

# Hessian symmetries of multifield cosmological models

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## Definition

An  $n$ -dimensional **scalar triple** is an ordered system  $(\mathcal{M}, \mathcal{G}, V)$ , where:

- $(\mathcal{M}, \mathcal{G})$  is a connected Riemannian  $n$ -manifold (called **scalar manifold**)
- $V \in C^\infty(\mathcal{M}, \mathbb{R})$  is a smooth function (called **scalar potential**).

## Assumptions

$(\mathcal{M}, \mathcal{G})$  is oriented and complete.

Each  $n$ -dimensional scalar triple  $(\mathcal{M}, \mathcal{G}, V)$  defines a **cosmological model** on  $\mathbb{R}^4$ , with Lagrange density:

$$\mathcal{L}_{\mathcal{M}, \mathcal{G}, V} = \left[ \frac{R(g)}{2} - \frac{1}{2} \text{Tr}_g \varphi^*(\mathcal{G}) - V(\varphi) \right] \text{vol}_g, \quad (1)$$

where the space-time metric  $g$  has 'mostly plus' signature. Take  $g$  to describe a simply-connected and spatially flat FLRW universe:

$$ds_g^2 := -dt^2 + a^2(t) d\vec{x}^2 \quad (x^0 = t, \quad \vec{x} = (x^1, x^2, x^3), \quad a(t) > 0 \quad \forall t) \quad (2)$$

and  $\varphi$  depends only on the cosmological time  $t$ :

$$\varphi = \varphi(t). \quad (3)$$

Substituting (2) and (3) in (1) and ignoring the integration over  $\vec{x}$  gives the **minisuperspace action**:

$$S_{\mathcal{M},g,v}[a,\varphi] = \int_{-\infty}^{\infty} dt L_{\mathcal{M},g,v}(a(t),\varphi(t),\dot{\varphi}(t)) \quad ,$$

where the **minisuperspace Lagrangian** is:

$$L_{\mathcal{M},g,v}(a,\varphi,\dot{\varphi}) \stackrel{\text{def.}}{=} -3a\dot{a}^2 + a^3 \left[ \frac{1}{2} \|\dot{\varphi}\|_g^2 - V(\varphi) \right] = a^3 \left[ -3H^2 + \frac{1}{2} \|\dot{\varphi}\|_g^2 - V(\varphi) \right] \quad .$$

Here  $\dot{\cdot} \stackrel{\text{def.}}{=} \frac{d}{dt}$  and  $H \stackrel{\text{def.}}{=} \frac{\dot{a}}{a}$  is the **Hubble parameter**. This Lagrangian describes a classical system with  $n+1$  degrees of freedom and configuration space  $\mathcal{N} \stackrel{\text{def.}}{=} \mathbb{R}_{>0} \times \mathcal{M}$ . The Euler-Lagrange equations are equivalent with:

$$\begin{aligned} 3H^2 + 2\dot{H} + \frac{1}{2} \|\dot{\varphi}\|_g^2 - V(\varphi) &= 0 \\ (\nabla_t + 3H)\dot{\varphi} + (\text{grad}_g V)(\varphi) &= 0 \quad . \end{aligned}$$

We must also impose the (non-holonomic) **Friedmann constraint**:

$$\frac{1}{2} \|\dot{\varphi}\|^2 + V \circ \varphi = 3H^2 \quad ,$$

which amounts to the zero energy condition.

## Proposition

When supplemented with the Friedmann constraint, the Euler-Lagrange equations of  $L_{\mathcal{M},g,V}$  are equivalent with the *cosmological equations*:

$$\begin{aligned}\nabla_t \dot{\varphi} + 3H\dot{\varphi} + (\text{grad}_g V) \circ \varphi &= 0 \\ \dot{H} + 3H^2 - V \circ \varphi &= 0 \\ \dot{H} + \frac{1}{2} \|\dot{\varphi}\|_g^2 &= 0 .\end{aligned}$$

## Remark

One can eliminate  $H$  algebraically from the cosmological equations as:

$$H(t) = \frac{1}{\sqrt{6}} \epsilon(t) \left[ \|\dot{\varphi}(t)\|_g^2 + 2V(\varphi(t)) \right]^{1/2}, \text{ where } \epsilon(t) \stackrel{\text{def.}}{=} \text{sign} H(t) ,$$

thereby obtaining the *reduced cosmological equation*:

$$\nabla_t \dot{\varphi}(t) + \sqrt{\frac{3}{2}} \epsilon(t) \left[ \|\dot{\varphi}(t)\|_g^2 + 2V(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\text{grad}_g V)(\varphi(t)) = 0 ,$$

which defines a (dissipative) *geometric dynamical system* on  $T\mathcal{M}$ .

We have a natural decomposition  $T\mathcal{N} = T_1\mathcal{N} \oplus T_2\mathcal{N}$ , where:

$$T_1\mathcal{N} \stackrel{\text{def.}}{=} p_1^*(T\mathbb{R}_{>0}) \ , \quad T_2\mathcal{N} \stackrel{\text{def.}}{=} p_2^*(T\mathcal{M}) \quad (p_1 : T\mathcal{N} \rightarrow \mathbb{R}_{>0}, \ p_2 : T\mathcal{N} \rightarrow \mathcal{M}) \ .$$

## Theorem

A vector field  $X \in \mathcal{X}(\mathcal{N})$  is a time-independent Noether symmetry iff:

$$X(a, \varphi) = X_{\Lambda, Y}(a, \varphi) = \frac{\Lambda(\varphi)}{\sqrt{a}} \partial_a + Y(\varphi) - \frac{4}{a^{3/2}} (\text{grad}_g \Lambda)(\varphi) \ ,$$

where  $\Lambda \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$  satisfies the  **$\Lambda$ -system**:

$$\text{Hess}_g(\Lambda) = \frac{3}{8} g \Lambda \quad , \quad \langle dV, d\Lambda \rangle_g = \frac{3}{4} V \Lambda$$

and  $Y \in \mathcal{X}(\mathcal{M})$  satisfies the  **$Y$ -system**:

$$\mathcal{K}_g(Y) = 0 \quad , \quad Y(V) = 0 \ .$$

The two systems can also be written as follows:

$$\begin{aligned} \left( \partial_i \partial_j - \Gamma_{ij}^k \partial_k \right) \Lambda &= \frac{3}{8} g_{ij} \Lambda \quad , \quad \nabla_i Y_j + \nabla_j Y_i = 0 \\ g^{ij} \partial_i V \partial_j \Lambda &= \frac{3}{4} V \Lambda \quad , \quad Y^i \partial_i V = 0 \ . \end{aligned}$$

## Definition

A time-independent Noether symmetry  $X = X_{\Lambda, Y}$  is called:

- **visible** if  $\Lambda = 0$ .
- **Hessian** if  $Y = 0$ .

The scalar triple  $(\mathcal{M}, \mathcal{G}, V)$  and cosmological model are called **visibly-symmetric** or **Hessian** if they admit visible or Hessian symmetries, respectively.

Let  $N_H(\mathcal{M}, \mathcal{G}, V)$ ,  $N_V(\mathcal{M}, \mathcal{G}, V)$  and  $N(\mathcal{M}, \mathcal{G}, V)$  denote the linear spaces of Hessian, visible and time-independent Noether symmetries.

## Proposition

*There exists a linear isomorphism*  
 $N(\mathcal{M}, \mathcal{G}, V) \simeq_{\mathbb{R}} N_H(\mathcal{M}, \mathcal{G}, V) \oplus N_V(\mathcal{M}, \mathcal{G}, V).$

## Remark

*Existence of a Hessian symmetry simplifies various cosmological problems. For example, it gives the following formula for the number of e-folds:*

$$\left[ \frac{a(t)}{a(t_0)} \right]^{3/2} \Lambda(\varphi(t)) - \Lambda_0 = \left( \frac{3}{2} H_0 \Lambda_0 + (d_{\varphi_0} \Lambda)(\dot{\varphi}_0) \right) (t - t_0) \quad .$$

## Rescaling the metric. The Hesse and $\Lambda$ - $V$ -equations

Let  $\beta = \sqrt{3/8}$  and  $G = \beta^2 \mathcal{G}$ .

### Definition

The **rescaled scalar manifold** is the Riemannian manifold  $(\mathcal{M}, G)$ .

The  $\Lambda$ -system of  $(\mathcal{M}, \mathcal{G}, V)$  is equivalent with:

- $\text{Hess}_G(\Lambda) = \Lambda G$  (the **Hesse equation** of the rescaled scalar manifold)
- $\langle dV, d\Lambda \rangle_G = 2V\Lambda$  (the  **$\Lambda$ - $V$  equation** of the rescaled scalar manifold)

### Definition

Let  $(\mathcal{M}, G)$  be a complete Riemannian manifold. A **Hesse function** of  $(\mathcal{M}, G)$  is a smooth solution of the **Hesse equation** of  $(\mathcal{M}, G)$ :

$$\text{Hess}_G(\Lambda) = \Lambda G \quad .$$

Let  $\mathcal{S}(\mathcal{M}, G)$  be the linear space of Hesse functions of  $(\mathcal{M}, G)$ . The **Hesse index** of  $(\mathcal{M}, G)$  is defined through:

$$\mathfrak{h}(\mathcal{M}, \mathcal{G}) \stackrel{\text{def.}}{=} \dim \mathcal{S}(\mathcal{M}, G) \quad .$$

The complete Riemannian manifold  $(\mathcal{M}, G)$  is called **globally of Hesse type** if it admits non-trivial Hesse functions, i.e. if  $\mathfrak{h}(\mathcal{M}, G) > 0$ .

## Proposition

We have  $\mathfrak{h}(\mathcal{M}, G) \leq n + 1$ . We say that  $(\mathcal{M}, G)$  is *globally maximally Hesse* if equality is attained.

## Definition

The *Hesse pairing* of  $(\mathcal{M}, G)$  is the symmetric  $\mathbb{R}$ -bilinear map  $(\ , \ )_G : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M})$  defined through:

$$(f_1, f_2)_G \stackrel{\text{def.}}{=} f_1 f_2 - \langle df_1, df_2 \rangle_G = f_1 f_2 - \langle \text{grad}_G f_1, \text{grad}_G f_2 \rangle_G, \quad \forall f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}).$$

## Proposition

Let  $\Lambda_1, \Lambda_2 \in \mathcal{S}(\mathcal{M}, G)$  be two Hesse functions on  $(\mathcal{M}, G)$ . Then the Hesse pairing  $(\Lambda_1, \Lambda_2)_G$  is constant on  $\mathcal{M}$ .

Hence the Hesse pairing restricts to a symmetric  $\mathbb{R}$ -bilinear map:

$$(\ , \ )_G : \mathcal{S}(\mathcal{M}, G) \times \mathcal{S}(\mathcal{M}, G) \rightarrow \mathbb{R}, \quad (\Lambda_1, \Lambda_2)_G \stackrel{\text{def.}}{=} (\Lambda_1, \Lambda_2)_G$$

on the vector space  $\mathcal{S}(\mathcal{M}, G)$ .

## Theorem

*For any non-trivial Hesse function  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$ , any smooth solution of the  $\Lambda$ - $V$ -equation of  $(\mathcal{M}, G)$  takes the following form:*

$$V = \Omega \|d\Lambda\|_G^2 = \Omega \left[ \Lambda^2 - (\Lambda, \Lambda)_G \right] , \quad (4)$$

*where  $\Omega \in \mathcal{C}^\infty(\mathcal{M} \setminus \text{Crit}(\Lambda))$  is constant along the gradient flow of  $\Lambda$ :*

$$\langle d\Omega, d\Lambda \rangle_G = 0 . \quad (5)$$

## Definition

The cosmological model defined by the scalar triple  $(\mathcal{M}, \mathcal{G}, V)$  is called *weakly Hessian* if the rescaled scalar manifold  $(\mathcal{M}, G)$  is globally of Hesse type. It is called *Hessian* if it admits non-trivial Hessian symmetries.

## Proposition

*The cosmological model defined by the scalar triple  $(\mathcal{M}, \mathcal{G}, V)$  is Hessian iff it is weakly Hessian and the scalar potential  $V$  has the form (4), with  $\Omega$  a solution of (5).*

## Theorem

Let  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  be a Hesse function of  $(\mathcal{M}, G)$  and suppose that  $V$  satisfies the  $\Lambda$ - $V$  equation with respect to  $\Lambda$ . Then the minisuperspace Lagrangian takes the following form in natural local coordinates  $(a_0, y, v)$  on the configuration space  $\mathcal{N}$ :

$$L(y, \dot{y}, v, \dot{v}) = -3(\Lambda, \Lambda)_G \dot{v}^2 - 3a_0 \dot{a}_0^2 + a_0^3 L_\lambda(y, \dot{y}) \quad , \quad (6)$$

where the reduced Lagrangian  $L_\lambda$  is given by:

$$L_\lambda(y, \dot{y}) \stackrel{\text{def.}}{=} \frac{1}{2\beta^2} \|\dot{y}\|_g^2 - V_\lambda(y) \quad . \quad (7)$$

with  $V_\lambda \stackrel{\text{def.}}{=} V|_{\mathcal{M}_\Lambda(\lambda)}$  the restriction of  $V$  to the level set  $\mathcal{M}_\Lambda(\lambda)$  and  $g$  the metric induced by  $G$  on this level set. We have:

$$V_\lambda(y) = \left[ \lambda^2 - (\Lambda, \Lambda)_G \right] \Omega(y) \quad , \quad (8)$$

with  $\Omega$  a smooth arbitrary function defined on  $\mathcal{M}_\Lambda(\lambda)$ .

## Definition

The **Hesse norm** of a Hesse function  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  is the non-negative number  $\kappa_\Lambda \stackrel{\text{def.}}{=} \sqrt{|(\Lambda, \Lambda)_G|}$ , while its **type indicator** is the sign factor  $\epsilon_\Lambda \stackrel{\text{def.}}{=} \text{sign}(\Lambda, \Lambda)_G$ . A non-trivial Hesse function  $\Lambda$  is called **timelike**, **spacelike** or **lightlike** when  $\epsilon_\Lambda$  equals  $+1$ ,  $-1$  or  $0$ .

## Proposition

Let  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  be a non-trivial Hesse function. Then:

1. If  $\Lambda$  is timelike, then its vanishing locus  $Z(\Lambda)$  is empty and  $\Lambda$  has constant sign (denoted  $\eta_\Lambda$ ) on  $\mathcal{M}$ . Moreover,  $\Lambda$  has exactly one critical point, with critical value  $\eta_\Lambda \kappa_\Lambda$ , which is a global minimum or maximum according to whether  $\eta_\Lambda = +1$  or  $-1$ .
2. If  $\Lambda$  is spacelike, then  $\text{Crit}(\Lambda) = \emptyset$ . Moreover, the vanishing locus of  $\Lambda$  is the following non-singular hypersurface in  $\mathcal{M}$ :

$$Z(\Lambda) = \{m \in \mathcal{M} \mid \|d_m \Lambda\|_G = \kappa_\Lambda\} \quad .$$

3. If  $\Lambda$  is lightlike, then  $Z(\Lambda) = \text{Crit}(\Lambda) = \emptyset$  and  $\Lambda$  has constant sign on  $\mathcal{M}$ , which we denote by  $\eta_\Lambda$ .

## Definition

A timelike or lightlike non-trivial Hesse function  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  is called **future (resp. past) pointing** when  $\eta_\Lambda = +1$  (resp.  $-1$ ).

Let  $\mathcal{M}_\Lambda(\lambda) \subset \mathcal{M}$  denote the  $\lambda$ -level set of  $\Lambda$ .

## Definition

Let  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  be a non-trivial Hesse function of  $\mathcal{M}$ . The **characteristic set** of  $\Lambda$  is the following closed subset of  $\mathcal{M}$ :

$$Q_\Lambda \stackrel{\text{def.}}{=} \begin{cases} \text{Crit}(\Lambda) , & \text{if } \Lambda \text{ is timelike} \\ Z(\Lambda) , & \text{if } \Lambda \text{ is spacelike} \\ \mathcal{M}_{|\Lambda|}(1) , & \text{if } \Lambda \text{ is lightlike} \end{cases} .$$

The **characteristic constant** of  $\Lambda$  is defined through:

$$C_\Lambda \stackrel{\text{def.}}{=} \begin{cases} \kappa_\Lambda , & \text{if } \epsilon = +1 \\ 0 , & \text{if } \epsilon = -1 \\ 1 , & \text{if } \epsilon = 0 \end{cases} .$$

Setting  $\mathcal{U}_\Lambda \stackrel{\text{def.}}{=} \mathcal{M} \setminus \text{Crit}(\Lambda)$ , we have:

$$Q_\Lambda = \{m \in \mathcal{U}_\Lambda \mid |\Lambda(m)| = C_\Lambda\} .$$

## Definition

The **characteristic sign function**  $\Theta_\Lambda : \mathcal{M} \rightarrow \mathbb{R}$  of a non-trivial Hesse function  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  is defined through:

$$\Theta_\Lambda(m) \stackrel{\text{def.}}{=} \begin{cases} 1, & \text{if } \epsilon_\Lambda = +1 \\ \text{sign}(\Lambda(m)), & \text{if } \epsilon_\Lambda = -1 \\ \text{sign}(|\Lambda(m)| - 1), & \text{if } \epsilon_\Lambda = 0 \end{cases}.$$

The  **$\Lambda$ -distance function**  $d_\Lambda : \mathcal{M} \rightarrow \mathbb{R}$  is defined through:

$$d_\Lambda(m) \stackrel{\text{def.}}{=} \Theta_\Lambda(m) \text{dist}_G(m, Q_\Lambda) .$$

## Theorem

*Let  $\Lambda \in \mathcal{S}(\mathcal{M}, G)$  be a non-trivial Hesse function. Then the following relation holds for all  $m \in \mathcal{M}$ :*

$$\Lambda(m) = \begin{cases} \text{sign}(\Lambda) \kappa_\Lambda \cosh d_\Lambda(m), & \text{if } \epsilon_\Lambda = +1 \\ \kappa_\Lambda \sinh d_\Lambda(m), & \text{if } \epsilon_\Lambda = -1 \\ \text{sign}(\Lambda) e^{d_\Lambda(m)}, & \text{if } \epsilon_\Lambda = 0 \end{cases}.$$

The proof uses properties of solutions to the eikonal equation of  $(\mathcal{M}, \mathcal{G})$ .

The isometry group  $\text{Iso}(\mathcal{M}, G)$  acts on  $\mathcal{C}^\infty(\mathcal{M})$  through:

$$\psi^*(f) \stackrel{\text{def.}}{=} f \circ \psi^{-1} , \quad \forall \psi \in \text{Iso}(\mathcal{M}, G) , \quad \forall f \in \mathcal{C}^\infty(\mathcal{M}) .$$

This action preserves the subspace of Hesse functions and hence it corestricts to the [Hesse representation](#):

$$\mathcal{H}_G(\psi)(f) \stackrel{\text{def.}}{=} \psi^*(\Lambda) = \Lambda \circ \psi^{-1} , \quad \forall \Lambda \in \mathcal{S}(\mathcal{M}, G) .$$

## Proposition

*The Hesse representation is  $(\ , \ )_G$ -orthogonal, i.e. any representation operator  $\mathcal{H}_G(\psi)$  preserves the Hesse pairing:*

$$(\mathcal{H}_G(\psi)\Lambda_1, \mathcal{H}_G(\psi)\Lambda_2)_G = (\Lambda_1, \Lambda_2)_G , \quad \forall \Lambda_1, \Lambda_2 \in \mathcal{S}(\mathcal{M}, G) .$$

The Hesse equation of  $(\mathcal{M}, G)$  is overdetermined elliptic, being equivalent with a system of Hessian equations which containing the Poisson and Monge-Ampere equations. Local solutions are called *local Hesse functions*; their germs form the *Hesse sheaf*  $\mathcal{H}_G$ , which has rank  $\leq n + 1$ . We say that  $(\mathcal{M}, G)$  is **locally of Hesse type** if  $\text{rk} \mathcal{H}_G > 0$  and **locally maximally Hesse** if  $\text{rk} \mathcal{H}_G = n + 1$ . Notice that  $(\mathcal{M}, G)$  is globally of Hesse type iff  $H^0(\mathcal{H}_G) \neq 0$ .

## Theorem

*A Riemannian manifold is locally maximally Hesse iff it is hyperbolic.*

## Theorem

*Let  $(\mathcal{M}, G)$  be a complete Riemannian manifold. The following are equivalent:*

- *$(\mathcal{M}, G)$  is locally maximally Hesse and globally of Hesse type.*
- *$(\mathcal{M}, G)$  is isometric with the Poincaré  $n$ -ball or with an elementary hyperbolic space form.*

*Moreover,  $(\mathcal{M}, G)$  is globally maximally Hesse iff it is isometric with the Poincaré  $n$ -ball.*

## Definition

An  $n$ -dimensional **elementary hyperbolic space form** is a complete hyperbolic  $n$ -manifold uniformized by a non-trivial torsion-free elementary discrete subgroup  $\Gamma \subset \mathrm{SO}_o(1, n)$ .

Any torsion-free elementary discrete subgroups of  $\mathrm{SO}_o(1, n)$  is:

- **hyperbolic**, if it conjugates to a subgroup of the canonical squeeze group  $\mathcal{T}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1, n)}(E_n) \simeq \mathrm{SO}(1, n-1)$ . In this case,  $\Gamma$  is a hyperbolic cyclic group.
- **parabolic** if it conjugates to a subgroup of the canonical shear group  $\mathcal{P}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_o(1, n)}(E_0 + E_n) \simeq \mathrm{ISO}(n)$ . In this case,  $\Gamma$  is a free Abelian group of rank at most  $n-1$ .

## Definition

An elementary hyperbolic space form is said to be of hyperbolic or parabolic **type** according to the type of its uniformizing group.

**Example.** The two-dimensional elementary hyperbolic space forms are:

- The hyperbolic annuli  $\mathbb{A}(R)$  (hyperbolic type,  $\Gamma \simeq \mathbb{Z}$ ,  $h(\mathbb{A}(R)) = 1$ )
- The hyperbolic punctured disk  $\mathbb{D}^*$  (parabolic type,  $\Gamma \simeq \mathbb{Z}$ ,  $h(\mathbb{D}^*) = 1$ .)

## Theorem

*Any Hesse surface  $(\Sigma, G)$  is locally maximally Hesse and hence isometric with one of the following:*

- *The hyperbolic disk  $\mathbb{D} := \mathbb{D}^2$  (Hesse index 3)*
- *The hyperbolic punctured disk  $\mathbb{D}^*$  (Hesse index 1)*
- *A hyperbolic annulus  $\mathbb{A}(R)$  (Hesse index 1)*

## Theorem

*The two-field cosmological model defined by the two-dimensional scalar triple  $(\Sigma, \mathcal{G}, V)$  is weakly-Hessian iff its rescaled scalar manifold  $(\mathcal{M}, G)$  is isometric with the hyperbolic disk, the hyperbolic punctured disk or a hyperbolic annulus  $\mathbb{A}(R)$  of arbitrary modulus  $\mu = 2 \log R > 0$ . In this case, the model is Hessian iff the scalar potential  $V$  has the form  $V = \Omega[\Lambda^2 - (\Lambda, \Lambda)_G]$ , where  $\Lambda \in \mathcal{S}(\Sigma, G)$  is a non-trivial Hesse function and  $\Omega$  is a smooth function which is constant along the gradient flow of  $\Lambda$ .*

The space  $\mathcal{S}(\Sigma, G)$  can be determined in each of the three cases. This leads to an explicit classification of all Hessian two-field cosmological models.

## Theorem

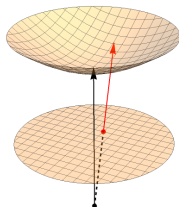
*The space of Hesse functions of the hyperbolic disk is 3-dimensional. A basis of this space is given by the classical Weierstrass coordinates  $\Xi^0, \Xi^1, \Xi^2$ , where  $\Xi = (\Xi^0, \Xi^1, \Xi^2) : D \rightarrow \mathbb{R}^3$  is the Weierstrass map:*

$$\Xi(u) \stackrel{\text{def.}}{=} \left( \frac{1 + |u|^2}{1 - |u|^2}, \frac{2\text{Re}u}{1 - |u|^2}, \frac{2\text{Im}u}{1 - |u|^2} \right) . \quad (9)$$

*Hence the general Hesse function on  $\mathbb{D}^2$  has the form:*

$$\Lambda_B(u) = (B, \Xi(u)) \quad \forall u \in D , \quad (10)$$

*where,  $B = (B^0, B^1, B^2) \in \mathbb{R}^3$  is any non-vanishing 3-vector and  $(\ , \ )$  is the Minkowski pairing of signature  $(1, 2)$  on  $\mathbb{R}^3$ .*



## Theorem

The following statements hold for the weakly-Hessian two-field cosmological model with scalar manifold  $\mathbb{D}_\beta = (\mathbb{D}, \mathcal{G})$ , where  $\mathcal{G}$  is the complete metric of constant negative curvature  $K = -\frac{3}{8}$ :

1. When  $B$  is timelike, the model admits the Hessian symmetry generated by (10) iff:

$$V_B(u) = \omega(n_B(u)) \left[ \frac{\Lambda_B(u)^2}{(B, B)} - 1 \right], \quad (11)$$

where  $\omega \in C^\infty(S^1)$  and  $n_B(u) = \frac{(B, B)\Xi(u) - (B, \Xi(u))B}{\sqrt{(B, B)(B, \Xi(u))^2 - (B, B)^2}} = \frac{(B, B)\Xi(u) - B\Lambda_B(u)}{\sqrt{(B, B)\Lambda_B(u)^2 - (B, B)^2}}$ . The model also admits visible symmetries iff  $\omega$  is constant, in which case visible symmetries form an elliptic subgroup of  $\text{PSU}(1, 1)$  conjugate to the canonical rotation subgroup  $\mathcal{R} \simeq \text{U}(1)$ .

2. When  $B$  is spacelike, the model admits the Hessian symmetry generated by (10) iff its scalar potential  $V$  has the form:

$$V_B(u) = \omega(n_B(u)) \left[ \frac{\Lambda_B(u)^2}{|(B, B)|} + 1 \right], \quad (12)$$

where  $\omega \in C^\infty(\mathbb{R})$  and  $n_B(u) = \frac{|(B, B)|\Xi(u) + (B, \Xi(u))B}{\sqrt{(B, B)^2 + |(B, B)|(B, \Xi(u))^2}} = \frac{|(B, B)|\Xi(u) + \Lambda_B(u)B}{\sqrt{(B, B)^2 + |(B, B)|\Lambda_B(u)^2}}$ . The model also admits visible symmetries iff  $\omega$  is constant, in which case visible symmetries form a hyperbolic subgroup of  $\text{PSU}(1, 1)$  conjugate to the canonical squeeze subgroup  $\mathcal{T} \simeq (\mathbb{R}, +)$ .

3. When  $B$  is lightlike, the model admits the Hessian symmetry generated by (10) iff:

$$V_B(u) = \omega(B_0 n_B(u)) \frac{\Lambda_B(u)^2}{B_0^2}, \quad (13)$$

where  $\omega \in C^\infty(\mathbb{R})$  and  $n_B(u) = \frac{2(B, \Xi(u))\Xi(u) - B}{2(B, \Xi(u))^2} = \frac{2\Lambda_B(u)\Xi(u) - B}{2\Lambda_B(u)^2}$ . The model also admits visible symmetries iff  $\omega$  is constant, in which case visible symmetries form a parabolic subgroup of  $\text{PSU}(1, 1)$  conjugate to the canonical shear subgroup  $\mathcal{P} \simeq (\mathbb{R}, +)$ .

## Corollary

The explicit forms of the scalar potential are:

- ① For timelike  $B$  (i.e.  $(B, B) \stackrel{\text{def.}}{=} B_0^2 - B_1^2 - B_2^2 > 0$ ):

$$V_B(x, y) = \omega(\tilde{\theta}(x, y)) \frac{P}{(B, B)(1 - \rho^2)^2}, \quad (14)$$

where:

$$P = (B_1^2 + B_2^2)(1 + \rho^4) + 2(B_1^2 - B_2^2)(x^2 - y^2) + 4B_0^2\rho^2 + 4B_0(1 + \rho^2)(B_1x + B_2y) + 8B_1B_2xy \quad (15)$$

and:

$$\tilde{\theta}(x, y) = \arg \left[ \frac{\text{sign}(B_0)(B_1 - iB_2)(x + iy) + (|B_0| - \sqrt{(B, B)})}{(|B_0| - \sqrt{(B, B)})(x + iy) + \text{sign}(B_0)(B_1 + iB_2)} \right]. \quad (16)$$

- ② For spacelike  $B$  (i.e.  $B_0^2 - B_1^2 - B_2^2 < 0$ ):

$$V_B(x, y) = \omega(\tilde{\tau}(x, y)) \frac{P}{|(B, B)|(1 - \rho^2)^2}, \quad (17)$$

where  $P$  is given by (15) and:

$$\tilde{\tau}(x, y) = \text{arcsinh} \left[ \frac{2\sqrt{|(B, B)|}(B_1y - B_2x)}{\sqrt{(B_1^2 + B_2^2)P}} \right]. \quad (18)$$

- ③ For lightlike  $B$  (i.e.  $B_0^2 - B_1^2 - B_2^2 = 0$ ):

$$V_B(x, y) = \omega(\tilde{\tau}(x, y)) \frac{(B_0(1 + \rho^2) + 2B_1x + 2B_2y)^2}{B_0^2(1 - \rho^2)^2}, \quad (19)$$

where:

$$\tilde{\tau}(x, y) = \frac{2(B_1y - B_2x)}{B_0(1 + \rho^2) + 2B_1x + 2B_2y}. \quad (20)$$

- One can describe explicitly the space of Hesse functions on the Poincaré  $\mathbb{D}^n$  of any dimension  $n$  and all scalar potentials on  $\mathbb{D}^n$  which solve the  $\Lambda$ - $V$  equation for any non-trivial Hesse function  $\Lambda$ . This leads to an explicit classification of Hessian  $n$ -field models on the Poincaré ball.
- Deeper analysis allow one to characterize all Hessian  $n$ -field models.

The stabilizer of a nontrivial  $(n + 1)$ -vector  $X \in \mathbb{R}^{n+1} \setminus \{0\}$  in the fundamental representation of  $SO_o(1, n)$  conjugates to one of the **canonical subgroups**:

- $X$  timelike:  $\text{Stab}_{SO_o(1,n)}(X) \sim \mathcal{R}_n \stackrel{\text{def.}}{=} \text{Stab}_{SO_o(1,n)}(E_0) \simeq SO(n)$  (elliptic, rotation).
- $X$  spacelike:  $\text{Stab}_{SO_o(1,n)}(X) \sim \mathcal{T}_n \stackrel{\text{def.}}{=} \text{Stab}_{SO_o(1,n)}(E_n) \simeq SO(1, n - 1)$  (hyperbolic, squeeze)
- $X$  lightlike:  $\text{Stab}_{SO_o(1,n)}(X) \sim \mathcal{P}_n \stackrel{\text{def.}}{=} \text{Stab}_{SO_o(1,n)}(E_0 + E_n) \simeq ISO(n)$  (parabolic, shear).

## Definition

A discrete subgroup  $\Gamma$  of  $SO_o(1, n)$  is called **elementary** if its action on the closure of the Poincaré ball fixes at least one point in  $\overline{D}^n$ .

An elementary discrete subgroup  $\Gamma \subset SO_o(1, n)$  is:

- **elliptic** if it conjugates to a subgroup of  $\mathcal{R}_n$ . In this case,  $\Gamma$  is finite.
- **hyperbolic** if it conjugates to a subgroup of  $\mathcal{T}_n$ . In this case,  $\Gamma$  contains a hyperbolic cyclic group of finite index, to which it reduces iff  $\Gamma$  is torsion-free.
- **parabolic** if it conjugates to a subgroup of  $\mathcal{P}_n$ . In this case,  $\Gamma$  is a finite extension of a free Abelian group of rank at most  $n - 1$ .