The symplectic structure of four-dimensional supergravity

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Context and background

Main goal

The main goal of this talk is to give a pedagogical introduction to the global geometric formulation of four-dimensional ungauged supergravity, developed recently in Rev. Math. Phys. 30 no. 5 (2018) and J. Geom. Phys. 128 (2018), with a view towards applications in differential geometry and topology. Special importance will be given to geometrically understand its *symplectic structure* and *Dirac quantization*.

Ungauged supergravity in four dimensions

Four-dimensional ungauged supergavity is a supersymmetric theory of gravity that, aside from being phenomenological interesting by itself, plays a prominent role in the description of the effective dynamics to Type-II string theory compactified on a Calabi-Yau three-fold.

Mathematical ingredients of ungauged supergravity

Four-dimensional ungauged supergravity involves elements of relevance in modern mathematics, such as:

- Kähler manifolds and moduli spaces in algebraic geometry.
- Projective special Kähler geometry.
- Quaternionic-Kähler manifolds.
- Maps of special type into complex and Quaternionic-Kähler manifolds.
- Homogeneous spaces and exceptional Lie groups.
- Spin geometry and generalized Killing spinors.
- Gauge-theoretic moduli problems.
- Dynamical systems.

These ingredients have been shown to appear in the local formulation of supergravity, and give rise to numerous open mathematical problems and potential applications in differential geometry and topology.

Mathematical formulation

In order to explore the mathematical problems proposed by ungauged supergravity and its potential mathematical applications it is necessary to develop first the mathematical formulation of the theory in a differential-geometric context.

Current status

- The local structure of supergravity has been extensively studied in the literature since the 70's.
- The local structure of all ungauged supergravities has been classified by dimension and amount supersymmetry preserved.
- Various formalisms, such as (exceptional) generalized geometry and double field theory, have been developed to further explore the local structure of supergravity, uncovering interesting mathematical structures.
- Local supersymmetric solutions of supergravity have been systematically studied in the literature and classified to some extent.
- The global mathematical formulation of supergravity remains to be developed, together with the study of its supersymmetric solutions, associated moduli spaces and applications in differential geometry.

Obtaining the mathematical formulation of supergravity on a differentiable manifold requires:

1. Determining its configuration space in terms of connections and global sections of the appropriate fiber bundles (submersions, gerbes...) equipped with the appropriate geometric structures.
2. Determining the equations of motion and Killing spinor equations of the theory in terms of global differential operators acting on the corresponding spaces of sections.
3. Determining the automorphism group of the system of partial differential equations defining the theory.

The local Lagrangian of ungauged supergravity

Fix contractible open set $U \subset \mathbb{R}^4$. The bosonic sector of unaguged four-dimensional supegravity on U is defined by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mathrm{R}(g) - \frac{1}{2}\mathcal{G}_{AB}(\varphi)\partial_\mu\varphi^A\partial^\mu\varphi^B - \gamma_{ij}(\varphi)\, F^i_{\mu\nu}F^{j\mu\nu} - \theta_{ij}(\varphi)\, F^i_{\mu\nu} *_g F^{j\,\mu\nu} ,$$

where $\varphi^A \colon U \to \mathbb{R}$ are functions (scalar fields) and:

$$F^i = \mathrm{d}A^i \in \Omega^2(U)$$

are the field strengths of the local gauge fields $A^i \in \Omega^1(U)$.

Here:

$$\gamma\colon V \to \mathrm{Mat}(n_v, \mathbb{R})\,, \quad \theta\colon V \to \mathrm{Mat}(n_v, \mathbb{R})\,, \quad \gamma(\varphi) \stackrel{\mathrm{def.}}{=} \gamma\circ\varphi\,, \quad \theta(\varphi) \stackrel{\mathrm{def.}}{=} \theta\circ\varphi\,,$$

with $\varphi(U) \subset V$, and $\mathcal{G}_{AB}$ and $\gamma_{ij}(\varphi)$ positive definite. Variables of the theory: (g, φ, A^i).

Remarks on the local Lagrangian

Therefore, the local bosonic sector of extended supergravity is uniquely determined by a choice of Riemannian metric $\mathcal{G}$ on V, and matrix valued functions γ and θ as described above. The matrix γ generalizes the inverse of the squared coupling constant appearing in ordinary four-dimensional gauge theories, whereas θ generalizes the *theta angle* of quantum chromodynamics. The functional S_l can be naturally written as a sum of three pieces:

$$\mathrm{S}_l = \mathrm{S}_l^e + \mathrm{S}_l^s + \mathrm{S}_l^v \,,$$

where:

- $\mathrm{S}_l^e \leftrightarrow$ Einstein-Hilbert term $\leftrightarrow$ Theory of Einstein metrics.
- $\mathrm{S}_l^s \leftrightarrow$ scalar sector $\leftrightarrow$ Theory of wave (harmonic) maps.
- $\mathrm{S}_l^v \leftrightarrow$ gauge sector $\leftrightarrow$ Yang-Mills-like theory.

Hence, the bosonic extended supergravity simultaneously involves three classical theories in differential geometry, coupling the Einstein-Hilbert theory to a non-linear sigma model with Riemannian target space $(V, \mathcal{G}_{ij})$ and to a given number of *abelian gauge fields*.

Interludio: scalar manifolds four-dimensional supergravity.

The scalar manifolds that can appear in ungauged four-dimensional supergravity are highly constrained by supersymmetry, as the following table indicates.

Number of supersymmetries	Isometry type of $(V, \mathcal{G}_{ij})$
$\mathcal{N} = 1$	$\mathcal{M}_{KH}$
$\mathcal{N} = 2$	$\mathcal{M}_{PSK} \times \mathcal{M}_{QK}$
$\mathcal{N} = 3$	$\mathrm{SU}(3, n)/\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(n))$
$\mathcal{N} = 4$	$\mathrm{SU}(1,1)/\mathrm{U}(1) \times \mathrm{SO}(6, n)/\mathrm{S}(\mathrm{O}(6) \times \mathrm{O}(n))$
$\mathcal{N} = 5$	$\mathrm{SU}(1,5)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(5))$
$\mathcal{N} = 6$	$\mathrm{SO}^*(12)/\mathrm{U}(1) \times \mathrm{SU}(6)$
$\mathcal{N} = 8$	$\mathrm{E}_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$

Table: Local isometry type of the scalar manifolds of four-dimensional supergravity

Equivalent formulation of the gauge sector

We can equivalently formulate the theory in terms of the *complexified field strengths*:

$$F^+ \overset{\text{def.}}{=} F - i * F \,, \qquad F^- \overset{\text{def.}}{=} F + i * F \,,$$

in terms of which the gauge sector of the theory is conveniently written, using matrix notation, as follows:

$$\mathrm{S}^v_l[\phi, A] \overset{\text{def.}}{=} \frac{i}{4} \int_U \left\{ F^{+T} \mathcal{N} F^+ - F^{-T} \mathcal{N}^* F^- \right\} \nu_{g_U} \,,$$

where:

$$\mathcal{N} \overset{\text{def.}}{=} \theta + i\gamma \colon V \to \mathbb{SH}(n_v) \,,$$

is a function on V valued in Siegel upper space $\mathbb{SH}(n_v)$ of square $n_v \times n_v$ complex matrices with positive definite imaginary part. For ease of notation, we define:

$$\mathcal{N}(\phi) \overset{\text{def.}}{=} \mathcal{N} \circ \phi \colon \theta(\phi) + i\gamma(\phi) \colon U \to \mathbb{SH}(n_v) \,,$$

to which we will refer as the *scalar valued* period matrix.

Scalar-coupled Maxwell equations

The *Bianchi identities* and *Maxwell equations* are equivalent to:

$$\mathrm{d}F^i = 0\,, \qquad \mathrm{d}G_i = 0\,,$$

where:

$$G_i(\phi) \stackrel{\text{def.}}{=} \theta_{ij}(\phi)F^j - \mathcal{I}_{ij}(\phi) * F^j \in \Omega^2(U)\,.$$

which in turn can be written simply as $\mathrm{d}\mathcal{V} = 0$, where $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n})$ is the following vector of two-forms:

$$\mathcal{V} = (F, G)^T\,.$$

Defining:

$$G^+ \stackrel{\text{def.}}{=} G - i*G\,, \qquad G^- \stackrel{\text{def.}}{=} G + i*G\,, \qquad \mathcal{V}^+ \stackrel{\text{def.}}{=} \mathcal{V} - i*\mathcal{V}\,, \qquad \mathcal{V}^- \stackrel{\text{def.}}{=} \mathcal{V} + i*\mathcal{V}\,,$$

we obtain:

$$G^+ = \mathcal{N}F^+\,, \qquad G^- = \mathcal{N}^*F^-\,, \qquad \mathcal{V}^+ = (F^+, \mathcal{N}F^+)^T\,.$$

Global symmetries of the local theory

We want to understand the global symmetries of the local Lagrangian of ungauged supergravity: we do not consider diffeomorphisms of M. We observe:

- The scalar sector is invariant under isometries of the scalar manifold $(V, \mathcal{G}_{AB})$.
- The Maxwell equations of motion are invariant under *constant matrix transformations*:

$$\mathrm{d}\mathcal{V} = 0 \;\Rightarrow \mathrm{d}(\mathcal{T}\,\mathcal{V}) = 0\,, \quad \forall\, \mathcal{T} \in \mathrm{Gl}(2n_v, \mathbb{R})\,.$$

We want to exploit and merge these two facts together in order to extend isometries of $(V, \mathcal{G}_{AB})$ to global symmetries of the complete theory, obtaining as a result a generalization of *electromagnetic duality*.

Global symmetries of the local theory II

We verify that if $F \in \mathrm{Iso}(V, \mathcal{G}_{AB})$ satisfies:

$$\mathcal{N} \circ F = \iota_F \cdot \mathcal{N}\,, \qquad \iota_F \in \mathrm{Sp}(2n, \mathbb{R})\,.$$

then the equations of motion of the complete theory are preserved and the transformed fields admit a Lagrangian formulation of the same type. Furthermore, if we transform $\mathcal{V}$ with a transformation in $\mathrm{Sp}(2n, \mathbb{R})$ then $\mathcal{N}$ transform as prescribed by the corresponding fractional transformation. This can be shown to hold in extended supergravity, implying that the isometry group of the scalar manifold extends to a symmetry of the complete local theory through *electromagnetic duality transformations*, also called *U-duality transformations*. This is a generalization of the notion of electromagnetic already happening in Maxwell's theory of electrodynamics!

Geometric supergravity: preliminaries.

We need to obtain a global *geometric model* (system of partial differential equations and associated geometric structures, which called *geometric supergravity*) which recovers locally the structure and symmetries previously introduced. Key point:

- Geometric bosonic supergravity needs to implement the electromagnetic U-duality groups in the sense that it must be possible to understand the theory as being the result of *gluing* the local theories *á la Čech*. This point is especially important for the resulting theory to describe *supergravity U-folds* in a geometric context.

We will assume for simplicity that the theory is coupled to a standard non-linear sigma model, instead of the more general possibility of a *section* sigma model.

Geometric supergravity: the electromagnetic structure.

Instead of going through the process of constructing geometric bosonic supergravity we will present it in its final form. Geometric bosonic supergravity is uniquely defined by the following data:

- A complete Riemannian manifold $(\mathcal{M}, \mathcal{G})$, the so-called *scalar manifold* of the theory.
- A tuple $\Delta \stackrel{\text{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$ consisting on a flat symplectic vector bundle $\mathcal{S}$, with symplectic structure ω and flat symplectic connection $\mathcal{D}$.
- A compatible taming $\mathcal{J}$ on $(\mathcal{S}, \omega, \mathcal{D})$, that is, an almost complex structure on $\mathcal{S}$ satisfying:

$$\omega(\mathcal{J}s_1, \mathcal{J}s_2) = \omega(s_1, s_2)\,, \qquad Q(s,s) \stackrel{\text{def.}}{=} \omega(s, \mathcal{J}s) > 0\,,$$

Definition

Electromagnetic structure $\Theta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$. Scalar-electromagentic structure $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$.

Geometric supergravity: an observation.

Isomorphism classes of duality structures on a fixed scalar manifold $\mathcal{M}$ are in general not unique and depend on the fundamental group of $\mathcal{M}$. Standard theory of flat vector bundles shows that isomorphism classes of duality structure are in one to one correspondence with a character variety:

$$\mathcal{X}_d = \mathrm{Hom}(\pi_1(\mathcal{M}), \mathrm{Sp}(2n, \mathbb{R}))/\mathrm{Sp}(2n, \mathbb{R}) .$$

The fact that character varieties yield in general *continuous* moduli spaces suggests the possibility of constructing an uncountable infinity of inequivalent geometric bosonic supergravities which become equivalent locally!

Geometric supergravity: twisted duality.

For every Lorentzian metric g on M and scalar map φ we define a map of vector bundles:

$$\star_{g,J^\varphi} : \Lambda T^*M \otimes \mathcal{S}^\varphi \to \Lambda T^*M \otimes \mathcal{S}^\varphi \,,$$

by $\star_{g,J^\varphi}(\alpha \otimes s) = *_g \alpha \otimes J^\varphi(s)$ on homogeneous elements. Restricted to two-forms: $\star^2_{g,J^\varphi} = 1$: new notion of (anti) self-duality in four-Lorentzian dimensions. Hence, we can split the bundle of two-forms taking values in $\mathcal{S}^+$ in eigenbundles of $\star_{g,J^\varphi}$:

$$\Lambda^2 T^*M \otimes \mathcal{S}^\varphi = (\Lambda^2 T^*M \otimes \mathcal{S}^\varphi)_+ \oplus (\Lambda^2 T^*M \otimes \mathcal{S}^\varphi)_- \,,$$

where the subscript denotes the corresponding eigenvalue.

Definition

Elements of $\Omega^2_+(M, \mathcal{S}^\varphi)$ will be called **twisted self-dual two-forms** and elements of $\Omega^2_-(M, \mathcal{S}^\varphi)$ will be called **twisted anti-self-dual two-forms**.

Geometric supergravity: the configuration space.

Once a choice of scalar-electromagnetic structure Φ has been made, geometric bosonic supergravity is uniquely determined and its configuration space is given by:

- A Lorentzian metric g on M.
- A *scalar map* $\varphi\colon M \to \mathcal{M}$.
- A positively-polarized two-form $\mathcal{V} \in \Omega^2_+(M, \mathcal{S}^\varphi)$ with values in $\mathcal{S}^\varphi$.

This gives a global definition of the configuration space $\mathrm{Conf}(M, \Phi)$ of the theory: crucial for understanding moduli spaces!

$$\mathrm{Conf}(M, \Phi) \stackrel{\mathrm{def.}}{=} \left\{(g, \varphi, \mathcal{V}) \mid g \in \mathrm{Lor}(M)\,,\ \varphi \in C^\infty(M, \mathcal{M})\,,\ \mathcal{V} \in \Omega^2_+(M, \mathcal{S}^\varphi)\right\} \,.$$

Note the *coupled nature* of the configuration space!

Geometric supergravity: the fundamental form.

The definition of electromagnetic structure $\Theta = (\mathcal{S}, \omega, \mathcal{D}, \mathcal{J})$ does not require $\mathcal{D}$ to be compatible with $\mathcal{J}$, which is crucial to describe supergravity.

Definition

Let $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$ be an scalar-electromagnetic structure. The fundamental form Ψ_Θ of Θ is defined as the following one-form on $\mathcal{M}$ taking values in $\mathrm{End}(\mathcal{S})$:

$$\Psi_\Theta \stackrel{\text{def.}}{=} \mathcal{D}\mathcal{J} \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{S}))\,.$$

Definition

Let Φ be a scalar-electromagnetic structure. Geometric bosonic supergravity on M is defined by the following set of partial differential equations:

- The Einstein equations:

$$\mathrm{Ric}^g - \frac{g}{2} R^g = \frac{g}{2}\,\mathrm{Tr}_g(\mathcal{G}^\varphi) - \mathcal{G}^\varphi + 2\mathcal{V} \oslash \mathcal{V}\,,$$

- The scalar equations:

$$\nabla \mathrm{d}\varphi = \frac{1}{2}(*\mathcal{V}, \Psi^\varphi \mathcal{V})\,.$$

- The electromagnetic (or Maxwell) equations:

$$\mathrm{d}_{D^\varphi} \mathcal{V} = 0\,,$$

with variables $(g, \varphi, \mathcal{V}) \in \mathrm{Conf}(M, \Phi)$.

Recall that:

$$\mathrm{d}_{D^\varphi} : \Omega^2(M, \mathcal{S}^\varphi) \to \Omega^3(M, \mathcal{S}^\varphi)\,, \quad \mathrm{d}\varphi \in \Omega^1(M, T\mathcal{M}^\varphi)\,.$$

Local structure of a scalar electromagnetic bundle

Let Φ be a scalar-electromagnetic structure on M and let $\mathcal{E} = (e_1, \dots, e_n, f^1, \dots, f^n)$ be a local (flat) symplectic frame on $(\mathcal{S}, \omega, \mathcal{J})$. $\mathcal{E}$ and $\mathcal{J}$ determine a unique map $\mathcal{N} = \theta + i\gamma \colon V \to \mathbb{SH}_n$. Write:

$$\mathcal{V} = F^i \, e_i + G_i \, f^i \,, F^i, G_i \in \Omega^2(U) \,.$$

Then, $\mathcal{V}$ is positively polarized if and only if, locally:

$$G^+ = \mathcal{N} F^+ \,,$$

and the Maxwell equations $\mathrm{d}_{D^\varphi} \mathcal{V}$ are satisfied if and only if, locally:

$$\mathrm{d} F^i = 0 \,, \qquad \mathrm{d} G_i = 0 \,,$$

This recovers the equations and structures of the local gauge sector of bosonic supergravity.

Global U-duality group

We identify the global U-duality group of geometric bosonic supergravity associated to the scalar-electromagnetic structure $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$ with the following subgroup of the automorphisms of $\mathcal{S}$:

$$\mathrm{U}(\Phi) \stackrel{\text{def.}}{=} \{u \in \mathrm{Aut}(\mathcal{S}) \mid \omega^u = \omega\,,\ \mathcal{D}^u = \mathcal{D}\,,\ f_u \in \mathrm{Iso}(\mathcal{M}, \mathcal{G})\,,\ \mathcal{J}^u = \mathcal{J}\}\,,$$

which fits in the short exact sequence:

$$1 \to \mathrm{Aut}_b(\Theta) \to \mathrm{U}(\Phi) \to \mathrm{Iso}_\Theta(\mathcal{M}, \mathcal{G}) \to 1\,,$$

whence $\mathrm{U}(\Phi)$ is finite-dimensional and it can markedly differ from the U-duality group of the local theory computed in the literature!

Theorem

$\mathrm{U}(\Phi)$ *preserves* $\mathrm{Sol}^g(M, \Phi)$.

This theorem sets on firm grounds the ideology behind the U-duality group being *solution-generating* in supergravity.

Simple example: from global to local.

Consider geometric bosonic supergravity associated to a symplectically trivial and *holonomy trivial* scalar-electromagnetic structure on M. Then:

$$\mathrm{U}(\Phi) \simeq \{(F, \mathcal{T}) \in \mathrm{Iso}(\mathcal{M}, \mathcal{G}) \times \mathrm{Sp}(2n, \mathbb{R}) \mid \mathcal{N} \circ F = \mathcal{T} \cdot \mathcal{N}\} \ ,$$

where we have globally identified the taming $\mathcal{J}$ with a smooth map $\mathcal{N} \colon \mathcal{M} \to \mathbb{SH}_n$. Hence we recover the local expression that we flashed before regarding the global symmetries of the local Lagrangian.

Dirac quantization I.

- Fix a bundle Λ of full lattices inside $\mathcal{S}$, preserved by the parallel transport of the flat connection D and ω integer-valued when restricted to Λ. Pairs (Δ, Λ): *integral duality structures*.

- Integral duality structures correspond to local systems T_Δ valued in the groupoid of $\mathrm{Symp}_{\mathbb{Z}}$ of *integral symplectic spaces*.

- A taming $\mathcal{J}$ of $(\mathcal{S}, \omega)$ defines an *integral electromagnetic structure* (Θ, Λ).

- We can encode the data of an integral electromagnetic structure using a (smooth) bundle $\mathcal{X}(\Theta)$ of polarized Abelian varieties, endowed with a flat Ehresmann connection whose transport preserves the symplectic structure of the torus fibers but need not preserve their complex structure.

Dirac quantization II.

The Dirac quantization condition requires that $[\mathcal{V}] \in H^2_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)$ be *integral*: it belongs to the image of the universal coefficient map of the second cohomology of M with coefficients in the $\mathrm{Symp}_{\mathbb{Z}}$-valued local system T_{Δ^φ}.

Let $H(M, \Delta^\varphi)$ be the twisted singular cohomology of M with coefficients in the local system T_{Δ^φ}. Since $\mathcal{S}^\varphi = \Lambda^\varphi \otimes_{\mathbb{Z}} \mathbb{R}$, the coefficient sequence gives a map:

$$j_* : H(M, \Delta^\varphi) \to H_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)\,,$$

whose image is a lattice $H_{\Lambda^\varphi}(M, \Delta^\varphi) \subset H(M, \mathcal{S}^\varphi)$.

Definition

An electromagnetic field $\mathcal{V} \in \Omega^2(M, \mathcal{S}^\varphi)$ is called *Dirac quantized* if its D^φ-twisted cohomology class $[\mathcal{V}] \in H^2_{d_{D^\varphi}}(M, \mathcal{S}^\varphi)$ belongs to $H^2_{\Lambda^\varphi}(M, \Delta^\varphi)$:

$$[\mathcal{V}] \in H^2_{\Lambda^\varphi}(M, \Delta^\varphi) = j_*(H^2(M, \Delta^\varphi))\,.$$

Dirac quantization III.

Since $H^2_{\Lambda^\varphi}(M, \mathcal{S}^\varphi)$ does not capture $\mathrm{Tor}\, H^2(M, \Lambda^\varphi)$, the integrality condition is weaker than the expected condition. In this case, the natural model for Dirac-quantized fields is provided by a twisted version of differential cohomology giving finally the mathematical description of the gauge fields of ungauged four-dimensional supergravity as connections in the appropriate fiber bundle: work in progress!

Further directions and open problems

1. Characterize the U-duality group of a general geometric supergravity theory and compute it in explicit important cases.
2. Implement Dirac quantization on a general geometric supergravity, developing the appropriate model in differential cohomology.
3. Explore more general submersions as possible models of locally non-geometric U-folds.
4. Supersymmetrize geometric bosonic theories $\Rightarrow$ geometric supergravities: $\mathcal{N} = 1$ case already developed in collaboration with V. Cortés.
5. Study the higher-dimensional origin of geometric bosonic supergravity by using the appropriate notion of reduction.
6. Explore a geometric model for Freudenthal duality in terms of the taming $\mathcal{J}$.
7. Classify all supersymmetric solutions of geometric supergravity on a geodesically complete and simply connected four-manifold.
8. Study geometric supergravity on globally hyperbolic Lorentzian four-manifolds and compute the associated flow equations.
9. Study the moduli space of supersymmetric solutions.

Thanks!

Definition

The *twisted exterior pairing* $(\ ,\ ) := (\ ,\ )_{g,\boldsymbol{Q}^s}$ is the unique pseudo-Euclidean scalar product on the twisted exterior bundle $\wedge_M(\boldsymbol{\mathcal{S}}^s)$ which satisfies:

$$(\rho_1 \otimes \xi_1, \rho_2 \otimes \xi_2)_{g,\boldsymbol{Q}^s} = (\rho_1, \rho_2)_g \boldsymbol{Q}^s(\xi_1, \xi_2)\,,$$

for any $\rho_1, \rho_2 \in \Omega(M)$ and any $\xi_1, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$. Here $\boldsymbol{Q}(\xi_1, \xi_2) = \boldsymbol{\omega}(\mathbf{J}\xi_1, \xi_2)$ and the superscript denotes pull-back by s.

For any vector bundle W, we trivially extend the twisted exterior pairing to a W-valued pairing, which for simplicity we denote by the same symbol, between the bundles $W \otimes (\wedge_M(\mathcal{S}^\varphi))$ and $\wedge_M(\mathcal{S}^\varphi)$. Thus:

$$(e \otimes \eta_1, \eta_2)_{g,\boldsymbol{Q}^s} \stackrel{\text{def.}}{=} e \otimes (\eta_1, \eta_2)_{g,\boldsymbol{Q}^s}\,, \quad \forall\, e \in \Omega^0(M, W)\,, \quad \forall\, \eta_1, \eta_2 \in \wedge_M(\boldsymbol{\mathcal{S}}^s)\,.$$

The *inner g-contraction of two-tensors* is the bundle morphism $\oslash_g : (\otimes^2 T^* M)^{\otimes 2} \to \otimes^2 T^* M$ uniquely determined by the condition:

$$(\alpha_1 \otimes \alpha_2) \oslash_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_3)_g \alpha_1 \otimes \alpha_4 \,, \quad \forall\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Omega^1(M) \,.$$

We define the *inner g-contraction of two-forms* to be the restriction of $\oslash_g$ to $\wedge^2 T^* M \otimes \wedge^2 T^* M \stackrel{\oslash_g}{\to} \otimes^2 T^* M$.

Definition

We define the *twisted inner contraction* of $\boldsymbol{\mathcal{S}}^s$-valued two-forms to be the unique morphism of vector bundles $\oslash \colon \wedge^2_M (\boldsymbol{\mathcal{S}}^s) \times_M \wedge^2_M (\boldsymbol{\mathcal{S}}^s) \to \otimes^2 (T^* M)$ which satisfies:

$$(\rho_1 \otimes \xi_1) \oslash (\rho_2 \otimes \xi_2) = \boldsymbol{Q}^s(\xi_1, \xi_2) \rho_1 \oslash_g \rho_2 \,,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $\xi_2, \xi_2 \in \Omega^0(M, \boldsymbol{\mathcal{S}}^s)$.

- The twisted inner contraction is necessary in order to globally write the equations of motion of supergravity!

Let $(\pi : (E, h) \to (M, g), \bar{\Phi})$ be an *integrable* bundle of scalar data of type $(\mathcal{M}, \mathcal{G}, \Phi)$. We consider a special trivializing atlas of π defined by the geodesically convex open sets $(U_\alpha)_{\alpha \in I}$ (which cover M). Since π is integrable, the appropriate trivializing maps give isometries $q_\alpha : (E_\alpha, h_\alpha) \xrightarrow{\sim} (U_\alpha \times \mathcal{M}, g_\alpha \times \mathcal{G})$. For any pair of indices $\alpha, \beta \in I$ such that $U_{\alpha\beta} \stackrel{\text{def.}}{=} U_\alpha \cap U_\beta$ is non-empty, the composition:

$$q_{\alpha\beta} \stackrel{\text{def.}}{=} q_\beta \circ q_\alpha^{-1} : U_{\alpha\beta} \times \mathcal{M} \to U_{\alpha\beta} \times \mathcal{M}\,,$$

has the form $q_{\alpha\beta}(m, p) = (m, \mathbf{g}_{\alpha\beta}(p))$, where:

$$\mathbf{g}_{\alpha\beta} \in \mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi)\,.$$

Setting $\mathbf{g}_{\alpha\beta} = \mathrm{id}_\mathcal{M}$ for $U_{\alpha\beta} = \emptyset$, the collection $(\mathbf{g}_{\alpha\beta})_{\alpha,\beta \in I}$ satisfies the cocycle condition:

$$\mathbf{g}_{\beta\delta}\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\delta}\,, \quad \forall\, \alpha, \beta, \delta \in I\,. \tag{1}$$

For any section $s \in \Gamma(\pi)$, the restriction $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ corresponds through q_α to the graph $\mathrm{graph}(\varphi_\alpha) \in \Gamma(\pi^0_\alpha)$ of a uniquely-defined smooth map $\varphi_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$:

$$s_\alpha = q_\alpha^{-1} \circ \mathrm{graph}(\varphi_\alpha) \ \text{ i.e. } \ q_\alpha(s_\alpha(m)) = (m, \varphi_\alpha(m))\,, \quad \forall m \in U_\alpha\,. \tag{2}$$

Composing the first relation from the left with $p^0_\alpha : E^0_\alpha \simeq U_\alpha \times \mathcal{M} \to \mathcal{M}$ gives:

$$\varphi^\alpha = \hat{q}_\alpha \circ s_\alpha \quad .$$

Which in turn implies:

$$\varphi^\beta(m) = \mathbf{g}_{\alpha\beta}\varphi^\alpha(m) \;\; \forall m \in U_{\alpha\beta} \quad , \tag{3}$$

where juxtaposition in the right hand side denotes the tautological action of the group $\mathrm{Isom}(\mathcal{M}, \mathcal{G}, \Phi)$ on $\mathcal{M}$. Conversely, any family of smooth maps $\{\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})\}_{\alpha \in I}$ satisfying (3) defines a smooth section $s \in \Gamma(\pi)$ whose restrictions to U_α are given by (2). The equation of motion for s is equivalent with the condition that each φ^α satisfies the equation of motion of the ordinary sigma model defined by the scalar data $(\mathcal{M}, \mathcal{G}, \Phi)$ on the space-time (U_α, g_α):

$$\tau^v(h, s) = -(\mathrm{grad}\bar{\Phi})^s \Leftrightarrow \tau(g_\alpha, \varphi^\alpha) = -(\mathrm{grad}\Phi)^{\varphi^\alpha} \;\; \forall \alpha \in I \quad . \tag{4}$$

Thus global solutions s of the equations of motion are *glued* from local solutions $\varphi^\alpha \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$ of the equations of motion of the ordinary sigma model using the $\mathrm{Iso}(\mathcal{M}, \mathcal{G}, \Phi)$-valued constant transition functions $\mathbf{g}_{\alpha\beta}$ which satisfy the cocycle condition (1).

Let $\boldsymbol{\mathcal{D}} = (\pi, \bar{\Phi}, \Xi)$ be an *integrable* scalar-electromagnetic bundle and let $(U_\alpha, \mathbf{q}_\alpha)_{\alpha \in I}$ be a special trivializing atlas. Let $s \in \Gamma(\pi)$ and and $\mathcal{V} \in \Omega^2(M, \boldsymbol{\mathcal{S}}^s)$. Let $s_\alpha \stackrel{\text{def.}}{=} s|_{U_\alpha}$ and $\mathcal{V}_\alpha \stackrel{\text{def.}}{=} \mathcal{V}|_{U_\alpha} \in \Omega^2(U_\alpha, \boldsymbol{\mathcal{S}}^s)$. The diffeomorphisms $q_\alpha : E_\alpha \to E^0_\alpha = U_\alpha \times \mathcal{M}$ and their linearizations $\mathbf{q}_\alpha : \boldsymbol{\mathcal{S}}_\alpha \to \boldsymbol{\mathcal{S}}^0_\alpha \stackrel{\text{def.}}{=} \mathcal{S}^{p^{\mathbf{0}}_\alpha}$ identify s_α with maps $\varphi_\alpha \stackrel{\text{def.}}{=} p^0_\alpha \circ q_\alpha(s_\alpha) \in \mathcal{C}^\infty(U_\alpha, \mathcal{M})$. The isomorphism of electromagnetic structures $\mathbf{q}_\alpha : (\boldsymbol{\mathcal{S}}_\alpha, \boldsymbol{\omega}_\alpha, \mathbf{D}_\alpha, \mathbf{J}_\alpha) \xrightarrow{\sim} (\boldsymbol{\mathcal{S}}^0_\alpha, \boldsymbol{\omega}^0_\alpha, \mathbf{D}^0_\alpha, \mathbf{J}^0_\alpha)$ pulls back to an isomorphism:

$$\mathbf{q}^s_\alpha : (\boldsymbol{\mathcal{S}}^s_\alpha, \boldsymbol{\omega}^s_\alpha, \mathbf{D}^s_\alpha, \mathbf{J}^s_\alpha) \xrightarrow{\sim} (\mathcal{S}^{\varphi^\alpha}, \omega^{\varphi^\alpha}, D^{\varphi^\alpha}, J^{\varphi^\alpha})\,,$$

This isomorphism identifies $\boldsymbol{\mathcal{V}}_\alpha$ with a $\mathcal{S}^{\varphi^\alpha}$-valued two-form defined on U_α through:

$$\mathcal{V}^\alpha \stackrel{\text{def.}}{=} \mathbf{q}^s_\alpha \circ \boldsymbol{\mathcal{V}}_\alpha \in \Omega^2(U_\alpha, \mathcal{S}^{\varphi^\alpha})\,, \tag{5}$$

and we have:

$$\begin{aligned} \boldsymbol{\mathcal{V}}_\alpha \in \Omega^{2+,s}_{g_\alpha, \boldsymbol{\mathcal{S}}^s_\alpha, \mathbf{J}^s_\alpha}(U_\alpha) \quad &\text{iff} \quad \mathcal{V}^\alpha \in \Omega^{2+,\varphi^\alpha}_{g_\alpha, \mathcal{S}^{\varphi^\alpha}, J^{\varphi^\alpha}}(U_\alpha) \\ \mathrm{d}_{\boldsymbol{\mathcal{D}}^s} \boldsymbol{\mathcal{V}}_\alpha = 0 \quad &\text{iff} \quad \mathrm{d}_{D^{\varphi^\alpha}} \mathcal{V}^\alpha = 0\,. \end{aligned}$$

$$(s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}^g_{\boldsymbol{\mathcal{D}}}(M) \;\text{ iff }\; (\varphi^\alpha, \mathcal{V}^\alpha) \in \mathrm{Sol}^{g_\alpha}_{\mathcal{D}}(U_\alpha) \;\; \forall \alpha \in I \;\;.$$

Relations (5) imply the gluing conditions:

$$\mathcal{V}^{\beta}|_{U_{\alpha\beta}} = \mathbf{f}^{s}_{\alpha\beta}\mathcal{V}^{\alpha}|_{U_{\alpha\beta}} \quad , \tag{6}$$

which accompany the gluing conditions (3) for φ^{α}.
Conversely, any pair of families $(\varphi_{\alpha})_{\alpha\in I}$ and $(\mathcal{V}^{\alpha})_{\alpha\in I}$ of solutions of the equations of motion of the ordinary scalar sigma model with Abelian gauge fields of type $\mathcal{D} = (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, \omega, D, J)$ defined on the open sets $U_{\alpha} \subset M$ of a special open cover for (M, g) which satisfy conditions (3) and (6) corresponds to a global solution $(s, \mathcal{V})$ of the equations of motion of the section sigma model coupled to Abelian gauge fields defined by the scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$. Hence such global solutions are obtained by gluing local solutions of the ordinary sigma model coupled to Abelian gauge fields using scalar-electromagnetic symmetries of the latter.

Definition

Let $\boldsymbol{\mathcal{D}}$ be a scalar-electromagnetic structure. A classical locally-geometric U-fold of type $\boldsymbol{\mathcal{D}}$ is a global solution $(g, s, \boldsymbol{\mathcal{V}}) \in \mathrm{Sol}_{\boldsymbol{\mathcal{D}}}(M)$ of the equations of motion a GESM theory associated to an scalar-electromagnetic bundle $\boldsymbol{\mathcal{D}}$ of type $\mathcal{D}$ with integrable associated Lorentzian submersion.

Consider an integral symplectic space $(\mathcal{S}_0, \omega_0, \Lambda_0)$ defined on M. One can define a version of differential cohomology valued in such objects; an explicit construction can be given for example using Cheeger-Simons characters valued in the (affine) symplectic torus $\mathcal{S}_0/\Lambda_0$ defined by $(\mathcal{S}_0, \omega_0, \Lambda_0)$. This can be promoted to a twisted version $\check{H}^k(N, T)$, whereby the coefficient object is replaced by a local system $T : \Pi_1(N) \to \mathrm{Symp}_0^{\times}$. Given an integral electromagnetic structure $\boldsymbol{\Xi}$ whose underlying integral duality structure $\boldsymbol{\Delta}$ corresponds to the local system T, the correct model for the set of semiclassical Abelian gauge fields is provided by a certain subset of $\check{H}^2(N, T)$; the curvature $\mathcal{V} = \mathrm{curv}(\alpha)$ of any element α of this subset is a polarized closed 2-form which satisfies the integrality condition. When (M, g) is globally hyperbolic, the initial value problem for the twisted electromagnetic field is well-posed (because the twisted d'Alembert operator is normally hyperbolic) and restriction to a Cauchy hypersurface allows one to describe explicitly the space of semiclassical fields. In that case, one can to quantize the electromagnetic theory in a manner which reproduces this model for the space of semiclassical fields. It turns out that elements $\alpha \in \check{H}^2(N, T)$ classify affine symplectic T^{2n}-bundles[1] with connection. This allows one to represent $\mathcal{V}$ as the curvature of a connection on such a bundle, which gives the geometric interpretation of semiclassical Abelian gauge fields.

[1] Non-principal fiber bundles with fiber a symplectic $2n$-torus and whose structure group reduces to the affine symplectic group of such a torus.

The U-duality group of a GESM theory

Let $\mathcal{D} \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega, J)$.

Definition

The *scalar-electromagnetic symmetry group* of $\mathcal{D}$ is the subgroup $\mathrm{Aut}(\mathcal{D})$ of $\mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega)$ defined through:

$$\mathrm{Aut}(\mathcal{D}) \stackrel{\text{def.}}{=} \{f \in \mathrm{Aut}^{\mathrm{ub}}(\mathcal{S}, D, J, \omega) \mid f_0 \in \mathrm{Aut}(\mathcal{M}, \mathcal{G}, \Phi)\} .$$

An element of this group is called a *scalar-electromagnetic symmetry*.

Theorem

For all $f \in \mathrm{Aut}(\mathcal{D})$, we have:

$$f \diamond \mathrm{Sol}^{g}_{\mathcal{D}}(M) = \mathrm{Sol}^{g}_{\mathcal{D}}(M) \quad .$$

Thus $\mathrm{Aut}(\mathcal{D})$ consists of symmetries of the equations of motion of a GESM theory.

It is natural then to define the U-duality group of a GESM theory as its scalar-electromagnetic symmetry group!

Theorem

Let $\Sigma \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi)$, $\Delta \stackrel{\text{def.}}{=} (\mathcal{S}, D, \omega)$, $\Xi \stackrel{\text{def.}}{=} (\mathcal{S}, D, J, \omega)$ *and* $\mathcal{D}_0 \stackrel{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Phi, \mathcal{S}, D, \omega)$. *We have short exact sequences:*

$$
\begin{aligned}
&1 \to \mathrm{Aut}(\Delta) \hookrightarrow \mathrm{Aut}(\mathcal{D}_0) \longrightarrow \mathrm{Aut}^{\Delta}(\Sigma) \to 1 \\
&1 \to \mathrm{Aut}(\Xi) \hookrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow \mathrm{Aut}^{\Xi}(\Sigma) \to 1 \quad ,
\end{aligned}
$$

where $\mathrm{Aut}(\Delta)$ *and* $\mathrm{Aut}(\Xi)$ *are the groups of based symmetries of* Δ *and* Ξ. *The groups appearing in the right hand side consist of those automorphisms of the scalar structure* Σ *which respectively admit lifts to scalar-electromagnetic dualities of* $\mathcal{D}_0 = (\Sigma, \Delta)$ *and scalar-electromagnetic symmetries of* $\mathcal{D} = (\Sigma, \Xi)$. *Fixing a point* $p \in \mathcal{M}$, *we can identify* $\mathrm{Aut}(\Delta)$ *with the commutant of* Hol_D^p *inside the group* $\mathrm{Aut}(\mathcal{S}_p, \omega_p) \simeq \mathrm{Sp}(2n, \mathbb{R})$. *In particular, the exact sequences above show that* $\mathrm{Aut}(\mathcal{D}_0)$ *and* $\mathrm{Aut}(\mathcal{D})$ *are Lie groups.*

Definition

The *fundamental matrix* of (V, J) with respect to $\mathcal{E}$ and $\mathcal{W}$ is the matrix $\Pi := \Pi_{\mathcal{E}}^{\mathcal{W}} \in \mathrm{Mat}(n, 2n, \mathbb{C})$ defined through:

$$e_\alpha = \sum_{k=1}^{n} \Pi_{k,\alpha} w_k = \sum_{k=1}^{n} [(\mathrm{Re}\Pi_{k,\alpha}) w_k + (\mathrm{Im}\Pi_{k,\alpha}) J w_k] \quad (\alpha = 1 \dots 2n) \quad . \tag{7}$$

Siegel upper half space $\mathbb{SH}_n$:

$$\mathbb{SH}_n \stackrel{\mathrm{def.}}{=} \{\tau \in \mathrm{Mat}_s(n, \mathbb{C}) | \mathrm{Im}\tau \text{ is strictly positive definite}\} \quad ,$$

When endowed with the natural complex structure induced from the affine space $\mathrm{Mat}_s(n, \mathbb{C})$, the space $\mathbb{SH}_n$ is a complex manifold of complex dimension $\frac{n(n+1)}{2}$ which is biholomorphic with the simply-connected bounded complex domain:

$$\{Z \in \mathrm{Mat}_s(n, \mathbb{C}) | I - \bar{Z} Z^T > 0\}$$

A symplectic basis of (V, ω) is a basis of the form $\mathcal{E} = (e_1 \dots e_n, f_1 \dots f_n)$, whose elements satisfy the conditions:

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad , \quad \omega(e_i, f_j) = -\omega(f_i, e_j) = \delta_{ij}$$

Theorem

Let $\mathcal{E} = (e_1 \dots e_n, f_1 \dots f_n)$ *be a symplectic basis of* (V, ω)*. Then the vectors* $\mathcal{E}_2 \stackrel{\text{def.}}{=} (f_1, \dots, f_n)$ *form a basis of the complex vector space* (V, J) *over* $\mathbb{C}$ *and the fundamental matrix of* (V, J) *with respect to* $\mathcal{E}$ *and* $\mathcal{E}_2$ *has the form:*

$$\Pi_{\mathcal{E}}^{\mathcal{E}_2} = [\boldsymbol{\tau}^{\mathcal{E}}, I_n]^T ,$$

where $\boldsymbol{\tau}^{\mathcal{E}} \in \mathrm{Mat}(n, n, \mathbb{C})$ *is a complex-valued square matrix of size* n*. Moreover,* J *is a taming of* (V, ω) *iff* $\boldsymbol{\tau}_{\mathcal{E}}$ *belongs to* $\mathbb{SH}_n$*. In this case, the matrix of the Hermitian form* h *with respect to the basis* $\mathcal{E}_2$ *equals* $(\mathrm{Im}\boldsymbol{\tau}^{\mathcal{E}})^{-1}$*.*

Theorem

Let $\boldsymbol{\tau} = \boldsymbol{\tau}_R + i\boldsymbol{\tau}_I \in \mathbb{SH}_n$ *be a point of the Siegel upper half space, where* $\boldsymbol{\tau}_R, \boldsymbol{\tau}_I \in \mathrm{Mat}_s(n, n, \mathbb{R})$ *are the real and imaginary parts of* $\boldsymbol{\tau}$*:*

$$\boldsymbol{\tau}_R \stackrel{\text{def.}}{=} \mathrm{Re}\boldsymbol{\tau} \quad , \quad \boldsymbol{\tau}_I \stackrel{\text{def.}}{=} \mathrm{Im}\boldsymbol{\tau} \quad .$$

Then the matrix of the taming $J(\boldsymbol{\tau}) \stackrel{\text{def.}}{=} \boldsymbol{\tau}_{\mathcal{E}}^{-1}(\boldsymbol{\tau})$ *in the symplectic basis* $\mathcal{E}$ *is given by:*

$$\hat{J}(\boldsymbol{\tau}) = \begin{bmatrix} \boldsymbol{\tau}_I^{-1}\boldsymbol{\tau}_R & \boldsymbol{\tau}_I^{-1} \\ -\boldsymbol{\tau}_I - \boldsymbol{\tau}_R\boldsymbol{\tau}_I^{-1}\boldsymbol{\tau}_R & -\boldsymbol{\tau}_R\boldsymbol{\tau}_I^{-1} \end{bmatrix} \quad .$$