Tensor models and the Sachdev-Ye-Kitaev model

ADRIAN TANASĂ

Univ. Bordeaux & IUF Paris & DFT, IFIN-HH Măgurele

in collaboration with:
V. Bonzom, S. Dartois, E. Fusy, R. Gurău,
L. Lionni, M. Raasakka, V. Rivasseau, D. Youmans

Bucuresti, 6 septembrie 2019

Plan

- Introduction: Quantum Field Theory (QFT) in a nutshell
- $0-$dimensional scalar QFT
- Matrix models and 2D quantum gravity
- The large N and double scaling limit of matrix models
- Tensor models
- Large N and double scaling limit of tensor models
- Sachdev-Ye-Kitaev (SYK) model
- Conclusion and perspectives

Introduction - QFT

QFT - quantum description of particles and their interactions
$\hookrightarrow$ the Standard Model of elementary particles

the QFT formalism also applies to:

- statistical physics (statistical QFT)
- condensed matter physics
- *etc.*

C. Itzykson and J.-B. Zuber, "QFT" (1980)
J. Zinn-Justin, "QFT and Critical Phenomenon" (1989)
M. Peskin and Scroeder, "An Introduction to QFT"

The Φ^4 model

the action (functional in the field)

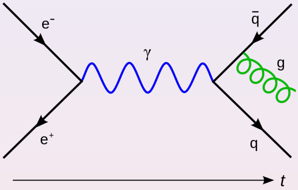
$$S[\Phi(x)] = \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2} \sum_{\mu=1}^{4} \left(\frac{\partial}{\partial x_\mu} \Phi(x) \right)^2 + \frac{1}{2} m^2 \Phi^2(x) + \frac{\lambda}{4!} \Phi^4(x) \right]$$

m - the mass of the particle,
λ - the coupling constant

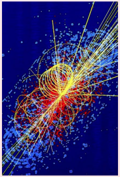
- quadratic part - propagation - edges
- non-quadratic part - interaction potential $V[\Phi(x)] = \frac{\lambda}{4!}\Phi^4(x)$ - vertices

$\Longrightarrow$ (Feynman) graphs of valence 4 - perturbation theory (in λ)

Elementary particle interactions



discovery of the Higgs boson (at CERN's LHC)



0−dimensional scalar QFT

the scalar field ϕ is not a function of space-time (there is no space-time)!
$\phi \in \mathbb{R}$

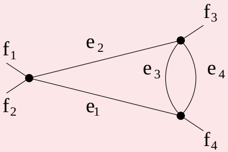
partition function:

$$Z = \int_{\mathbb{R}} d\phi \, e^{-\frac{1}{2}\phi^2 + \frac{\lambda}{4!}\phi^4}.$$

perturbation theory - formal series in λ

$\rightarrow$ (abstract) Feynman graphs and Feynman integrals

example:



One (still) needs to evaluate integrals of type

$$\frac{\lambda^n}{n} \int d\phi \, e^{-\phi^2/2} \left(\frac{\phi^4}{4!} \right)^n .$$

one can (still) use standard QFT techniques:

$$\int d\phi \, e^{-\phi^2/2} \phi^{2k} = \frac{\partial^{2k}}{\partial J^{2k}} \int d\phi \, e^{-\phi^2/2 + J\phi} |_{J=0} = \frac{\partial^{2k}}{\partial J^{2k}} e^{J^2/2} |_{J=0}.$$

J - the source

$0-$dimensional QFT - interesting "laboratories" for testing theoretical physics tools

V. Rivasseau and Z. Wang, *J. Math. Phys.* (2010), arXiv:1003.1037

From scalars to matrices

Random matrices in Physics

- nuclear physics - spectra of heavy atoms

  Wigner, *Annals Math.* (1955)
- QCD with a large number of colors

  't Hooft, *Nucl. Phys.* **B** (1974)
- 2D quantum gravity
- string theory
- *etc.*

- spacing between perched birds (parked cars)

Quantum gravity (QG)

General Relativity - Einstein-Hilbert action

"Spacetime tells matter how to move; matter tells spacetime how to curve." J. Wheeler

GR and QFT effects are of the same order of magnitude at the Planck scale

$$\ell_{\mathrm{Planck}} = \sqrt{Kh/c^3} \propto 10^{-35}\mathrm{m}$$

Quantum gravity (QG)

Quantum theory of gravity - Grail of modern theoretical physics

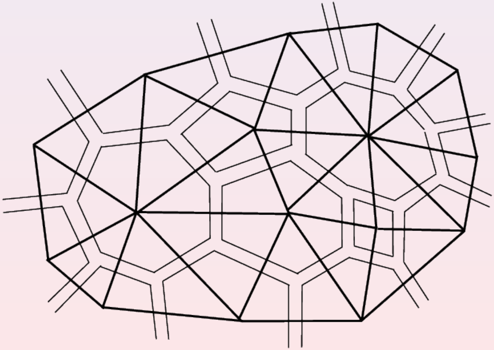
several approaches:

- string theory
- loop quantum gravity
- causal dynamical triangulations
- noncommutative geometry (see yesterady's talks)
- matrix models - 2D QG
- *etc.*

- holography

Matrix models

The Feynman graphs arising from the perturbative expansion of the partition function of matrix models are **dual graphs to triangulated 2D surfaces**.



- The model defines a certain statistical ensemble over discrete geometries - connection with 2D quantum gravity.
- An important technique is the **large N expansion**, which is controlled by the **genus** of the ribbon Feynman graphs; the leading order contribution to the partition function is given by **planar graphs** (pave the *2D sphere* S^2).
- By simultaneous scaling of N and the coupling constant, the **double-scaling limit** allowed to define a continuum limit, where all topologies contribute, connected to 2D gravity.

Matrix models

Ph. Di Francesco *et. al., Phys. Rept.* (1995), hep-th/9306153, L. Alvarez-Gaumé, Lausanne lectures (1990), V. Kazakov, *Proc. Cargèse workshop* (1990), E. Brézin, *Proc. Jerusalem winter school* (1991), F. David, *Lectures Les Houches Summer School* (1992) *etc.*

M - $N \times N$ hermitian matrix

the partition function:

$$Z = \int dM e^{-\frac{1}{2}\mathrm{Tr} M^2 + \frac{\lambda}{\sqrt{N}}\mathrm{Tr} M^3}.$$

$dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re} M_{ij}\,\mathrm{Im} M_{ij}$ (the measure)
diagrammatic expansion - Feynman ribbon graphs

generates *random triangulations*

sum over random triangulations - discretized analogue of the integral over all possible geometries

$0-$dimensional string theory (a pure theory of surfaces with no coupling to matter on the string worldsheet)

Large N expansion of matrix models

the matrix amplitude can be combinatorially computed - in terms of number of vertices (V), edges and faces (F) of the graph
change of variables: $M \to M\sqrt{N}$ (easy to count powers of N)

$$\mathcal{A} = \lambda^V N^{-\frac{1}{2}V+F} = \lambda^V N^{2-2g}$$

(since $E = \frac{3}{2}V$)
the partition function (and the free energy) supports a $1/N$ expansion:

$$Z = N^2 Z_0(\lambda) + Z_1(\lambda) + \ldots = \sum_g N^{2-2g} Z_g(\lambda)$$

Z_g gives the contribution from surfaces of genus g

large N limit, only planar surfaces survive - dominant graphs *(triangulations of the sphere $\mathcal{S}^2$)*

V. A. Kazakov, *Phys. Lett.* **B** ('85), F. David, *Nucl. Phys.* **B** ('85), E. Brézin *et al., Commun. Math. Phys.* ('78)

The double scaling limit for matrix models

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
the leading singular piece of Z_g:

$$Z_g(\lambda) \propto f_g(\lambda_c - \lambda)^{(2-\gamma_{\rm str})\chi/2} \text{ with } \gamma_{\rm str} = -\frac{1}{2} \text{ (pure gravity)}$$

contributions from higher genera ($\chi < 0$) are enhanced as $\lambda \to \lambda_c$

$\kappa^{-1} := N(\lambda - \lambda_c)^{(2-\gamma_{\rm str})/2}$

the partition function expansion:

$$Z = \sum_g \kappa^{2g-2} f_g$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ
coherent contribution from all genus surfaces

M. Douglas and S. Shenker, *Nucl. Phys.* **B** ('90), E. Brézin and V. Kazakov, *Phys. Lett.* **B**, *Nucl. Phys.* **B** ('90),

D. Gross and M. Migdal, *Phys. Rev. Lett.*, *Nucl. Phys.* **B** ('90)

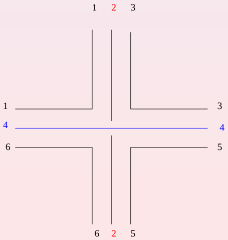
From matrices to tensors

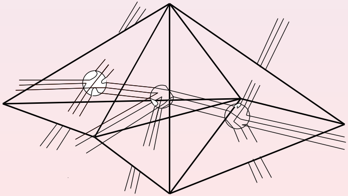
Tensor models were introduced already in the 90's - replicate in dimensions higher than 2 the success of **random matrix models**:
J. Ambjorn *et. al.*, *Mod. Phys. Lett.* ('91),

N. Sasakura, *Mod. Phys. Lett.* ('91), M. Gross *Nucl. Phys. Proc. Suppl.* ('92)

natural generalization of matrix models

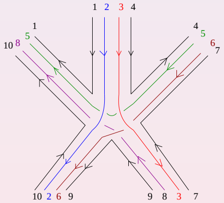
matrix $\rightarrow$ rank three tensor





4−dimensional models

4D vertex (dual image of a 4−simplex (5−cell)):



QFT-inspired simplification - the colored tensor model

highly non-trivial combinatorics and topology
$\rightarrow$ a QFT simplification of these models - colored tensor models

(R. Gurău, Commun. Math. Phys. (2011), arXiv:0907.2582)

a quadruplet of complex fields $\left(\phi^0, \phi^1, \phi^2, \phi^3\right)$;

$$
\begin{aligned}
S[\{\phi^i\}] &= S_0[\{\phi^i\}] + S_{int}[\{\phi^i\}] \\
S_0[\{\phi^i\}] &= \frac{1}{2}\sum_{p=0}^{3}\sum_{i,j,k,=1}^{N} \overline{\phi^p_{ijk}}\phi^p_{ijk} \\
S_{int}[\{\phi^i\}] &= \frac{\lambda}{4}\sum_{i,j,k,i',j',k'=1}^{N} \phi^0_{ijk}\phi^1_{i'j'k}\phi^2_{i'jk'}\phi^3_{k'j'i} + \text{ c. c.},
\end{aligned} \tag{1}
$$

the indices $0, \ldots, 3$ - color indices.
extra property: the faces of the Feynman graphs of this model have always exactly two (alternating) colors.

R. Gurau, *"Random Tensors"*, Oxford Univ. Press (2016)

Various QFT developments for colored(-like) tensor models

- large N expansion

  R. Gurau, Annales Henri Poincare (2011), [arXiv:1011.2726 [gr-qc]]
  R. Gurau and V. Rivasseau, Europhys. Lett. (2011),

- double-scale limit

  G. Schaeffer and R. Gurău, arXiv:1307.5279, S. Dartois *et. al.*, JHEP (2013),

- renormalizability

  J. Ben Geloun and V. Rivasseau, Commun. Math. Phys. (2013), arXiv:1111.4997 [hep-th].
  D. O. Samary and F. Vignes-Tourneret, Commun. Math. Phys. (2014), arXiv:1211.2618 [hep-th],

- Hopf algebraic reformulation of tensor renormalizability

  M. Raasakka and A. Tanasă, *Sém. Loth. Comb.* (2014)

- Dyson-Schwinger equation study

  T. Krajewski, arXiv:1211.1244 [math-ph], D. O. Samary *et. al.*, arXiv:1411.7231 [hep-th]

- loop vertex expansion of the perturbative series

  T. Krajewski & R. Gurau, arXiv:1409.1705, *Annaless IHP D - Combinatorics, Phys. & their Interactions*

  (2015)

Question:

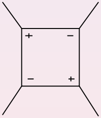
How much of these large N scaling properties of colored tensor models generalize to larger family of tensor graphs?

A QFT simplification of tensor models

$\rightarrow$ a QFT simplification of these models - multi-orientable models

A. Tanasă, J. Phys. **A** (2012), arXiv:1109.0694

proposal made within the GFT framework



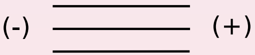
edge going from a $+$ to a $-$ corner

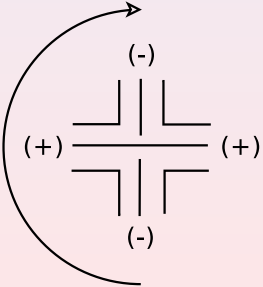
non-commutative QFT inspired idea

F. Vignes-Tourneret, Annales Henri Poincaré (2007),
R. Gurău and A. Tanasă, Annales Henri Poincaré (2008),
V. Rivasseau and A. Tanasă, Commun. Math. Phys. (2008)

The action: the propagator and the vertex

$$S[\phi] = S_0[\phi] + S_{int}[\phi], \tag{2}$$
$$S_0[\phi] = \frac{1}{2}\sum_{i,j,k=1}^{N} \bar\phi_{ijk}\phi_{ijk}, \quad S_{int}[\phi] = \frac{\lambda}{4}\sum_{i,j,k,i',j',k'=1}^{N} \phi_{ijk}\bar\phi_{kj'i'}\phi_{k'ji'}\bar\phi_{k'j'i}.$$





Multi-orientable tensor Feynman graphs

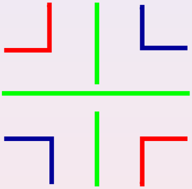
The set of Feynman graphs generated by the colored action (1) is a strict subset of the set of Feynman graphs generated by the MO action (2).

Some tools - jacket ribbon subgraphs

In the colored case the $1/N$ expansion relies on the notion of **jacket ribbon subgraphs**, which are associated to the cycle of colors up to orientation.

Generalization of the notion of jackets for MO graphs

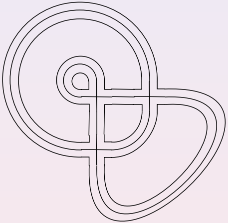
three pairs of opposite corner strands

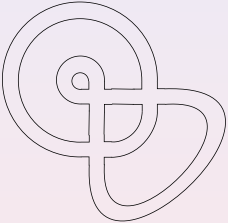


A **jacket of an MO graph** is the graph made by excluding one type of strands throughout the graph. The *outer* jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands (the green ones); jacket $\bar{a}$ excludes all strands of type a (the red ones) and jacket $\bar{b}$ excludes all strands of type b (the blue ones).
$\hookrightarrow$ such a splitting is always possible

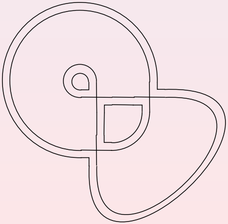
Example of jacket subgraphs

A MO graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$

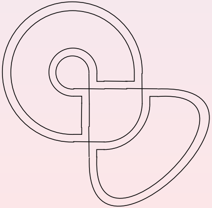




$\bar{c}$



$\bar{a}$

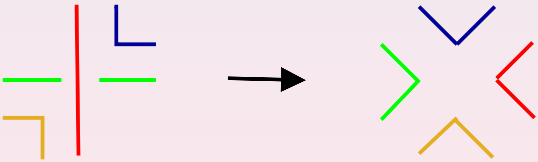


$\bar{b}$

Is such a jacket subgraph a ribbon subgraph?

Any jacket of a MO graph is a (connected vacuum) ribbon graph (with uniform degree 4 at each vertex).

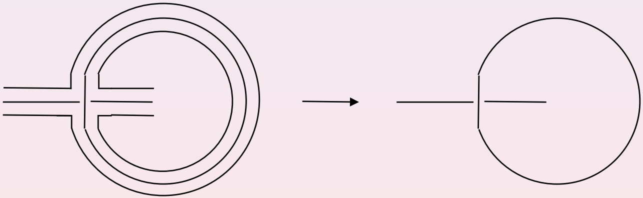
untwisting vertex procedure:



may introduce twists on the edges

this does not hold for any, non-MO, tensor graph

Example: Deleting a pair of opposite corner strands in this tadpole does not lead to a ribbon graph.



Euler characteristic & degree of MO tensor graphs

ribbon graphs can represent orientable or non-orientable surfaces.

Euler characteristic formula:

$$\chi(\mathcal{J}) = V_{\mathcal{J}} - E_{\mathcal{J}} + F_{\mathcal{J}} = 2 - k_{\mathcal{J}},$$

$k_{\mathcal{J}}$ is the non-orientable genus,
$V_{\mathcal{J}}$ is the number of vertices,
$E_{\mathcal{J}}$ the number of edges and
$F_{\mathcal{J}}$ the number of faces.

If the surface is orientable, k is even and equal to twice the orientable genus g

Given an MO graph $\mathcal{G}$, its degree $\varpi(\mathcal{G})$ is defined by

$$\varpi(\mathcal{G}) = \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2} = 3 + \frac{3}{2} V_{\mathcal{G}} - F_{\mathcal{G}},$$

the sum over $\mathcal{J}$ running over the three jackets of $\mathcal{G}$.

Large N expansion of the MO tensor model

Feynman amplitude calculation - each tensor graph face contributes with a factor N, N being the size of the tensor

$\Longrightarrow$ one needs to count the number of faces of the tensor graph

this can be achieved using the graph's jackets (ribbon subgraphs)

The Feynman amplitude of a general MO tensor graph $\mathcal{G}$ writes:

$$A(\mathcal{G}) = \lambda^{V_{\mathcal{G}}} N^{3-\varpi(\mathcal{G})}.$$

The free energy writes as a formal series in $1/N$:

$$F(\lambda, N) = \sum_{\varpi \in \mathbb{N}/2} C^{[\varpi]}(\lambda) N^{3-\varpi},$$

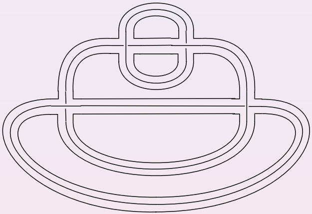
$$C^{[\varpi]}(\lambda) = \sum_{\mathcal{G}, \varpi(\mathcal{G})=\varpi} \frac{1}{s(\mathcal{G})} \lambda^{v_{\mathcal{G}}}.$$

Dominant graphs

dominant graphs:

$$\varpi = 0.$$

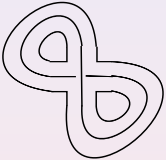
An example of a dominant tensor graph

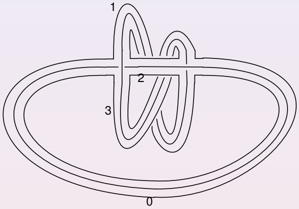


- outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 0$.
- the two remaining jackets also have vanishing genus $g_2 = g_3 = 0$ (can be directly computed using Euler's characteristic formula)

$\Longrightarrow$ vanishing degree ($\varpi = 0$) $\Leftrightarrow$ dominant graph

Two examples of non-dominant tensor graphs





double tadpole:
$\varpi = 0 + \frac{1}{2} + 0 = \frac{1}{2}$.

"twisted sunshine" (bipartite 4−edge colorable graph):
Its outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 1$.
The two remaining jackets are isomorphic and have non-orientable genus $k_2 = k_3 = 1$.
$\Longrightarrow \varpi = 2$.

Dominant graphs of the tensor large N expansion

The colored and MO models admit a $1/N$ expansion whose dominant graphs are the "melonic" ones.

More on "melonic" tensor graphs

1. they maximize the number of faces for a given number of vertices.
2. they correspond to a particular class of triangulations of the sphere $\mathcal{S}^3$.

The double scaling limit of the MO tensor model

R. Gurău, A. Tanasă, D. Youmans, Europhys. Lett. (2015)

The dominant configurations in the double scaling limit are the dominant schemes

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
contributions from higher degree are enhanced as $\lambda \to \lambda_c$

$\kappa^{-1} := N^{\frac{1}{2}}(1 - \lambda/\lambda_c)$
the partition function expansion:

$$Z = \sum_{\bar{\omega}} N^{3-\bar{\omega}} f_{\bar{\omega}}$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ

contribution from all degree tensor graphs

A further generalization

- generalisation to $O(N)^3$ invariant tensor models

  S. Carrozza and A. T., *Lett. Math. Phys.* (2016), arXiv: 1512.06718

First results on symmetric tensors - large N expansion

$O(N)$ symmetry

- numeric check (up to order 8 in λ)

  I. Klebanov and G. Tarnopolsky, JHEP (2017), arXiv:1706.00839[hep-th]

- analytic proof, for a real model with 2 tensor fields

  R. Gurău, arXiv:1706.05328[hep-th] *Commun. Math. Phys.* (2018)

- analytic proof (at any order in perturbation)

  D. Benedetti *et. al.*, arXiv:1712.00249, *Commun. Math. Phys.* (in press)

Back to quantum gravity - holography

Holography - property of quantum gravity

Holography: in a theory with gravity one must have a correspondence between a volume of space and the boundary enclosing it.

example of holography : the AdS/CFT correspondence

Maldacena

The Sachdev-Ye model

Sachdev and Ye, *Phys. Rev. Lett.* (1993)

toy-model of $N >> 1$ spins

$$\mathcal{H} = \frac{1}{M} \sum_{j,k=1}^{N} J_{jk}\, \mathbf{S_j} \cdot \mathbf{S_k}$$

where the J_{ij} are drawn from the distribution $P(J_{ij}) = e^{-\frac{J_{ij}^2}{2J^2}}$ and the spins are in some representation of $SU(M)$

representation of the spins in terms of fermions:

$S_\mu^\nu = c_\mu^\dagger c^\nu$, $\sum_{\mu=1}^{M} c_\mu^\dagger c^\mu = n_b$

n_b - the number of columns in the Young tableaux characterizing the representation of $SU(M)$

$$\mathcal{H} = \frac{1}{\sqrt{M}} \sum_{j,k=1}^{N} \sum_{\mu,\nu=1}^{M} J_{ij}\, c_{i\mu}^\dagger c_{j\nu}^\dagger c_i^\nu c_j \mu$$

2−point function in the large N limit A. Georges and O. Parcollet *Phys. Rev.* **B** (1999)

The Sachdev-Ye-Kitaev model

- Kitaev (2015) - the Sachdev-Ye-Kitaev (SYK) model

$$\mathcal{H} = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l$$

where χ_j are Majorana fermions
the model has quenched disorder with the couplings drawn from the distribution $P(J_{ijkl}) = e^{-\frac{N^3 J_{ijkl}^2}{12 J^2}}$

Kitaev proposed this model as holographic toy-model huge amount of interest from the AdS/CFT community

Maldacena, Polchinski, Gross, Gaiotto *etc.*

167 high-energy physics articles with *SYK* in the title (in InSPIRES)

- study of the 2− and 4−point function of the model *via* a careful analysis of the Dyson-Schwinger equation

  J. Maldacena and D. Stanford, *Phys. Rev.* **D** (2016)

  J. Polchinski and V. Rosenhaus, *JHEP* (2016), arXiv:1601.06768

- generalization to include f flavors of fermions

  D. Gross and V. Rosenhaus, *JHEP* (2017), arXiv:1610.01569[hep-th]

- supersymmetric version

  W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, *Phys. Rev.* **D** (2017), arXiv:1610.08917[hep-th]

- *etc.*

The SYK model and tensor models

1. Witten arXiv:1610.09758
   dominant graph in the large N - the melon graphs (both in SYK and tensor models!

   proof of melonic dominance in the SYK model

   V. Bonzom *et. al.*, arXiv:1808.10314 [math-ph] , *Lett. Math. Phys.* (in press)

   re-expression of an SYK-like model in terms of a colored tensor model (same structure of the Dyson-Schwinger equation)

   $\hookrightarrow$ the Gurău-Witten model

2. Klebanov and Tarnopolsky *Phys. Rev.* **D** (2017), arXiv:1611.08915
   re-expression of SYK-like models in terms of
   1. an $O(N)^D$ invariant tensor model or
   2. an MO tensor model

Why tensor SYK?

- the coupling constant is usual - usual quantum mechanics (or QFT) model
- number of fields does not proliferate as $N \to \infty$; as in QCD, it is the gauge group which grows
- the large N limit is better controlled

Further developments

- a supersymmetric SYK-like tensor model

  C. Peng, M. Spradlin, A. Volovich, *JHEP* (2017), arXiv: 1612.06330 [hep-th]

- numerical simulations C. Krishnan *et. al.* *JHEP* (2017), arXiv:1612.06330[hep-th]

- lattice models P. Narayan and J. Yoon, *JHEP* (2017) arXiv:1705.01554[hep-th]

- *etc.*

Diagrammatics of sub-leading orders of the $1/N$ expansion

V. Bonzom, L. Lionni, A. T., *J. Math. Phys.* (2017) arXiv: 1702.06944[hep-th]

comparison between the SYK model and the Gurău-Witten model
- $2-$ and $4-$point function

Feynman diagrams which contribute to the same order in the SYK model can contribute to different orders in the Gurău-Witten model

the Gurău-Witten model sort of lifts a degeneracy of Feynman diagrams (at least at low orders)

Other recent results

- study of the graphs appearing at the general order of the large N expansion in the colored SYK model

  É. Fusy *et. al.* arXiv:1810.02146

- identifying and compairing the graphs appearings at various orders (not just the dominant one) of the large N expansion in MO SYK-like tensor model

  V. Bonzom *et. al.*, arXiv:1903.01723 (in press)

- the effect of non-Gaussian average over the SYK random couplings - a modification of the variance of the Gaussian distribution of couplings at leading order in N

  T. Krajewski, *et. al.*, arXiv:1812.03008 [hep-th], *Phys. Rev.* **D** (2019)

Perspectives

utilization of the tensor model "technology" for the study of the SYK model:

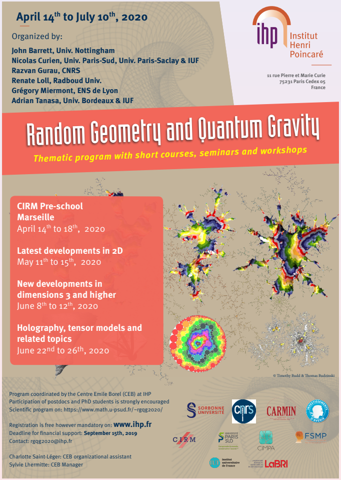
- study of the Dyson-Schwinger equation for SYK-like tensor models
- computations of Feynman amplitudes at sub-leading orders R. Gurau, R. Pascalie, A. T., work in progress
- *etc.*

Bottom line

Current uses of random tensors:

- Study of random geometries in arbitrary dimension
- Holography
- Data analysis
- *etc.*

IHP trimester - Rand. Geom. & Quantum Grav. 2020



registration: before Sept. 15 (for financial aid for housing in Paris)

Thank you for your attention!

Vă mulțumesc pentru atenție!

Why SYK?

Remarkable features:

1. solvable at strong coupling
2. maximally chaotic
3. emergent conformal symmetry

"The amount of theoretical work one has to cover before being able to solve problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."
P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930