PERTURBATIVE QFT: THE CAUSAL APPROACH

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Outline

CAUSAL PERTURBATION THEORY

The purpose of perturbative quantum field theory (pQFT) is the construction of the scattering matrix as a formal perturbative series. The coefficients of the series are the chronological products. In other words, to give a mathematical status of Feynman integrals.

We try to obtain in a natural way pQFT. The evolution operator in non-relativistic quantum mechanics verifies

$$\frac{d}{dt}U(t,s) = -iV_{\rm int}(t)U(t,s); \qquad U(s,s) = I. \tag{1}$$

In terms of the interaction potential and can be expressed as follows:

$$U(t,s) \equiv \sum \frac{(-i)^n}{n!} \int dt_1 \cdots dt_n T(t_1,\ldots,t_n) \tag{2}$$

where the *chronological products* $T_n(t_1,\ldots,t_n)$ verify the following properties:

- **Initial condition**

$$T(t_1) = V_{\rm int}(t_1) \tag{3}$$

- **Symmetry**

$$T_2(t_1, t_2) = (1 \leftrightarrow 2) \tag{4}$$

- **Causality**:

$$\boxed{T_2(t_1, t_2) = T_1(t_1)\ T_1(t_2), \qquad \text{for} \quad t_1 > t_2} \tag{5}$$

  and a similar formula in general.

- **Unitary**

$$U(t,s)^\dagger\ U(t,s) = I \tag{6}$$

In terms of the chronological products, define the *anti-chronological products*

$$\bar{T}_2(t_1, t_2) = T_1(t_1)\ T_1(t_2) + T_1(t_2)\ T_1(t_1) - T_2(t_1, t_2) \tag{7}$$

and we have

$$\bar{T}_2(t_1, t_2) = T_2(t_1, t_2)^\dagger \tag{8}$$

and similar formulas in arbitrary order.

- **Invariance properties**
  If the interaction potential is translation invariant then we have

$$T_n(t_1 + \tau, \ldots, t_n + \tau) = T_n(t_1, \ldots, t_n) \tag{9}$$

We can write an explicit formula

$$T_2(t_1, t_2) = \theta(t_1 - t_2)\ V_{int}(t_1)\ V_{int}(t_2) + \theta(t_2 - t_1)\ V_{int}(t_2)\ V_{int}(t_1). \tag{10}$$

The purpose is to generalize this idea in the relativistic context especially the causality property.

BOGOLIUBOV AXIOMS

One writes the scattering matrix

$$S(g) \equiv I + i \int dx g(x) T(x)$$
$$+ \frac{i^2}{2} \int dx\ dy\ g(x)\ g(y)\ T(x,y) + \cdots \tag{11}$$

where $g : \mathcal{M} \rightarrow R$ is some test function over the Minkowski space and the sum is formal. The expressions of the type $T(x,y)$ are called *chronological products*. They are distribution-valued operators in a Hilbert space $\mathcal{H}$.

One can justfy as above the following set of axioms:

- **Initial condition**
  The expression $T(x_1)$ is an input of the theory, called the *interaction Lagrangian*.
- **Symmetry**

$$T(x_1, x_2) = (1 \leftrightarrow 2) \tag{12}$$

- **Causality**
  We have a more refined causality property involving the light cones from the Minkowski space:

$$\boxed{T(x, y) = T(x)T(y)} \tag{13}$$

  for $x \succ y$ i.e. $y \cap (x + \bar{V}^{+}) = \emptyset$ i.e. the point x succeeds causally the point y .
- **Poincaré invariance**
  We must have a natural action of the Poincaré group in $\mathcal{H}$ and we impose that for all elements $g \in inSL(2, \mathbb{C})$ of the universal covering group of the Poincaré group:

$$U_g T(x_1, x_2) U_g^{-1} = T(g \cdot x_1, g \cdot x_2) \tag{14}$$

- **Unitarity**: If we define the *anti-chronological products* according to

$$\bar{T}(x_1, x_2) \equiv T(x_1)T(x_2) + T(x_2)T(x_1) - T(x_1, x_2) \tag{15}$$

then the unitarity axiom is:

$$\bar{T}(x_1, x_2) = T(x_1, x_2)^\dagger. \tag{16}$$

The formula

$$T_2(x_1, x_2) = \theta(x_1^0 - t_2^0)\ T(x_1)\ T(x_2) + \theta(x_2^0 - x_1^0)\ T(x_2)\ T(x_1). \tag{17}$$

involves an illegal operation - the multiplication of distributions - and is not true in general. We have to find more sophisticated ways to construct the chronological products.

One can generalize these axioms for arbitrary chronological products $T(x_1, \ldots, x_n)$ and even more for expressions of the type

$$T(A_1(x_1), \ldots, A_n(x_n))$$

where $A_1(x_1), \ldots, A_n(x_n)$ are some operators in the Hilbert space $\mathcal{H}$. From the causality axiom one can derive that we must have

$$\boxed{[A_j(x), A_k(y)] = 0, \quad (x - y)^2 < 0.} \tag{18}$$

i.e. for causally separated points.

It is a non-trivial problem to find solutions of this axiom. We remark that there is no non-relativistic analogue of the axiom. This is the first instance where we see the difficulty of unifying relativity with quantum mechanics.

There are some interesting solutions of the preceding relation: the free fields and the associated Wick monomials.

A quantum field is a distribution-valued operator

$$\Phi(x) \tag{19}$$

acting in some Hilbert space. For a free field the Hilbert space is of Fock type. There are various methods to construct quantum free fields.

- Using creation and annihilation operators in a Fock space; the fields are linear combinations of them
- Using Weyl algebras
- Using Borchers algebra; It is based essentially on the reconstruction theorem of Wightman: this theorem says that from the vacuum averages

  $$< \Omega, \Phi(x_1) \cdots \Phi(x_n)\Omega > \tag{20}$$

  one can reconstruct the quantum field $\Phi(x)$.
- Using GNS construction

We illustrate the last method considering for simplicity a real scalar field. We consider the algebra

$$\mathcal{S} = \sum \mathcal{S}_n; \tag{21}$$

Here $\mathcal{S}_n$ are distributions in the variables $x_1, \dots, x_n \in \mathcal{M}$.
One defines a quasi-free state on $\mathcal{S}$ (which is endowed with a structure of $*$-unital algebra) according to

$$\omega(f \otimes g) = \int D_m^{(+)}(x - y)\; f(x)\; g(y) \tag{22}$$

where $D_m^{(+)}(x - y)$ is the positive part of the Pauli-Villars distribution. In general

$$\omega(f_1 \otimes \cdots \otimes f_n) = \sum \omega(f_{i_1}, f_{i_2}) \cdots \omega(f_{i_{n-1}}, f_{i_n}) \tag{23}$$

for n even and null for n odd; in an equivalent language, the state is defined by imposing that the associated truncated state is zero for more that three entries. Then one generates the GNS representation.

As a consequence, we can prove that we have obtained the *(quantum) free scalar field*: the attribute free is due to the verification of Klein-Gordon equation

$$(\Box + m^2)\ \Phi = 0. \tag{24}$$

We also have the causal commutation relation (CCR):

$$\boxed{[\Phi(x), \Phi(y)] = -i\ D_m(x-y) \cdot \mathbf{I}} \tag{25}$$

and:

$$\Phi^\dagger = \Phi \tag{26}$$

The quantum (real) scalar field constructed above is used to describe *Higgs* Bosons. Scalar fields are also used in MOND theories: modified gravitation theories describing dark matter.

The same construction works for other cases:

- Generalized quantum fields: in this case one puts in the right hand side of the preceding equation:

$$D_m(x-y) \to \int \rho(\lambda) D_\lambda(x-y) \tag{27}$$

  so we loose Klein-Gordon equation. These fields are used for *regularization* (Pauli-Villars, dimensional, etc.)
- Dirac fields: the fields are: $\psi_\alpha, \ \alpha = 1, \dots, 4$ verifying Dirac equation and associated to the representation $[m, 1/2]$ of the Poincaré group. These fields are used to describe *leptons* (the electron, miuons) and *quarks*.
- Thermal fields: one multiplies by a Boltzmann factor the Fourier transform of the Pauli-Jordan distribution (this breaks Lorentz covariance!).
- Quantum fields on Riemann manifolds.
  In the last case one has to replace the positive part of Pauli-Jordan distribution

$$D_m^{(+)}(x_1 - x_2) \tag{28}$$

  by a distribution of Hadamard type

$$\omega_2(x_1, x_2). \tag{29}$$

  Such distributions are not unique so there is no unique vacuum. Accordingly, there is no unique decomposition of the fields in the creation and annihilation parts.

A similar construction makes sense of monomials in one or more variables $:\Phi(x)^n:$, $:\Phi(x)^n\Phi(y)^m:$ etc. These are the *Wick* monomials. It is a non-trivial result that such expressions make sense and verify a causality property as above. If the interaction Lagrangian is a Wick monomial then one can supplement Bogoliubov axioms as follows:

- We start from *Wick theorem*

$$:\Phi(x)^m::\Phi(y)^n:=\sum_{k=l} C_m^k C_n^l < \Omega, :\Phi(x)^k::\Phi(y)^l:\Omega > \times \\ :\Phi(x)^{m-k}\Phi(y)^{n-l}: \tag{30}$$

  we impose **Wick expansion** property:

$$T(:\Phi(x)^m:,:\Phi(y)^n:) = \sum_{k=l} C_m^k C_n^l < \Omega, T(:\Phi(x)^k:,:\Phi(y)^l:)\Omega > \times \\ :\Phi(x)^{m-k}\Phi(y)^{n-l}: \tag{31}$$

  We note the first appearence of a loop expansion and of a Hopf algebra structure.

- **Power counting** There is an upper bound on the degree of singularity of the distributions

$$t^{k,l}(x,y) \equiv < \Omega, T(:\Phi(x)^k:, :\Phi(y)^l:)\Omega > \tag{32}$$

appearing above:

$$\omega(t^{k,l}) \leq k + l - 4. \tag{33}$$

If we take $m = n < 4, m = n = 4, m = n > 4$ we obtain super-renormalizable, renormalizable, and non-renormalizable models. It is assumed that the last case it is not physical; however it can be treated as an effective field theory.

SECOND ORDER CHRONOLOGICAL PRODUCTS

We go to the second order of perturbation theory using the *causal commutator*

$$D^{A,B}(x,y) \equiv D(A(x), B(y)) = [A(x), B(y)] \tag{34}$$

where $A(x), B(y)$ are arbitrary Wick monomials. These type of distributions are translation invariant i.e. they depend only on $x - y$ and the support is inside the light cones:

$$\boxed{supp(D) \subset V^+ \cup V^-.} \tag{35}$$

A theorem from distribution theory guarantees that one can causally split this distribution:

$$D(A(x), B(y)) = A(A(x), B(y)) - R(A(x), B(y)). \tag{36}$$

where:

$$supp(A) \subset V^+ \qquad supp(R) \subset V^-. \tag{37}$$

The expressions $A(A(x), B(y)), R(A(x), B(y))$ are called *advanced* resp. *retarded* products.

They are not uniquely defined: one can modify them with *quasi-local terms* i.e. terms proportional with $\delta(x-y)$ and derivatives.
There are some limitations on these redefinitions coming from Lorentz invariance, and *power counting*: this means that we should not make the various distributions appearing in the advanced and retarded products too singular.
Then we define the *chronological product* by:

$$\begin{aligned} T(A(x), B(y)) &= A(A(x), B(y)) + B(y)A(x) \\ &= R(A(x), B(y)) + A(x)B(y). \end{aligned} \tag{38}$$

The expression $T(x, y)$ corresponds to the choice

$$T(x, y) \equiv T(T(x), T(x)). \tag{39}$$

The expressions D, A, R, T admit a loop decomposition, according to the number of Wick contractions from the Wick expansion property.
The preceding relations are inspired by the relations well-knowed relations:

$$\begin{gathered} D = D^{\mathrm{adv}} - D^{\mathrm{ret}} \\ D^{\mathrm{adv}} = \theta(x^0)\ D, \qquad D^{\mathrm{ret}} = -\theta(-x^0)\ D \end{gathered} \tag{40}$$

$$\begin{aligned} D^F &= D^{\mathrm{ret}} + D^{(+)} = D^{\mathrm{adv}} - D^{(-)} \\ \bar{D}^F &= D^{(+)} - D^{\mathrm{adv}} = -D^{\mathrm{ret}} - D^{(-)} \end{aligned} \tag{41}$$

$$D^F + \bar{D}^F = D^{(+)} - D^{(-)} \tag{42}$$

The Fock space is constructed starting from some Hilbert space (called the *one-particle* Hilbert space and associating the symmetric (resp. antisymmetric) tensor algebra for Bosons (resp. Fermions). There is a theorem *spin-statistics* saying that for integer spins we should use Bose statistics (symmetric case) and for half-odd spins we should use Fermi statistics (the antisymmetric case).
One takes the one-particle Hilbert space to carry a (projective) irreducible unitary representation of the Poincaré group. These are defined up to unitary equivalence. The choice of the explicit representation is a sort of educated guess.
The construction of a free field associated to some quantum particle $[m, s]$ is not an unique operation.
Some choices are not very good: for higher spin fields one is tempted to use only the physical degrees of freedom. For instance the photon or the gluon are described by the representation $[0, 1]$ and using the Wigner description of this representation one is lead to a model for which the degree of singularity grows with the order of the perturbation theory. This leads to a growing number of free constants in the chronological products: the theory seems to be non-renormalizable.

The way out of this is suggested by the Faddeev - Popov trick. One looks for new representations of the Fock space associated to $[m, s]$. Basically, the Fock space is generated by physical and un-physical degree of freedom, so one has to select the subspace of physical states. So, we take a set of fields to $B_j(x), F_A(x)$ of Bose (resp. Fermi) statistics and looks for an operator to Q such that:

$$[Q, B(x)] \sim F(x), \quad [Q, F(x)] \sim B(x), \quad Q\Omega = 0 \tag{43}$$

and

$$Q^2 = 0. \tag{44}$$

Moreover we want that the cohomology space

$$Ker(Q)/Ran(Q) \tag{45}$$

should be isomorphic to the Fock space of the physical states. The construction of such a *gauge structure* is usually suggested by the classical field version of the theory. The interaction Lagrangian is a Wick polynomial in the fields $B_j(x), F_A(x)$ which should leave invariant the subspace of physical states. A natural way to do it is to impose

$$d_Q T(x) \equiv [Q, T(x)] = 0 \tag{46}$$

One easily discovers that this requirement is too strong: there are no solutions! So, according to Stora, we can relax the preceding axiom to:

$$[Q, T(x)] \sim \text{total divergence} \tag{47}$$

in such a way that if we multiply this identity with a test function the right hand side becomes smaller and smaller as the test function becomes flatter and flatter. However, the flat limit - called *the adiabatic limit* - cannot be performed always. It means that the preceding relation has the status of a new axiom: the gauge invariance condition.
We have an associated relative cohomology structure for the observables. The observables should be operators verifying the preceding relations (the relative cocycle condition) and the observables of the type

$$T(x) = [Q, B(x)] + \text{total divergence} \tag{48}$$

are coboundaries They are trivial from the physical point of view: they give a null contribution when averaged on physical states (in the formal adiabatic limit).

Examples:

- **Photons and gluons**
  They correspond to the representation $[0, 1]$ of the Poincaré group. In the case of the photon we have:

$$B \sim v^\mu, \qquad F \sim u, \ \tilde{u}$$

where all fields are of null mass. The Fermi fields are scalars but of wrong statistics, so they cannot be physical. Also in the Boson vector field there are non-physical degrees of freedom. In the case of the r gluons we have r copies of the preceding construction

$$B \sim v_a^\mu, \qquad F \sim u_a, \ \tilde{u}_a, \quad a = 1, \dots, r.$$

Explicitly, the gauge charge operator is given by

$$\begin{aligned} Q\Omega = 0, \qquad Q^\dagger = Q, \\ [Q, v_a^\mu] = i\partial^\mu u_a, \\ \{Q, u_a\} = 0, \qquad \{Q, \tilde{u}_a\} = -i\ \partial_\mu v_a^\mu. \end{aligned} \tag{49}$$

One can prove that the gauge invariance condition given above restricts drastically the possible form of T i.e. every such expression is, up to a trivial Lagrangian, equivalent to the QCD Lagrangian:

$$T = f_{abc} \left(\frac{1}{2} : v_{a\mu} v_{b\nu} F_c^{\nu\mu} : - : v_a^\mu u_b \partial_\mu \tilde{u}_c : \right) \tag{50}$$

where the (real) constants f_{abc} are completely antisymmetric. Here

$$F_a^{\mu\nu} = \partial^\mu v_a^\nu - (\mu \leftrightarrow \nu). \tag{51}$$

This is the tri-linear part of the classical expression for the YM Lagrangian.

- **Massive vector fields** $W^{\pm}, Z$
  They correspond to the representation $[m, 1], m > 0$ of the Poincaré group. In this case we have:
  $$B \sim v^{\mu}, \quad \Phi, \qquad F \sim u, \ \tilde{u}$$
  where all fields have mass m.
  These type of vector particles, together with Dirac fields (describing matter) and a scalar field (Higgs) are the building blocks of the Standard Model.
- **Gravity**
  It corresponds to the representation $[0, 2]$ of the Poincaré group. In this case we have:
  $$B \sim h_{\mu\nu}, \qquad F \sim u_{\mu}, \ \tilde{u}_{\nu}$$
  where all fields are of null mass the first tensor is symmetric.
  There is a possibility to generalize the formalism to massive gravity which corresponds to a theory with non-zero cosmological constant.
  One can investigate if a multi-graviton theory is possible. The answer is negative: is there are more that one species of particles of the type $[0, 2]$ there is no coupling between them. There is no analogue of YM Lagrangian.

- **Supersymmetry**
  It is an extension of the Poincaré algebra. Besides the generators $P_\mu, L_{\mu\nu}$ of the Lie algebra (corresponding to the translation and Lorentz transformations resp.) we have some new generators Q_α of odd character such that

$$\{Q, Q^\dagger\} \sim P, \quad [Q, P] = 0, \quad [Q, L] \sim Q \tag{52}$$

  The irreducible representations of this super-algebra can classified (Haag).
  For instance the Wess-Zumino model corresponds to $[m, 0] \oplus [m, 1/2], \quad m > 0$ and the vector model to $[m, 1] \oplus [m, 3/2], \quad m > 0.$ The vector model is used to build the supersymmetric extensions of the Standard model.

QUANTUM ANOMALIES

Using the relation

$$d_Q T(x) = i\partial_\mu T^\mu(x) \tag{53}$$

expressing first-order gauge invariance we can obtain

$$d_Q T(T(x), T(y)) - i\frac{\partial}{\partial x^\mu} T(T^\mu(x), T(y)) - i\frac{\partial}{\partial y^\mu} T(T(x), T^\mu(y)) = 0 \tag{54}$$

for $x \neq y$. In general we have *anomalies* i.e a non-trivial expression in the right hand side. In a condensed notation for the left hand side:

$$sT(T(x), T(y)) = A(x, y) \tag{55}$$

where the right hand side is a quasi-local expression i.e. it is localized in the set $x = y$.
One can compute the anomalies in various ways (for instance using the off-shell formalism).
The elimination of the anomalies ensures that the cronological products leave invariant the physical Fock space.

If the anomaly is a coboundary

$$\begin{gathered} A(x,y) = sN(x,y) = \\ d_Q N(x,y) - i\frac{\partial}{\partial x^\mu} N^\mu(x,y) - i\frac{\partial}{\partial y^\mu} N^\mu(y,x) \end{gathered} \tag{56}$$

with N, N^μ quasi-local, then we can redefine the chronological products and eliminate it. We can fix second order gauge invariance by the redefinitions of the chronological products

$$\begin{gathered} T(T(x),T(y)) \to T(T(x),T(y)) + N(x,y), \\ T(T(x),T^\mu(y)) \to T(T(x),T^\mu(y)) + N^\mu(x,y) \end{gathered} \tag{57}$$

where

$$N(x,y) = \delta(x-y)\ N(x), \quad N^\mu(x,y) = \delta(x-y)\ N^\mu(x) \tag{58}$$

The anomalies are restricted by some equations following from the gauge invariance condition: the so-called Wess-Zumino consistency conditions.

For YM models, like QCD we have in second order an anomaly in the tree contribution from the loop expansion. It can be eliminated *iff* we have Jacobi identity

$$f_{abe}f_{cde} + (a \leftrightarrow c) + (b \leftrightarrow c) = 0. \tag{59}$$

If we add all particles of the standard model we get more identites of this type. The expression N is the quadri-linear term from the classical YM Lagrangian.
Going in the third order of the perturbation theory we get anomalies in the tree contribution and in the one-loop contribution. The elimination of the first anomaly gives the Higgs potential. The one-loop anomaly has an axial part and its elimination gives constraints on the Dirac part of the interaction Lagrangian.
An interesting supplementary axiom for a gauge model is to impose *super-renormalizablility* i.e. to require:

$$T(x,y) = T_0(x,y) + \text{trivial operator} \tag{60}$$

where the first term contains distributions less singular than the power counting.

CONCLUSIONS

There are other ways to do construct the chronological products:

- **Hepp**:
  - uses Feynman amplitudes (essentially vacuum averages of chronological products)
  - uses a set of axioms involving only numerical distributions (but implying Bogoliubov axioms)
  - needs a regularization procedure (this means to works with generalized free fields)
  - one can use the naive Feynman rules if one adds counter-terms in the interaction Lagrangian
  - forest formula (Zimmermann) and its modern version (Kreimer, Connes): the separation of the finite part amounts to the Birkhoff decomposition of Laurent series.
- **Polchinski**:
  - works with the Green functions, so one must take the masses positive (to avoid infrared divergences).
  - needs a regularization procedure (ultra-violet cut-off)
  - flow equations in the ultra-violet cut-off
  - recursive procedure
  - clever choice of the initial condition

Advantages:
- a close contact with classical field theory, using a loop expansion so the 0 -loop contribution is the classical Lagrangian.

Problems:
- complicated proofs for the elimination of ultraviolet divergences (Hepp),
- infrared divergences,
- the BRST transformation is non-linear so it can be defined only for classical fields and for the asymtotic quantum fields (Kugo, Ojima)
- no axiomatic scheme; recent work of Fredenhagen and collab.

Causal Method:

Main results:

1. A systematic study of lower orders of perturbation theory (interaction Lagrangians, constraints on the constants), multi-Higgs models, multi-graviton models;
2. Derivation of the Wess-Zumino consistency for anomaly, anomalies in the third order;
3. Super-renormalizablility; one-loop contribution in linear gravity.

In the **the causal method** one constructs directly the chronological products.
Advantages:
- the construction of the QFT is done directly quantum mechanically (no quantization procedure needed),
- proofs for the elimination of ultraviolet divergences is much simpler (Epstein-Glaser, Steinmann)
- the are no infrared problems,
- BRST transformation is linear so it can be defined for quantum fields,
- there exists an axiomatic approach,
- one can investigate if the singular behavior is better than power counting (i.e. we have some sort of super-renormalizablility); for one and two-loops contributions this is true in the second order of pQFT for YM and for massless gravity,
- there are some discrepancies with respect to the BRST and Polchinski approaches: anomalies, SUSY.

Problems: Apparently the proof of gauge invariance is much more difficult.
The equivalence with the functional method (BRST gauge invariance) is not rigorously proved.
The open problems are rather difficult and completely outside of the main stream. This could change!

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