((\Pi, E), (\phi, \omega))$ on a smooth manifold M is transitive then M is either a locally conformal symplectic manifold with background or a twisted coorientable contact one.

Theorem

The characteristic distribution of a Jacobi pair with background is completely integrable with the characteristic leaves either locally conformal symplectic manifolds with background or twisted coorientable contact ones.

Relaxing twisted Jacobi pairs: Jacobi pair with background

Let $((\Pi, E), (\phi, \omega))$ be a Jacobi pair with background on the smooth manifold M. Then, to each everywhere non-vanishing smooth function

$$a \in \mathcal{F}(M)$$

we can associate the Jacobi pair with background $((\Pi^a, E^a), (\phi^a, \omega^a))$, where

$$\Pi^a = a\Pi, \quad E^a = aE + \Pi^\sharp \mathrm{d}a, \tag{28}$$

$$\phi^a = \frac{1}{a}\phi + \mathrm{d}\left(\frac{1}{a}\right) \wedge \omega, \quad \omega^a = \frac{1}{a}\omega. \tag{29}$$

It can be shown that the brackets associated with the above Jacobi pairs with backgrounds are related via

$$\{f, g\}^a = \frac{1}{a}\{af, ag\}. \tag{30}$$

Relaxing twisted Jacobi pairs: Jacobi pair with background

The 'Poissonization' procedure also works for Jacobi pairs with background.

Definition

Let smooth manifold $(M, \mathcal{A}_M)$ endowed with a pair (Π, ϕ) consisting in

$$\Pi \in \mathfrak{X}^2(M), \quad \phi \in \Omega^3(M),$$

which verify

$$[\![\Pi, \Pi]\!] = 2\Pi^\sharp \phi \tag{31}$$

is called a Poisson manifold with background. If in addition there exists the vector field Z such that

$$\mathcal{L}_Z \Pi \equiv [\![Z, \Pi]\!] = -\Pi, \quad \mathcal{L}_Z \phi = \phi \tag{32}$$

then the Poisson manifold with background is said to be homogeneous.

Relaxing twisted Jacobi pairs: Jacobi pair with background

Theorem

If $((\Pi, E,), (\phi, \omega))$ *is a Jacobi pair with background, then the manifold*

$$\tilde{M} = M \times \mathbb{R} \tag{33}$$

can be naturally organized as a 'homogeneous' Poisson manifold with background defined by

$$\tilde{\Pi} = \mathrm{e}^{-\tau}\left(\Pi + \partial_\tau \wedge E\right), \quad \tilde{\phi} = \mathrm{e}^{\tau}\left(\phi + \omega \wedge \mathrm{d}\tau\right), \quad \tilde{Z} = \partial_\tau. \tag{34}$$

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Jacobi-like pairs as distinguished elements of a Lie algebroid

We start from the Lie algebroid

$$(TM \times \mathbb{R}, [\![\bullet, \bullet]\!], \rho)$$

with

$$[\![(X, f), (Y, g)]\!] \equiv ([X, Y], Xg - Yf), \quad \rho(X, f) \equiv X.$$

By means of the isomorphisms

$$\Gamma\left(\Lambda^{r+1}(TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{r+1}(M) \times \mathfrak{X}^{r}(M),$$

its Gerstenhaber algebra $(\Gamma(\wedge^{\bullet}(TM \times \mathbb{R}), [\![\bullet, \bullet]\!]))$ reads

$$[\![(P, Q), (R, S)]\!] = ([P, R], [P, S] + (-)^{r} [Q, R]).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Moreover, the differential of its de Rham complex $(\Gamma(\wedge^\bullet(TM\times\mathbb{R})^*), \mathbf{d})$, $\mathbf{d}$ can be written, by means of the isomorphisms

$$\Gamma\left(\Lambda^{r+1}(TM\times\mathbb{R})^*\right)\simeq\Omega^{r+1}(M)\times\Omega^{r}(M),$$

in terms of the standard de Rham differential, d as

$$\mathbf{d}(\omega,\alpha)\equiv(\mathrm{d}\omega,-\mathrm{d}\alpha)\,,\quad(\omega,\alpha)\in\Lambda^{k}(M)\times\Lambda^{k-1}(M)\,.$$

We consider the $\mathbf{d}$-cocycle $(0,1)\in\Omega^1(M)\times\mathcal{F}(M)$,

$$\mathbf{d}(0,1)=0$$

and construct the Lie algebroid with 1-cocycle

$$\left(TM\times\mathbb{R},[\![\bullet,\bullet]\!]^{(0,1)},\rho\right).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

The previous bracket has the concrete expression

$$[\![(P,Q),(R,S)]\!]^{(0,1)} = (I, II)\,,$$

where

$$I \equiv [P,R] + p\,(-)^r\,P \wedge S - rQ \wedge R,$$

$$II \equiv [P,S] + (-)^r\,[Q,R] + (p-r)\,Q \wedge S.$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Within the previous context, the equations governing the Jacobi pair (Π, E) on the smooth manifold M simply read

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 0.$$

Also, the equations exhibiting the twisted Jacobi pair $((\Pi, E), \omega)$ reduce to

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 2\,(\Pi, E)^{\sharp}\,(\mathrm{d}\omega, \omega).$$

Finally, the equations displaying the Jacobi pair with background $((\Pi, E), (\phi, \omega))$ are

$$[\![(\Pi, E), (\Pi, E)]\!]^{(0,1)} = 2\,(\Pi, E)^{\sharp}\,(\phi, \omega).$$

Jacobi-like pairs as distinguished elements of a Lie algebroid

Previously, we denoted by $(\Pi, E)^\sharp$ the $\mathcal{F}(M)$-module morphism

$$(\Pi, E)^\sharp : \Gamma(\Lambda^k(T^*M \times \mathbb{R})) \to \Gamma(\Lambda^k(TM \times \mathbb{R})), \tag{35}$$

which is the linear extension of

$$\Omega^1(M) \times \mathcal{F}(M) \ni (\beta, f) \to (\Pi^\sharp \beta + fE, -i_E \beta) \in \mathfrak{X}^1(M) \times \mathcal{F}(M)$$

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Jacobi-like line bundles

By definition, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a bracket

$$\{\bullet, \bullet\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L),$$

that enjoys the properties:

- It is $\mathbb{R}$-linear and skew-symmetric;
- It verifies the Jacobi identity i.e.

$$\{s_1, \{s_2, s_3\}\} + \text{circular} = 0, \quad s_1, s_2, s_3 \in \Gamma(L) \tag{36}$$

- It is local i.e.

$$\text{supp}\{s_1, s_2\} \subset \text{supp} s_1 \cap \text{supp} s_2, \quad s_1, s_2 \in \Gamma(L) \tag{37}$$

Jacobi-like line bundles

It is immediate that Jacobi pairs are equivalent to trivial Jacobi bundles. Indeed, any Jacobi pair (Π, E) on a given manifold endows the trivial line bundle

$$\mathbb{R}_M \equiv \mathbb{R} \times M$$

$\mathcal{F}(M)$-module of smooth sections

$$\Gamma\left(\mathbb{R}_M\right) = \mathcal{F}\left(M\right)$$

with the bracket

$$\{f, g\} \equiv i_{\Pi}\mathrm{d}f \wedge \mathrm{d}g + i_E\left(f\mathrm{d}g - g\mathrm{d}f\right) \tag{38}$$

and conversely, any Jacobi structure on the trivial line bundle displays a bracket in $\mathcal{F}\left(M\right)$.

Jacobi-like line bundles

We consider the Lie algebroid $(DL, [\bullet, \bullet], \sigma)$ whose sections

$$\mathcal{D}(L) \equiv \Gamma(DL)$$

are nothing but the derivations of the module [over $\mathcal{F}(M)$] $\Gamma(L)$ i.e. $\mathbb{R}$-linear maps $\triangle$ which enjoy the existence of a [unique] vector field $X_{\triangle}$ such that

$$\triangle(fs) = (X_{\triangle} f)\, s + f \triangle s, \quad s \in \Gamma(L), f \in \mathcal{F}(M). \tag{39}$$

Previously, the bracket is given by

$$[\triangle, \triangle'] \equiv \triangle\triangle' - \triangle'\triangle,$$

while the anchor returns symbols of derivations

$$\sigma(\triangle) \equiv X_{\triangle}.$$

Jacobi-like line bundles

Now, associated with the tautological representation of DL on L

$$\nabla : \Gamma(DL) \longrightarrow \Gamma(DL), \quad \nabla_{\square}\lambda \equiv \square\lambda, \quad \lambda \in \Gamma(L), \tag{40}$$

the Jacobi algebroid (DL, L) is at hand. This is equivalent to

- the Gerstenhaber-Jacobi algebra consisting in the module

$$\mathcal{D}^{\bullet}L \equiv \Gamma\left(\wedge^{\bullet}J_1 L \otimes L\right) \quad \text{over the algebra} \quad \Gamma\left(\wedge^{\bullet}J_1 L\right) \tag{41}$$

- der-complex consisting in the module

$$\Omega_L^{\bullet} \equiv \Gamma\left(\wedge^{\bullet}(DL)^* \otimes L\right) \quad \text{over the algebra} \quad \Gamma\left(\wedge^{\bullet}(DL)^*\right) \tag{42}$$

In the above we used the notation

$$J_1 L \equiv \left(J^1 L\right)^* \tag{43}$$

and also the vector bundle isomorphism

$$J_1 L \simeq DL \otimes L^*. \tag{44}$$

Jacobi-like line bundles

The homogeneous elements in the algebra (41) consists in skew-symmetric, first-order differential operators

$$\triangle : \Gamma(L) \times \cdots \times \Gamma(L) \to \mathcal{F}(M) \equiv \Gamma(\mathbb{R}_M) \tag{45}$$

while those of the module (41) are the skew-symmetric, first-order differential operators

$$\square : \Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L) \tag{46}$$

The bracket in the previous Gerstenhaber-Jacobi algebra reads

$$[\![\square_1, \square_2]\!] \equiv (-)^{k_1 k_2} \square_1 \circ \square_2 - \square_2 \circ \square_1, \quad \square_a \in \mathcal{D}^{k_a+1} L, \tag{47}$$

with $\circ$ the Gerstenhaber multiplication

$$\square_1 \circ \square_2 (s_1, \cdots, s_{k_1+k_2+1}) \equiv$$
$$\sum_{\tau \in S_{k_1+1,k_2}} (-)^{\tau} \square_1 \left(\square_2 \left(s_{\tau(1)}, \cdots, s_{\tau(k_1+1)}\right), s_{\tau(k_1+2)}, \cdots, s_{\tau(k_1+k_2+1)}\right)$$

Concerning der-complex, $\Omega_L^\bullet$, it is endowed with a homological derivation, d_D, which symbol is nothing but de Rham differential, d_{DL}, associated with the Lie algebroid DL

$$\langle \mathrm{d}_D \lambda, \square \rangle = \langle \square, j^1 \lambda \rangle \tag{48}$$

$$\mathrm{d}_D (\omega \wedge \Omega) = (\mathrm{d}_{DL} \omega) \wedge \Omega + (-)^{|\omega|} \omega \wedge \mathrm{d}_D \Omega, \omega \in \Gamma \left(\wedge^\bullet (DL)^* \right), \Omega \in \Omega_L^\bullet \tag{49}$$

It can be shown that the cohomology of d_D in the der-complex is always trivial i.e.

$$\mathrm{d}_D \Omega_{k>0} = 0 \Longleftrightarrow \Omega_{k>0} = \mathrm{d}_D \Theta_{k-1}. \tag{50}$$

Jacobi-like line bundles

In this unified context, a Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

that verifies Maurer-Cartan equation

$$[\![J, J]\!] \equiv -2J \circ J = 0. \tag{51}$$

The connection between the bracket and the bi-differential operator J simply reads

$$\{s_1, s_2\} \equiv J\left(s_1, s_2\right), \quad s_1, s_2 \in \Gamma\left(L\right). \tag{52}$$

Jacobi-like line bundles

By means of the vector bundle morphism

$$\hat{J} : J^1L \wedge J^1L \to L, \quad \langle \hat{J}, j^1\lambda \wedge \rho \rangle \equiv J(\lambda, \rho), \tag{53}$$

the Jacobi bundle $(L \to M, J)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Example

Let $\mathcal{K}$ be a contact structure on M ,i.e.,

$$\omega_\mathcal{K} : \mathcal{K} \times \mathcal{K} \to TM/\mathcal{K}, \quad \langle \omega_\mathcal{K}, X \wedge Y \rangle \equiv [X, Y] \mod \mathcal{K}$$

is non-degenerate. It defines a unique Jacobi bundle $(TM/\mathcal{K} \to M, J_\mathcal{K})$ which is transitive.

Jacobi-like line bundles

Example

An lcs structure on a given line bundle $L \to M$ is a pair (∇, Ω) consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle and a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ which is closed with respect to the homological degree 1 derivation d_∇ associated with the Jacobi algebroid structure $([\bullet, \bullet], \nabla)$ on the pair (TM, L),

$$d_\nabla \Omega = 0.$$

It defines a unique transitive Jacobi bundle $(L \to M, J)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Jacobi-like line bundles

Moreover, a twisted Jacobi bundle consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

which 'nilpotency' (51) is 'twisted' via the closed Atiyah 3-form

$$\Phi \in \Omega_L^3, \quad \mathrm{d}_D \Phi = 0, \tag{54}$$

i.e.

$$[\![J, J]\!] = 2\hat{J}^\sharp \Phi. \tag{55}$$

Also here, the twisted Jacobi bundle $(L \to M, J, \Phi)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Jacobi-like line bundles

Example

A hyperplane distribution $\mathcal{K}$ together with a 2-form $\psi \in \Gamma\left(\wedge^2 \mathcal{K}^* \otimes L\right)$, $L \equiv TM/\mathcal{K}$ is said to be a twisted contact structure on M if

$$\omega_{\mathcal{K}} + \psi \in \Gamma\left(\wedge^2 \mathcal{K}^* \otimes L\right)$$

is non-degenerate. It defines a unique twisted Jacobi bundle $(L \to M, J_{\mathcal{K},\psi}, \Omega_{\mathcal{K},\psi})$ which is transitive.

Jacobi-like line bundles

Example

A twisted lcs structure on a given line bundle $L \to M$ is pair $((\nabla, \Omega), \omega)$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ and an L-valued 2-form $\omega \in \Omega^2(M; L)$ which verify the compatibility condition

$$d_\nabla \Omega = d_\nabla \omega.$$

It defines a unique transitive twisted Jacobi bundle $(L \to M, J, \mathrm{d}_D \sigma^* \omega)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Finally, a Jacobi bundle with background consists in a line bundle $L \to M$ endowed with a first-order bi-differential operator

$$J \in \mathcal{D}^2 L$$

which ‘nilpotency’ (51) is ‘broken’ via an Atiyah 3-form

$$\Phi \in \Omega_L^3, \tag{56}$$

i.e.

$$[\![J, J]\!] = 2\hat{J}^\sharp \Phi. \tag{57}$$

Also here, Jacobi bundle with background $(L \to M, J, \Phi)$ is said to be transitive if

$$\mathrm{Im}\left(\sigma \circ \hat{J}^\sharp\right) = TM.$$

Jacobi-like line bundles

Example

An lcs structure with background on a given line bundle $L \to M$ is pair $((\nabla, \Omega), (\phi, \omega))$ consisting in a representation ∇ of the tangent Lie algebroid $TM \to M$ on a line bundle, a non-degenerate L-valued 2-form $\Omega \in \Omega^2(M; L)$ an L-valued 3-form $\phi \in \Omega^3(M; L)$ and an L-valued 2-form which verify the compatibility condition

$$d_\nabla \Omega = d_\nabla \omega + \phi.$$

It defines a unique transitive Jacobi bundle with background$(L \to M, J, \mathrm{d}_D \sigma^* \omega + \sigma^* \phi)$ with

$$J(\lambda, \mu) \equiv \langle \Omega, \Omega^\sharp(\mathrm{d}_\nabla \mu) \wedge \Omega^\sharp(\mathrm{d}_\nabla \lambda) \rangle.$$

Outline

- From Jacobi pairs to twisted Jacobi pairs
- Relaxing twisted Jacobi pairs: Jacobi pair with background
- Jacobi-like pairs as distinguished elements of a Lie algebroid
- Jacobi-like line bundles
- Jacobi-like line bundles encompass Jacobi-like pairs

Jacobi-like line bundles encompass Jacobi-like pairs

When the line bundle $L \to M$ is trivial

$$L \equiv \mathbb{R}_M,$$

by means of the isomorphism

$$D\mathbb{R}_M = TM \times \mathbb{R}, \tag{58}$$

the homogeneous elements of the Gerstenhaber-Jacobi algebra (41) reduce to

$$\mathcal{D}^0\mathbb{R}_M = \mathcal{F}(M) \quad \mathcal{D}^k\mathbb{R}_M = \mathfrak{X}^k(M) \times \mathfrak{X}^{k-1}(M), \quad k > 0 \tag{59}$$

while the Gerstenhaber-Jacobi bracket becomes

$$[\![\bullet, \bullet]\!]^{(0,1)}.$$

Jacobi-like line bundles encompass Jacobi-like pairs

In addition, the homogeneous elements of the (Atiyah)der-complex read

$$\Omega^0_{\mathbb{R}_M} = \mathcal{F}(M) \quad \Omega^k_{\mathbb{R}_M} = \Omega^k(M) \times \Omega^{k-1}(M), \quad k > 0. \tag{60}$$

Moreover, the homological derivation in the Atiyah complex can be written in terms of de Rham differential like

$$\mathrm{d}_D f \equiv \mathrm{d}f, \quad \mathrm{d}_D(\omega_k, \omega_{k-1}) \equiv (\mathrm{d}\omega_k, \omega_k - \mathrm{d}\omega_{k-1}), \quad k > 0. \tag{61}$$

With these identifications at hand, the bi-differential operator J is realised as

$$J \leftrightarrow (\Pi, E) \in \mathfrak{X}^2(M) \times \mathfrak{X}^1(M),$$

the Atiyah 3-form in the twisted Jacobi bundle (54) becomes

$$\Phi \leftrightarrow (\mathrm{d}\omega, \omega) = \mathrm{d}_D(\omega, 0) \in \Omega^3(M) \times \Omega^2(M),$$

while the Atiyah 3-form in the Jacobi bundle with background reads

$$\Phi \leftrightarrow (\phi, \omega) \in \Omega^3(M) \times \Omega^2(M).$$

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