Generalized Berezin quantization of almost Kähler–Cartan geometry and nonholonomic Ricci solitons and Einstein spaces

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Motivation and Goals

Methods of nonholonomic & almost Kähler geometry $\rightarrow$ unified formalism for geometric & DQ

of (supersymmetric) Ricci flows/ solitons and (non) commutative modified gravity theories

Goals

1. Geometry of nonholonomic complex manifolds endowed with standard complex structures and induced almost Kähler–Cartan geometric objects.
2. Elaborate a new geometric framework for quanting Ricci soliton and modified gravity models by generalizing the Berezin and Berezin–Toeplitz quantization for nonhlonomic real and complex manifolds.
3. A comparative study of DQ of Ricci solitons
4. Study a few explicit examples of (noncommutative) quantum almost Kähler – Ricci solitons and generic off–diagonal metrics

Former results on nonholonomic Ricci flows, DQ and NC exact solutions

- S. Vacaru, J. Math. Phys. **50** (2009) 073503 and **54** (2013) 073511; J. Geom. Phys. **60** (2010) 1289; Class. Quant. Grav. **27** (2010) 105003 and 20 similar papers
- C. Lazaroiu et all. JHEP **0809** (2008) 059 and **0905** (2009) 055

Outline

Canonical metric compatible connections for $(\mathbf{g}, \mathbf{N})$

Nonholonomic manifolds V and 2+2 splitting N : $T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$

h- / v-coordinates $u = (x, y)$, $u^{\alpha} = (x^i, y^a)$; $i, j, k, ... = 1, 2$; $a, b, c ... = 3, 4$;
frames $e_{\alpha'} = (e_{i'}, e_{a'})$, $e_{\alpha'} = e^{\alpha}_{\ \alpha'}(u)\partial_{\alpha}$, $\partial_{\alpha} = \partial/\partial u^{\alpha} = (\partial_i, \partial_a)$

Aim: state geometric principles when $(\mathbf{V}, \mathbf{g}, \mathbf{N}) \to$ N–adapted and metric compatible linear connections and almost symplectic structures

all values are determined by data $\left(\mathbf{g}_{\alpha\beta}; \mathbf{N} = N^a_i(x, y) dx^i \otimes \frac{\partial}{\partial y^a}\right)$

$$\mathbf{g} = {}_{\shortmid}\mathbf{g} \to \begin{cases} \nabla : & \nabla \mathbf{g} = 0;\ {}^{\nabla}\mathcal{T}^{\alpha} = 0, \text{ Levi–Civita connection ;} \\ {}_{\shortparallel}\mathbf{D} : & {}_{\shortparallel}\mathbf{D}\mathbf{g} = 0;\ h\,{}_{\shortparallel}\mathcal{T}^{\alpha} = 0, v\,{}_{\shortparallel}\mathcal{T}^{\alpha} = 0, \text{ canonical d–connection ;} \\ {}_{\shortmid}\mathbf{D} : & {}_{\shortmid}\mathbf{D}\mathbf{g} = 0;\ h\,{}_{\shortmid}\mathcal{T}^{\alpha} = 0, v\,{}_{\shortmid}\mathcal{T}^{\alpha} = 0, \text{ Cartan d–connection .} \end{cases}$$

$$\mathbf{g} \to \text{ distortions: } \quad {}_{\shortparallel}\mathbf{D} = \nabla + {}_{\shortparallel}\mathbf{Z}, \text{ for exact solutions;}$$
$$ {}_{\shortmid}\mathbf{D} = \nabla + {}_{\shortmid}\mathbf{Z}, \text{ for DQ}$$

Cartan d–connection ${}_{\shortmid}\mathbf{D}$ [in brief]

1. Prescribe function $\mathcal{L}(u)$ on $\mathbf{V}$, ${}_{\shortmid}h_{ab} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b}$, $\det|{}_{\shortmid}h_{ab}| \neq 0$.
2. Construct ${}_{\shortmid}\mathbf{N} = \{{}_{\shortmid}N_i^a(u) = \frac{\partial\ {}_{\shortmid}G^a}{\partial y^{2+i}}\}$ for ${}_{\shortmid}G^a = \frac{1}{4}\ {}_{\shortmid}h^{a\ 2+i}\left(\frac{\partial^2 \mathcal{L}}{\partial y^{2+i}\partial x^k} y^{2+k} - \frac{\partial \mathcal{L}}{\partial x^i}\right)$, semi–sprays are equivalent to Euler–Lagrange eqs for $\mathcal{L}(u)$.
3. N–elongated bases $[{}_{\shortmid}\mathbf{e}_\alpha, {}_{\shortmid}\mathbf{e}_\beta] = {}_{\shortmid}\mathbf{e}_\alpha {}_{\shortmid}\mathbf{e}_\beta - {}_{\shortmid}\mathbf{e}_\beta {}_{\shortmid}\mathbf{e}_\alpha = w_{\alpha\beta}^\gamma\ {}_{\shortmid}\mathbf{e}_\gamma$,

$$ {}_{\shortmid}\mathbf{e}_\alpha = ({}_{\shortmid}\mathbf{e}_i = \partial_i - {}_{\shortmid}N_i^a \partial_a, e_a = \partial_a),\quad {}_{\shortmid}\mathbf{e}^\alpha = (e^i = dx^i,\ {}_{\shortmid}\mathbf{e}^a = dy^a + {}_{\shortmid}N_i^a dx^i) $$

4. d–metric: ${}_{\shortmid}\mathbf{e}_\alpha = e_\alpha^{\alpha'} e_{\alpha'}$ and $\mathbf{g}_{\alpha'\beta'} e_\alpha^{\alpha'} e_\beta^{\beta'} = {}_{\shortmid}\mathbf{g}_{\alpha\beta}$,

$$ {}_{\shortmid}\mathbf{g} = {}_{\shortmid}g_{ij} dx^i \otimes dx^j + {}_{\shortmid}h_{ab}\ {}_{\shortmid}\mathbf{e}^a \otimes {}_{\shortmid}\mathbf{e}^b,\quad {}_{\shortmid}g_{ij} = {}_{\shortmid}h_{2+i\ 2+j}. $$

5. d–connection: $\mathbf{D} = (hD; vD) = \{\Gamma^\gamma_{\ \alpha\beta} = (L^i_{jk}, {}^vL^a_{bk}; C^i_{jc}, {}^vC^a_{bc})\}$, $\mathbf{D}\mathbf{g} = 0$, 1–form $\Gamma^\gamma_{\ \alpha} := \Gamma^\gamma_{\ \alpha\beta}\mathbf{e}^\beta$, torsion 2-form $\mathcal{T}^\alpha := \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^\alpha_{\ \beta} \wedge \mathbf{e}^\beta = \mathbf{T}^\alpha_{\ \beta\gamma}\mathbf{e}^\beta \wedge \mathbf{e}^\gamma$,
Cartan d–connection:

$$ \exists\ {}_{\shortmid}\mathbf{D} = \{{}_{\shortmid}\Gamma^\gamma_{\ \alpha\beta} = ({}_{\shortmid}L^i_{jk};\ {}_{\shortmid}C^a_{bc})\}:\ {}_{\shortmid}\mathbf{D}\ {}_{\shortmid}\mathbf{g} = 0;\ h\ {}_{\shortmid}\mathcal{T} = 0,\ v\ {}_{\shortmid}\mathcal{T} = 0 $$

(pseudo) Riemannian as a canonical almost Kähler

Canonical almost symplectic (Kähler) variables

1. Almost complex structure determined by $\mathbf{N}$, or $\mathcal{L}$:
   $\mathbf{J}(\mathbf{e}_i) = -e_{2+i}$ and $\mathbf{J}(e_{2+i}) = \mathbf{e}_i$, where $\mathbf{J} \circ \mathbf{J} = -\mathbb{I}$,
   $\mathbf{J} = \mathbf{J}^{\alpha}_{\ \beta}\, \mathbf{e}_{\alpha} \otimes \mathbf{e}^{\beta} = -\frac{\partial}{\partial y^i} \otimes dx^i + \left(\frac{\partial}{\partial x^i} - {}_{\shortmid}N_i^{2+j} \frac{\partial}{\partial y^j}\right) \otimes \left(dy^i + {}_{\shortmid}N_k^{2+i} dx^k\right)$
2. The Neijenhuis d-tensor, for $\mathbf{X} = X^{\alpha}\mathbf{e}_{\alpha} = X^i \mathbf{e}_i + X^a e_a$,
   ${}^{\mathbf{J}}\Omega(\mathbf{X}, \mathbf{Y}) := -[\mathbf{X}, \mathbf{Y}] + [\mathbf{JX}, \mathbf{JY}] - \mathbf{J}[\mathbf{JX}, \mathbf{Y}] - \mathbf{J}[\mathbf{X}, \mathbf{JY}]$
3. Almost symplectic structure $\mathbf{g} = {}_{\shortmid}\mathbf{g}, \mathbf{N} = {}_{\shortmid}\mathbf{N}, \mathbf{J} = {}_{\shortmid}\mathbf{J} \rightarrow {}_{\shortmid}\theta(\cdot, \cdot) := {}_{\shortmid}\mathbf{g}({}_{\shortmid}\mathbf{J}\cdot, \cdot)$,
   $$ {}_{\shortmid}\theta = {}_{\shortmid}g_{ij}(x,y)(dy^{2+i} + {}_{\shortmid}N_k^{2+i} dx^k) \wedge dx^j, $$
   $$ {}_{\shortmid}\theta = d\, {}_{\shortmid}\omega, \text{ for } {}_{\shortmid}\omega := \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^i} dx^i \rightarrow d\, {}_{\shortmid}\theta = dd\, {}_{\shortmid}\omega = 0;\ \theta_{\alpha'\beta'} e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} = {}_{\shortmid}\theta_{\alpha\beta} $$
4. the Cartan ${}_{\shortmid}\mathbf{D} = {}^{\theta}_{\shortmid}\mathbf{D}$ is a unique almost symplectic: ${}^{\theta}_{\shortmid}\mathbf{D}\, {}_{\shortmid}\theta = 0$ & ${}^{\theta}_{\shortmid}\mathbf{D}\, {}_{\shortmid}\mathbf{J} = 0$
5. Almost Kähler–Cartan spaces:
   $$ (\mathbf{g}, \mathbf{N}, \mathbf{D}) \approx ({}_{\shortmid}\mathbf{g}, {}_{\shortmid}\mathbf{N}, {}_{\shortmid}\mathbf{D}) \approx (\theta, \mathbf{J}, {}^{\theta}\mathbf{D}) \approx ({}_{\shortmid}\theta, {}_{\shortmid}\mathbf{J}, {}^{\theta}_{\shortmid}\mathbf{D}) $$

Nonholonomic Ricci solitons and modified gravity

Almost Kähler Ricci solitons

Gradient d–vector $\mathbf{X}_\beta = {}_{\shortmid}\mathbf{D}_\beta K(u)$ for some smooth potential function $K(x,y)$; gradient almost Kähler Ricci solitons are solutions of

$$ {}_{\shortmid}\mathbf{R}_{\alpha\beta} + {}_{\shortmid}\mathbf{D}_\alpha\, {}_{\shortmid}\mathbf{D}_\beta K = \lambda \mathbf{g}_{\alpha\beta}; $$

steady if $\lambda = 0$; shrinking, for $\lambda > 0$; expanding, for $\lambda < 0$

Ricci solitons, MG and effective EG; Lagrange density $R \to f(R,T)$; $f({}^s\mathbf{R})$,

$$ f_R \mathbf{R}_{\alpha\beta} - \frac{1}{2} f \mathbf{g}_{\alpha\beta} + (\mathbf{g}_{\alpha\beta}\mathbf{D}_\gamma \mathbf{D}^\gamma - \mathbf{D}_\alpha \mathbf{D}_\beta)\, f_R = 0 $$

associate to nonholonomic Ricci solitonic eqs $\mathbf{R}_{\alpha\beta} + \mathbf{D}_\alpha \mathbf{D}_\beta K = \lambda \mathbf{g}_{\alpha\beta}$, $K = f_R$

$$ \text{effective gravitational eqs } \mathbf{R}_{\alpha\beta} = \Lambda(x^i, y^a)\mathbf{g}_{\alpha\beta}, $$

polarized cosmological "constant" $\Lambda = \frac{\lambda + \mathbf{D}_\gamma \mathbf{D}^\gamma f_R - f/2}{1 - f_R}$; for massive gravity, the effective cosmological constant contains additional terms.
Off–diagonal configurations with Killing symmetry on $\partial/\partial y^4$, $\Lambda \approx \Lambda(x^i)$.

Almost Kähler Structures and Nonholonomic Complex Manifolds

N–connection and double almost complex structures

Local coordinates $z^{\alpha} = \check{u}^{\alpha} = (u^{\alpha}, iu^{\grave{\alpha}}) = (\check{x}^j = x^j + i\grave{x}^j, \check{y}^a = y^a + i\grave{y}^a)$. In brief, $z = (\check{x}, \check{y})$, $u = (x, y)$, $\grave{u} = (\grave{x}, \grave{y})$.
Complex conjugated coordinates $\overline{z}^{\beta} = (\overline{z}^j = x^j - i\grave{x}^j, \overline{z}^b = y^a - i\grave{y}^a)$; N–connection: ${}_{\natural}N^a_j = N^a_j - i\check{N}^a_j$ and ${}_{\natural}\overline{N}^a_j = N^a_j + i\check{N}^a_j$,

$$\begin{aligned}
\frac{\partial}{\partial z^{\beta}} \to {}_{\natural}\mathbf{e}_{\beta} &= [\,{}_{\natural}\mathbf{e}_j = \frac{1}{2}(\mathbf{e}_j - i\grave{\mathbf{e}}_j) = \frac{\partial}{\partial z^j} - {}_{\natural}N^a_j \frac{\partial}{\partial z^a},\ {}_{\natural}e_b = \frac{1}{2}(e_b - i\grave{e}_b) = \frac{1}{2}(\frac{\partial}{\partial y^b} - i\frac{\partial}{\partial \grave{y}^b})] \\
\frac{\partial}{\partial \overline{z}^{\beta}} \to {}_{\natural}\overline{\mathbf{e}}_{\beta} &= [\,{}_{\natural}\overline{\mathbf{e}}_j = \frac{1}{2}(\mathbf{e}_j + i\grave{\mathbf{e}}_j) = \frac{\partial}{\partial \overline{z}^j} - {}_{\natural}\overline{N}^a_j \frac{\partial}{\partial \overline{z}^a},\ {}_{\natural}\overline{e}_b = \frac{1}{2}(e_b + i\grave{e}_b) = \frac{1}{2}(\frac{\partial}{\partial y^b} + i\frac{\partial}{\partial \grave{y}^b})].
\end{aligned}$$

Definition: *A pair* $(\mathbf{Y}, {}_{\natural}\mathbf{N})$ *with* $T\mathbf{Y}^{\mathbb{C}} := T\mathbf{Y}\otimes_{\mathbb{R}}\mathbb{C}$ *and N–connection structure* ${}_{\natural}\mathbf{N} : T\mathbf{Y}^{\mathbb{C}} = h\mathbf{Y}^{\mathbb{C}}\oplus v\mathbf{Y}^{\mathbb{C}}$*, is referred to as an almost complex nonholonomic manifold.*
All real endomorphisms and N–adapted differential operators are extended from $T\mathbf{Y}$ to $T\mathbf{Y}^{\mathbb{C}}$ by $\mathbb{C}$–linearity. In local complex coordinate coefficient forms, ${}_{\natural}\mathbf{N} = \{{}_{\natural}N^a_j\}$ and ${}_{\natural}\overline{\mathbf{N}} = \{{}_{\natural}\overline{N}^a_j\}$.
The formulas for the *almost complex structure* are generalized in ${}_{\natural}\mathbf{N}$–adapted form following such formulas with ${}_{\natural}\mathbf{J} \circ {}_{\natural}\mathbf{J} = -\mathbb{I}$,

$$\begin{aligned}
{}_{\natural}\mathbf{J}({}_{\natural}\mathbf{e}_j) &= -\,{}_{\natural}e_{2+j} \text{ and } {}_{\natural}\mathbf{J}({}_{\natural}e_{2+j}) = {}_{\natural}\mathbf{e}_j, \\
{}_{\natural}\mathbf{J} &= {}_{\natural}\mathbf{J}^{\underline{\alpha}}_{\ \underline{\beta}} \frac{\partial}{\partial z^{\alpha}} \otimes dz^{\beta} = \check{\mathbf{J}}^{\alpha'}_{\ \beta'}\ {}_{\natural}\mathbf{e}_{\alpha'} \otimes {}^{\natural}\mathbf{e}^{\beta'} = -\,{}_{\natural}e_{2+j} \otimes {}^{\natural}e^j + {}_{\natural}\mathbf{e}_j \otimes {}^{\natural}\mathbf{e}^{2+j},
\end{aligned}$$

Almost Kähler Structures and Nonholonomic Complex Manifolds

Theorem: *An almost complex* $\mathbf{J}$ *for* $(\mathbf{Y}, \mathbf{J})$ *comes from a holomorphic structure if and only if* $T^{0,1}\mathbf{Y}$ *is integrable.*

Definition:

- *Let us call the natural complex structure the structure* ${}^{c}\mathbf{J}$ *arising from a holomorphic structure on a complex manifold* Y.
- *On* $\mathbf{Y}$, *alternative* ${}_{\natural}\mathbf{J}$ *exists, determined by any splitting* ${}_{\natural}\mathbf{N}$*; in particular, be induced by any real canonical* $\mathbf{N}$, *or arbitrary real* $\mathbf{N}$, *and corresponding* $\mathbf{J}$, *or* ${}_{\shortmid}\mathbf{J}$.

Lemma: *The almost complex and N–connection structures define*

$$ {}^{\mathbb{C}}\Lambda^{1}\mathbf{Y} = \Lambda^{1,0}\mathbf{Y} \oplus \Lambda^{0,1}\mathbf{Y}, \textit{ where } \Lambda^{1,0}\mathbf{Y} = h\Lambda^{1,0}\mathbf{Y} \oplus v\Lambda^{1,0}\mathbf{Y} \textit{ and } \Lambda^{0,1}\mathbf{Y} = h\Lambda^{0,1}\mathbf{Y} \oplus v\Lambda^{0,1}\mathbf{Y}. $$

Proof: explicit calculus with differential forms, $\Lambda^{1,0}\mathbf{Y} = \{\mathbf{a} - i\mathbf{a} \circ \mathbf{J} | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$, $\Lambda^{0,1}\mathbf{Y} = \{\mathbf{a} + i\mathbf{a} \circ \mathbf{J} | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$ and
$h\Lambda^{1,0}\mathbf{Y} = \{h(\mathbf{a} - i\mathbf{a} \circ \mathbf{J})\, | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$, $v\Lambda^{1,0}\mathbf{Y} = \{v(\mathbf{a} - i\mathbf{a} \circ \mathbf{J})\, | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$,
$h\Lambda^{0,1}\mathbf{Y} = \{h(\mathbf{a} + i\mathbf{a} \circ \mathbf{J})\, | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$, $v\Lambda^{1,0}\mathbf{Y} = \{v(\mathbf{a} + i\mathbf{a} \circ \mathbf{J})\, | \mathbf{a} \in \Lambda^{1}\mathbf{Y}\}$,
for instance, $h(\mathbf{a} - i\mathbf{a} \circ \mathbf{J})$ means that it is taken the h–part of the distinguished 1–form $\mathbf{a} - i\mathbf{a} \circ \mathbf{J}$. □

$$ \begin{aligned} df &= \partial f + \overline{\partial} f, \text{ with } \partial f = \frac{\partial f}{\partial z^{\alpha}} dz^{\alpha} \text{ and } \overline{\partial} f = \frac{\partial f}{\partial \overline{z}^{\alpha}} d\overline{z}^{\alpha}, \\ {}^{\natural}\mathbf{e} f &= {}^{\natural}\partial f + {}^{\natural}\overline{\partial} f, \text{ with } {}^{\natural}\partial f = ({}^{\natural}\mathbf{e}_{\alpha} f)\, {}^{\natural}\mathbf{e}^{\alpha} \text{ and } {}^{\natural}\overline{\partial} f = ({}^{\natural}\overline{\mathbf{e}}_{\alpha} f)\, {}^{\natural}\overline{\mathbf{e}}^{\alpha}. \end{aligned} $$

Almost Kähler Structures and Nonholonomic Complex Manifolds

Holomorphic nonholonomic vector bundles

We can define the differential operators $\partial : \mathcal{C}^{\infty}(\Lambda^{p,q}Y) \to \mathcal{C}^{\infty}(\Lambda^{p+1,q}Y)$ where $\overline{\partial} : \mathcal{C}^{\infty}(\Lambda^{p,q}Y) \to \mathcal{C}^{\infty}(\Lambda^{p,q+1}Y)$. Considering the operator d^2 for $d := \partial + \overline{\partial}$, we prove another

Lemma–Definitions: *One hold the identities* $\partial^2 = 0, \overline{\partial}^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0,$

when a) a vector field $X \subset \mathcal{C}^{\infty}(T^{1,0}Y)$ *is holomorhic if* $X(f)$ *is holomorphic for every locally defined holomorphic function* f;

b) a differential form S *of type* (p,q) *is holomorphic if* $\overline{\partial}S = 0$.

Theorem: *For a fixed splitting* ${}_{\natural}\mathbf{N} = \{ {}_{\natural}N_j^a \}$ *and* ${}_{\natural}\overline{\mathbf{N}} = \{ {}_{\natural}\overline{N}_j^a \}$ *with holomorphic coefficients for a* $(\mathbf{Y}, {}_{\natural}\mathbf{J})$, *there is a system of nonlinaer frame (vielbein) transforms and their duals with coefficients linear on* ${}_{\natural}N_j^a$ *and, respectively, on* ${}_{\natural}\overline{N}_j^a$ *preserving the holomorphic configurations.*

Proof: In explicit form, we can verify that ${}_{\natural}\mathbf{e}_{\alpha} = {}_{\natural}\mathbf{e}_{\alpha}^{\ \alpha'}(z,\overline{z})\frac{\partial}{\partial z^{\alpha'}}$ and ${}^{\natural}\mathbf{e}^{\beta} = {}^{\natural}\mathbf{e}^{\beta}_{\ \beta'}(z,\overline{z})dz^{\beta'}$,

$${}_{\natural}\mathbf{e}_{\alpha}^{\ \alpha'}(z,\overline{z}) = \begin{bmatrix} e_i^{\ \underline{i}}(z,\overline{z}) & -{}_{\natural}N_i^b(z,\overline{z})e_b^{\ \underline{a}}(z,\overline{z}) \\ 0 & e_a^{\ \underline{a}}(z,\overline{z}) \end{bmatrix} \text{ and } {}^{\natural}\mathbf{e}^{\beta}_{\ \beta'}(z,\overline{z}) = \begin{bmatrix} e^i_{\ \underline{i}}(z,\overline{z}) & {}_{\natural}N_k^b(z,\overline{z})e^k_{\ \underline{i}}(z,\overline{z}) \\ 0 & e^a_{\ \underline{a}}(z,\overline{z}) \end{bmatrix},$$

satisfy the conditions of this theorem. □

$d := \partial + \overline{\partial} \quad \longleftrightarrow \quad {}^{d}_{\natural}\mathbf{e} = {}_{\natural}\mathbf{e} + {}_{\natural}\overline{\mathbf{e}}$, with respective $\partial \longleftrightarrow {}_{\natural}\mathbf{e}, \overline{\partial} \longleftrightarrow {}_{\natural}\overline{\mathbf{e}}$,

for ${}_{\natural}\mathbf{e} = h\,{}_{\natural}\mathbf{e} + v\,{}_{\natural}\mathbf{e}$ and ${}_{\natural}\overline{\mathbf{e}} = h\,{}_{\natural}\overline{\mathbf{e}} + v\,{}_{\natural}\overline{\mathbf{e}}$.

f is holomorphic if $\quad \overline{\partial}f =$ and/or ${}_{\natural}\overline{\mathbf{e}}f = 0$.

Almost Kähler Structures and Nonholonomic Complex Manifolds

Using previous Lemma, we have $d(i\partial\overline{\partial}) = i(\partial + \overline{\partial})\partial\overline{\partial} = i(\partial^2\overline{\partial} - \partial\overline{\partial}^2) = 0$. This provides a proof for

Proposition: [The local $i\partial\overline{\partial}$–Lemma and its N–adapted version.] *A real 2–form Q of type* $(1,1)$ *on a compact manifold Y is closed if and only if in the vicinity of any point* $z \in Y$ *there is an open neighborhood U such that* $Q_{|U} = i\partial\overline{\partial}q$ *for some real function q on U. In N–adapted form, we have* $Q = i\, {}_{\natural}\mathbf{e}\, {}_{\natural}\overline{\mathbf{e}}q$.

Generalizing on nonholonomic manifolds (using N–elongated operators), we prove such an important result:

Lemma: *[$\overline{\partial}$–Poincaré Lemma in N–adapted form]. A $\overline{\partial}$–closed* $(0,1)$*–form* **A** *is locally $\overline{\partial}$–exact. For a nontrivial N–connection structure, the condition of exactness results in conventional* $h\, {}_{\natural}\overline{\mathbf{e}}\mathbf{A} = 0$ *and* $v\, {}_{\natural}\overline{\mathbf{e}}\mathbf{A} = 0$.

Definition: *A N–anholonomic holomorphic vector bundle* $(\mathbf{E}, {}_{\natural}\mathbf{N})$ *is defined as a holomorphic vector bundle* $\pi : \mathbf{E} \rightarrow V^{\mathbb{C}}$ *over a complex manifold* $V^{\mathbb{C}}$ *with typical fieber being a complex vector space and a N–connection:* ${}_{\natural}\mathbf{N} : T\mathbf{E} = h\mathbf{E} \oplus v\mathbf{E}$.
In particular, $\mathbf{E} = TV^{\mathbb{C}}$ *defines a N–anholonomic holomorphic tangent bundle.*

On a $(\mathbf{E}, {}_{\natural}\mathbf{N})$ $\exists$ ${}_{\natural}\overline{\mathbf{e}} : \mathcal{C}^{\infty}(\Lambda^{p,q}\mathbf{E}) \rightarrow \mathcal{C}^{\infty}(\Lambda^{p,q+1}\mathbf{E})$ satisfying the Leibniz property and defining a pseudo–holomorphic structure. For d–operators with $\overline{\partial} \longleftrightarrow {}_{\natural}\overline{\mathbf{e}} = h\, {}_{\natural}\overline{\mathbf{e}} + v\, {}_{\natural}\overline{\mathbf{e}}$ and $\overline{\partial}^2 = 0$, ${}_{\natural}\overline{\mathbf{e}}$ defines a nonholonomic holomorphic structure.

A section σ in a pseudo–holomorphic nonholonomic vector bundle $(\mathbf{E}, {}_{\natural}\mathbf{N})$ is called N–holomorphic if ${}_{\natural}\overline{\mathbf{e}}\sigma = 0$ and $\overline{\partial}\sigma = 0$.

Theorem: *A complex nonholonomic vector bundle* $(\mathbf{E}, {}_{\natural}\mathbf{N})$ *is holomorphic if and only if it has a holomorphic structure* $\overline{\partial}\sigma = 0$. *It is N-adapted and N–holomorphic if* ${}_{\natural}\overline{\mathbf{e}}\sigma = 0$.

Nonholonomic almost Hermitian and Kähler structures

We consider that V is a differential manifold, not necessarily complex, and that $\mathbf{E} \to V$ is a complex nonholonomic vector bundle over V and N–connection ${}_{\natural}\mathbf{N}$.

Distinguished connections on nonholonomic vector bundles

Definition: *A d–connection ${}_{\natural}\mathbf{D} = (h\,{}_{\natural}\mathbf{D}, v\,{}_{\natural}\mathbf{D})$ on $(\mathbf{E}, {}_{\natural}\mathbf{N})$ is a $\mathbb{C}$–linear connection preserving under parallelism the h- and v–decomposition determined by ${}_{\natural}\mathbf{N}$.*

We can associate to ${}_{\natural}\mathbf{D}$ a $\mathbb{C}$–linear differential operator ${}_{\natural}\mathbf{D} : \mathcal{C}^{\infty}(\mathbf{E}) \to \mathcal{C}^{\infty}(\Lambda^1\mathbf{E})$ satisfying

$$\begin{aligned} {}_{\natural}\mathbf{D}(f\sigma) &= {}_{\natural}\mathbf{e}f \otimes \sigma + f{}_{\natural}\mathbf{D}\sigma, \text{ where} \\ h\,{}_{\natural}\mathbf{D}(f\sigma) &= h\,{}_{\natural}\mathbf{e}f \otimes h\sigma + f\,h{}_{\natural}\mathbf{D}(h\sigma) \text{ and } v\,{}_{\natural}\mathbf{D}(f\sigma) = v\,{}_{\natural}\mathbf{e}f \otimes v\sigma + f\,h{}_{\natural}\mathbf{D}(h\sigma), \end{aligned}$$

$\forall f \in \mathcal{C}^{\infty}(V)$ and a section $\sigma = h\sigma + v\sigma \in \mathcal{C}^{\infty}(\mathbf{E})$.
The curvature of ${}_{\natural}\mathbf{D}$ the $End(\mathbf{E})$–valued 2–form ${}_{\natural}\mathcal{R}(\sigma) := {}_{\natural}\mathbf{D}({}_{\natural}\mathbf{D}\sigma)$.
With respect to N–adapted frames, ${}_{\natural}\mathbf{D}$ can be characterized by a 1–form ${}_{\natural}\Gamma^{\gamma}_{\ \alpha} := {}_{\natural}\Gamma^{\gamma}_{\ \alpha\beta}\ {}^{\natural}\mathbf{e}^{\beta}$.

In N–adapted form, the d–torsion, $\mathcal{T}^{\alpha} = \{\mathbf{T}^{\alpha}_{\ \beta\gamma}\}$, and d–curvature, $\mathcal{R}^{\alpha}_{\ \beta} = \{\mathbf{R}^{\alpha}_{\ \ \beta\gamma\delta}\}$ are

$$\begin{aligned} {}_{\natural}\mathcal{T}^{\alpha} :=& \quad {}_{\natural}\mathbf{D}\ {}^{\natural}\mathbf{e}^{\alpha} = d\ {}^{\natural}\mathbf{e}^{\alpha} + {}_{\natural}\Gamma^{\alpha}_{\ \beta} \wedge {}^{\natural}\mathbf{e}^{\beta} = {}_{\natural}\mathbf{T}^{\alpha}_{\ \beta\gamma}\ {}^{\natural}\mathbf{e}^{\beta} \wedge {}^{\natural}\mathbf{e}^{\gamma}, \\ {}_{\natural}\mathcal{R}^{\alpha}_{\ \beta} :=& \quad {}_{\natural}D\,{}_{\natural}\Gamma^{\alpha}_{\ \beta} = d\,{}_{\natural}\grave{}^{\alpha}_{\ \beta} - {}_{\natural}\Gamma^{\gamma}_{\ \beta} \wedge {}_{\natural}\Gamma^{\alpha}_{\ \gamma} = {}_{\natural}\mathbf{R}^{\alpha}_{\ \beta\gamma\delta}\ {}^{\natural}\mathbf{e}^{\gamma} \wedge {}^{\natural}\mathbf{e}^{\delta}. \end{aligned}$$

Nonholonomic almost Hermitian and Kähler structures

Almost Hermitian structures and Chern connection and d–connection

We shall work with nonholonomic $(\mathbf{E} \to V, {}_{\natural}\mathbf{N})$ when for every point $u \in V$ there is an *nonholonomic Hermitian structure* $\mathbf{H}$: $\mathbf{E}_u \times \mathbf{E}_u \to \mathbb{C}$ on the fibers of $\mathbf{E}$ with such properties $\forall \mathbf{X}, \mathbf{Z} \in \mathbf{E}_u$: a) $\mathbf{H}(\mathbf{X}, \mathbf{Z})$ is $\mathbb{C}$–linear in u; b) $\mathbf{H}(\mathbf{X}, \mathbf{Z}) = \overline{\mathbf{H}(\mathbf{Z}, \mathbf{X})}$; c) $\mathbf{H}(\mathbf{X}, \mathbf{Z}) > 0\ \forall \mathbf{Z} \neq 0$; d) $\mathbf{H}(,)$ is a smooth function on V for every smooth sections of $\mathbf{E}$. Every rank k complex vector bundles E admits Hermitian structures and this property is preserved if we endow such spaces with nonholonomic distributions.

Suppose that V is a complex manifold for a nonholonomic complex bundle $(\mathbf{E}, {}_{\natural}\mathbf{N})$, consider the projections $\pi^{1,0} : \Lambda^1(\mathbf{E}) \to \Lambda^{1,0}(\mathbf{E})$ and $\pi^{0,1} : \Lambda^1(\mathbf{E}) \to \Lambda^{0,1}(\mathbf{E})$ and introduce the corresponding $(1,0)$ and $(0,1)$–components of a chosen d–connection ${}_{\natural}\mathbf{D}$, when

${}_{\natural}\mathbf{D}^{1,0} := \pi^{1,0} \circ {}_{\natural}\mathbf{D} : \mathcal{C}^{\infty}(\Lambda^{p,q}(\mathbf{E})) \to \mathcal{C}^{\infty}(\Lambda^{p+1,q}(\mathbf{E}))$ and ${}_{\natural}\mathbf{D}^{0,1} := \pi^{0,1} \circ {}_{\natural}\mathbf{D} : \mathcal{C}^{\infty}(\Lambda^{p,q}(\mathbf{E})) \to \mathcal{C}^{\infty}(\Lambda^{p,q+1}(\mathbf{E}))$.

$\forall A \in \mathcal{C}^{\infty}(\Lambda^{p,q}(V))$ and $\sigma \in \mathcal{C}^{\infty}(\mathbf{E})$, such d–operators satisfy the Leibniz rule

${}_{\natural}\mathbf{D}^{1,0}(A \otimes \sigma) = {}_{\natural}\mathbf{e}A \otimes \sigma + (-1)^{p+q} A \wedge {}_{\natural}\mathbf{D}^{1,0}\sigma$ and ${}_{\natural}\mathbf{D}^{0,1}(A \otimes \sigma) = {}_{\natural}\overline{\mathbf{e}}A \otimes \sigma + (-1)^{p+q} A \wedge {}_{\natural}\mathbf{D}^{0,1}\sigma$.

Viewing $\mathbf{H}$ as a field of $\mathbb{C}$–valued real forms on a$\mathbf{E}$, we argue that ${}_{\natural}\mathbf{D}$ is a H–connection if $\mathbf{H}$ is parallel with respect to ${}_{\natural}\mathbf{D}$.

Theorem: [The Chern connection, ${}_{\natural}^{\mathbf{c}}D$, and the Chern d–connection, ${}_{\natural}^{\mathbf{c}}\mathbf{D}$] *For every nonholonomic Hermitian structure in a holomorphic N–anholonomic vector bundle* $(\mathbf{E}, {}_{\natural}\mathbf{N})$ *with N–holonomic structure* ${}_{\natural}\overline{\mathbf{e}}$, *there exists a d–connection* ${}_{\natural}^{\mathbf{c}}\mathbf{D}$ *which is such a unique* $\mathbf{H}$*–connection that* ${}_{\natural}\mathbf{D}^{0,1} = {}_{\natural}\overline{\mathbf{e}}$. *This is just the Chern connection (in this work denoted* ${}_{\natural}^{\mathbf{c}}D$*) such that* ${}_{\natural}^{\mathbf{c}}D^{0,1} = \overline{\partial}$ *if the nonholonomic structure became integrable or if we work with respect to holonomic frames.*

Nonholonomic almost Hermitian and Kähler structures

Hermitian and Kähler d–metrics

Definition: *A (nonholonomc) Hermitian d–metric on* $(\mathbf{Y},\ _{\natural}\mathbf{J})$ *is a d–metric* $\aleph(\mathbf{A},\mathbf{B}) = \aleph(\ _{\natural}\mathbf{J}\mathbf{A},\ _{\natural}\mathbf{J}\mathbf{B}),\ \forall \mathbf{A},\mathbf{B} \in T\mathbf{Y} = h\mathbf{Y} \oplus v\mathbf{Y}$. *The fundamental form of this Hermitian d–metric is* $\,_{\natural}^{\mathbf{e}}\theta(\mathbf{A},\mathbf{B}) := \aleph(\ _{\natural}\mathbf{J}\mathbf{A},\mathbf{B})$.

Let us consider a $(\mathbf{Y},\ _{\natural}\mathbf{J},\aleph)$, $\dim_{\mathbb{R}} = 2k$, with holomorphic local coordinates z^{α} when the coefficients of the Hermitian metric d–tensor and fundamental form are respectively $\aleph_{\alpha\overline{\beta}} := \aleph(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \overline{z}^{\alpha}})$ and $\,_{\natural}^{e}\theta = i\aleph_{\alpha\overline{\beta}} dz^{\alpha} \wedge d\overline{z}^{\beta}$.

Using N–adapted frame transforms, $\,_{\natural}^{\mathbf{e}}\theta = i\aleph_{\alpha'\overline{\beta}'}\ {}^{\natural}\mathbf{e}^{\alpha'} \wedge\ {}^{\natural}\overline{\mathbf{e}}^{\beta'}$, for $\aleph_{\alpha'\overline{\beta}'} =\ {}^{\natural}\mathbf{e}^{\alpha}_{\ \alpha'},\ {}^{\natural}\overline{\mathbf{e}}^{\beta}_{\ \beta'}, \aleph_{\alpha\overline{\beta}}$.

We can extract (almost) Kähler configurations if the fundamental form $\,_{\natural}^{\mathbf{e}}\theta \to \,_{\natural}^{\mathbf{K}}\theta$ is closed. Such a form can be expressed locally using a real function $q(z^{\alpha}, \overline{z}^{\alpha})$, when $\,_{\natural}^{K}\theta := i\partial\overline{\partial}q$, for $\aleph_{\alpha\overline{\beta}} = {}^{K}\aleph_{\alpha\overline{\beta}} = \frac{\partial^2 q}{\partial z^{\alpha}\partial \overline{z}^{\beta}}$, or $\,_{\natural}^{K}\theta := i\ _{\natural}\mathbf{e}\ _{\natural}\overline{\mathbf{e}}q$.

Definition:

- *A nonholonomic Hermitian d–metric* $\aleph$ *on* $(\mathbf{Y},\ _{\natural}\mathbf{J})$ *is called an almost Kähler d–metric if the fundamental form* $\,_{\natural}^{\mathbf{e}}\theta(\cdot,\cdot) := \aleph(\ _{\natural}\mathbf{J}\cdot,\cdot)$ *is closed, i.e.* $d\,_{\natural}^{\mathbf{e}}\theta = 0$, *but it is (in general) with non-vanishing Neijenhuis tensor* ${}^{\natural}{}^{\mathbf{J}}\Omega$.
- *Such a d–metric* $\aleph$ *is called a Kähler d–metric if* $\,_{\natural}\mathbf{J}$ *is a complex structure,* $\,_{\natural}^{\mathbf{e}}\theta = \,_{\natural}^{\mathbf{K}}\theta$ *is closed,* $d\,_{\natural}^{\mathbf{K}}\theta = \mathbf{0}$ *and* ${}^{\natural}{}^{\mathbf{J}}\Omega = 0$. *A local real function q is a local Kähler potential if* $\,_{\natural}^{\mathbf{K}}\theta = i\partial\overline{\partial}q$.

Theorem: *Prescribing on holomorphic manifold* $\mathbf{Y}$ *a fundamental generating holomorphic function* $\,_{\natural}\mathcal{L} = q$ *with nondegenerate real part, we can construct a canonical almost Kähler nonholonomic model with fundamental geometric objects determined by the almost Kähler d–metric* ${}^{K}\aleph_{\alpha\overline{\beta}}$.

Nonholonomic almost Hermitian and Kähler structures

Comparison of preferred d–connections

Theorem–Definitions: *Let data* $[\mathbf{Y}, {}_{\natural}\mathbf{N}, {}^{\mathbf{K}}_{\natural}\theta(\cdot,\cdot) := \aleph({}_{\natural}\mathbf{J}\cdot,\cdot)]$ *define an almost Kähler geometric model on a nonholonomic holomorphic manifold* $\mathbf{Y}^{\mathbb{C}}$*. There are preferred linear connections uniquely and completely defined by the metric,* $\aleph$*, and/or, equivalently, almost symplectic,* ${}^{\mathbf{K}}_{\natural}\theta$*, structures for a prescribed N–connection* ${}_{\natural}\mathbf{N}$ *following such geometric principles:*

1. *The Levi–Civita connection* ${}_{\natural}\nabla$ *(in brief,* ∇*) is determined by* $\aleph$*: a)* ${}_{\natural}\nabla\aleph = 0$*, and b) zero torsion.*
2. *The canonical d–connection* ${}^{\shortmid\shortmid}_{\natural}\mathbf{D}$ *is determined by data* $(\aleph, {}_{\natural}\mathbf{N})$ *by the conditions: a)* ${}^{\shortmid\shortmid}_{\natural}\mathbf{D}\aleph = 0$*, and b)* ${}^{\shortmid\shortmid}_{\natural}h\mathcal{T}^{\alpha} = 0$ *and* ${}^{\shortmid\shortmid}_{\natural}v\mathcal{T}^{\alpha} = 0$ *but* $\exists\, {}^{\shortmid\shortmid}_{\natural}hv\mathcal{T}^{\alpha} \neq 0$*; by unique distortion relation* ${}^{\shortmid\shortmid}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\shortmid\shortmid}_{\natural}\mathbf{Z}$
3. *The Cartan d–connection* ${}^{\shortmid}_{\natural}\mathbf{D}$ *determined by the data* $({}^{\shortmid}\aleph, {}^{\shortmid}_{\natural}\mathbf{N})$ *stated by a generating function* ${}_{\natural}\mathcal{L}$ *by extending on holonomorphic manifolds the constructions with the normal d–connection,* ${}^{\shortmid}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\shortmid}_{\natural}\mathbf{Z}$*.*
4. *The almost Kähler - Cartan d-connection,* ${}^{\theta}_{\natural}\mathbf{D} \simeq {}^{\shortmid}_{\natural}\mathbf{D}$*, is constructed as the Cartan d–connection but with fundamental generating holomorphic function* ${}_{\natural}\mathcal{L} = q$ *and geometric objects determined by the almost Kähler d–metric* ${}^{K}\aleph_{\alpha\overline{\beta}}$ *and almost symplectic nonholonomic variables.*
5. *The Chern d–connection* ${}^{\mathbf{c}}_{\natural}\mathbf{D}$*, or* ${}^{\shortmid\mathbf{c}}_{\natural}\mathbf{D}$*, is a unique* **H***–connection that* ${}_{\natural}\mathbf{D}^{0,1} = {}_{\natural}\overline{\mathbf{e}}$*, or* ${}^{\shortmid}_{\natural}\mathbf{D}^{0,1} = {}^{\shortmid}_{\natural}\overline{\mathbf{e}}$*, which can be N–adapted to a general* ${}_{\natural}\mathbf{N}$*, or* ${}^{\shortmid}_{\natural}\mathbf{N}$ *structure, with dependence on d–metric or fundamental 1–form encoded into respective distortion d–tensors,* ${}^{\mathbf{c}}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\mathbf{c}}_{\natural}\mathbf{Z}$ *and* ${}^{\shortmid\mathbf{c}}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\shortmid\mathbf{c}}_{\natural}\mathbf{Z}$*.*
6. *The Chern connection* ${}^{\mathbf{c}}_{\natural}D$*, or* ${}^{\shortmid\mathbf{c}}_{\natural}D$*, is a unique* **H***–connection that* ${}^{\mathbf{c}}_{\natural}D^{0,1} = \overline{\partial}$*, or* ${}^{\shortmid\mathbf{c}}_{\natural}D^{0,1} = \overline{\partial}$*,* ${}^{\mathbf{c}}_{\natural}D = {}_{\natural}\nabla + {}^{\mathbf{c}}_{\natural}Z$ *and* ${}^{\shortmid\mathbf{c}}_{\natural}D = {}_{\natural}\nabla + {}^{\shortmid\mathbf{c}}_{\natural}Z$*.*

Nonholonomic almost Hermitian and Kähler structures

Using the last Theorem, we can generalize for holomorphic nonholonomic manifolds:

Corollary: *The almost Kähler - Cartan d-connection* ${}^{\theta}_{\natural}\mathbf{D} \simeq {}^{\shortmid}_{\natural}\mathbf{D}$ *is a unique almost symplectic d–connection which satisfies the conditions* ${}^{\theta}_{\natural}\mathbf{D}\, {}^{\shortmid}_{\natural}\theta = 0$ *and* ${}^{\theta}_{\natural}\mathbf{D}\, {}^{\shortmid}_{\natural}\mathbf{J} = 0$ *and can be constructed if there are prescribed any data* $(\mathbf{g}, \mathbf{N}) \approx (\aleph, {}^{\shortmid}_{\natural}\mathbf{N}) \approx ({}_{\natural}\theta, {}_{\natural}\mathbf{J}) \approx ({}^{\shortmid}_{\natural}\theta, {}^{\shortmid}_{\natural}\mathbf{J})$ *on a holomorphic nonholonomic manifold.*

On curvatures on almost complex nonholonomic manifolds

$\forall\, {}_{\natural}\mathbf{D}$ on (real or holomorphic) $(\mathbf{Y}, {}_{\natural}\mathbf{N}, {}_{\natural}\mathbf{g})$, $dim_{\mathbb{C}}\mathbf{V} = k$, the curvature tensor is defined in standard form
${}_{\natural}\mathcal{R}(\mathbf{X}, \mathbf{A})\mathbf{Z} := ({}_{\natural}\mathbf{D}_{\mathbf{X}}\, {}_{\natural}\mathbf{D}_{\mathbf{A}} - {}_{\natural}\mathbf{D}_{\mathbf{A}}\, {}_{\natural}\mathbf{D}_{\mathbf{X}} - {}_{\natural}\mathbf{D}_{[\mathbf{X},\mathbf{A}]})\mathbf{Z}, \qquad \forall \mathbf{X}, \mathbf{A}, \mathbf{Z} \in \mathcal{C}^{\infty}({}_{\natural}T\mathbf{Y}).$

This tensor is identified with a h–projection on ${}_{\natural}T\mathbf{Y}$, ${}_{\natural}\mathcal{R}(\mathbf{X}, \mathbf{A}, \mathbf{Z}, \mathbf{B}) := {}_{\natural}h({}_{\natural}\mathcal{R}(\mathbf{X}, \mathbf{A})\mathbf{Z}, \mathbf{B}), \forall \mathbf{X}, \mathbf{A}, \mathbf{Z}, \mathbf{B} \in {}_{\natural}T\mathbf{Y}$.

The Ricci tensor of ${}_{\natural}\mathbf{D}$ is defined by ${}_{\natural}\mathcal{R}ic(\mathbf{X}, \mathbf{A}) := Tr\{\mathbf{B} \to {}_{\natural}\mathbf{R}(\mathbf{B}, \mathbf{X}), \mathbf{A}\}$.

Encode the modified Ricci soliton / Einstein equations for real theories are ${}_{\natural}\mathcal{R}ic(\mathbf{X}, \mathbf{A}) := \lambda(z, \overline{z})\mathbf{g}(\mathbf{X}, \mathbf{A})$.

For almost Kähler - Cartan models, a similar Ricci form ${}^{\shortmid}_{\natural}\rho(\mathbf{X}, \mathbf{A}) := {}^{\shortmid}_{\natural}\mathcal{R}ic({}^{\shortmid}_{\natural}\mathbf{J}\mathbf{X}, \mathbf{A}) = -i\, {}^{\shortmid}_{\natural}\mathbf{e}\, {}^{\shortmid}_{\natural}\overline{\mathbf{e}} \log\det |{}^{K}\aleph_{\alpha'\overline{\beta'}}|$,
for $\aleph_{\alpha'\overline{\beta}'} = {}^{\natural}\mathbf{e}^{\alpha}_{\ \alpha'}\, {}^{\natural}\overline{\mathbf{e}}^{\beta}_{\ \beta'} \aleph_{\alpha\overline{\beta}}$, which is closed in N–adapted form, ${}^{d}_{\natural}\mathbf{e}({}^{\shortmid}_{\natural}\rho) = 0$.

The main geometric idea is to perform a necessary type geometric quantization for certain data
$[{}^{\theta}_{\natural}\mathbf{D}, (\mathbf{g}, \mathbf{N}) \approx (\aleph, {}^{\shortmid}_{\natural}\mathbf{N}) \approx ({}_{\natural}\theta, {}_{\natural}\mathbf{J}) \approx ({}^{\shortmid}_{\natural}\theta, {}^{\shortmid}_{\natural}\mathbf{J})]$ and then to deform them into generalized ones with $[{}_{\natural}\mathbf{D}, {}_{\natural}\theta, {}_{\natural}\mathbf{J}, {}_{\natural}\aleph]$.

The distortion of the Riemann tensor is computed: ${}^{\shortmid}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\shortmid}_{\natural}\mathbf{Z}$ and ${}^{\shortparallel}_{\natural}\mathbf{D} = {}_{\natural}\nabla + {}^{\shortparallel}_{\natural}\mathbf{Z}$ into corresponding formula when
${}^{\shortmid}_{\natural}\mathcal{R} = {}^{\nabla}_{\natural}\mathcal{R} + {}^{\shortmid}_{\natural}\mathcal{Z}$ and ${}^{\shortparallel}_{\natural}\mathcal{R} = {}^{\nabla}_{\natural}\mathcal{R} + {}^{\shortparallel}_{\natural}\mathcal{Z}$

${}^{\shortmid}_{\natural}\mathcal{R}ic = {}^{\nabla}_{\natural}\mathcal{R}ic + {}^{\shortmid}_{\natural}\mathcal{Z}ic$ and ${}^{\shortparallel}_{\natural}\mathcal{R}ic = {}^{\nabla}_{\natural}\mathcal{R}ic + {}^{\shortparallel}_{\natural}\mathcal{Z}ic$.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

We reformulate some most important results about polarizations for quantum line bundles endowed with nonlinear connection structure.

Polarizations, almost Kähler nonholonom strs & quantum line bundles

Consider a connected compact nonholonomic complex manifold $\mathbf{Y}, \dim_{\mathbb{C}} \mathbf{Y} = k$, and define a polarization of this manifold as a positive holomorphic line bundle $\mathbf{L}$ over $\mathbf{Y}$.

Work with polarized complex N–anholonomic manifolds $(\mathbf{Y}, \mathbf{L})$ when $\mathbf{Y}$ can be presented as a projective algebraic variety by the Kodaira embedding determined by ${}^{s}\mathbf{L} := \mathbf{L}^{\otimes s}\ \forall s \geq s_0$, where s_0 is a positive integer.

Definition: *A Hodge nonholonomic manifold is defined by a pair* $(\mathbf{Y}, {}_{\natural}^{\shortmid}\theta)$ *with* ${}_{\natural}^{\shortmid}\theta$ *being integral, i.e. with its cohomology class* $[{}_{\natural}^{\shortmid}\theta] \in H^2(\mathbf{Y}, \mathbb{Z})$.

For a polarization $\mathbf{L}$ of $\mathbf{Y}$, the almost Kähler form is called nonholonomically L–polarized if $c_1(\mathbf{L}) = [{}_{\natural}^{\shortmid}\theta]$. We can define a triple $(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta)$ as a polarized Hodge nonholonomic manifold.

There is a standard result that any Hodge manifold admits Kähler polarizations.

Additional nonholonomic distributions with N–connection splitting do not change such a property which result in unique nonholonomic configurations when $\mathbf{Y}$ is simply connected. We can formulate an inverse statement that a polarized nonholonomic complex manifold $(\mathbf{Y}, \mathbf{L})$ admits Kähler metrics whose Kähler class equals $c_1(\mathbf{L})$.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

Proposition: *For any polarized nonholonomic manifold* $(\mathbf{Y}, \mathbf{L})$, *there exists a bijection between L–polarized admits Kähler metrics on* $\mathbf{Y}$ *and homothey (positive constant prefactor rescaling) classes of almost Hermitian nonholonomic bundle d–metrics on* $\mathbf{L}$.

Proof: Such bejections using standard ones for Kähler forms and than nonholonomically deforming the constructions.

- Taking a Hermitian d–metric $\aleph$ on $\mathbf{L}$, we construct a unique Kähler metric and d–metric $({}^{\mathbf{c}}_{\natural}D$, or ${}^{\shortmid\mathbf{c}}_{\natural}D$, and ${}^{\mathbf{c}}_{\natural}\mathbf{D})$.
  The formulas ${}^{\shortmid}_{\natural}\theta = \frac{i}{2\pi}\, {}^{\mathbf{c}}_{\natural}\mathcal{R}$, or ${}^{\shortmid}_{\natural}\theta = \frac{i}{2\pi}\, {}^{\shortmid\mathbf{c}}_{\natural}\mathcal{R}$, transform via ${}^{\shortmid}_{\natural}\mathcal{R} = {}^{\nabla}_{\natural}\mathcal{R} + {}^{\shortmid}_{\natural}\mathcal{Z}$ and ${}^{\mathbf{c}}_{\natural}\mathcal{R} = {}^{\nabla}_{\natural}\mathcal{R} + {}^{\mathbf{c}}_{\natural}\mathcal{Z}$ into

$$ {}^{\shortmid}_{\natural}\theta = \frac{i}{2\pi}({}^{\shortmid}_{\natural}\mathcal{R} - {}^{\shortmid}_{\natural}\mathcal{Z} + {}^{\mathbf{c}}_{\natural}\mathcal{Z}), \text{ or } {}^{\shortmid}_{\natural}\theta = \frac{i}{2\pi}({}^{\shortmid}_{\natural}\mathcal{R} - {}^{\shortmid}_{\natural}\mathcal{Z} + {}^{\shortmid\mathbf{c}}_{\natural}\mathcal{Z}), $$

  where both the left and write parts are parameterized in almost Kähler – Cartan variables and can be expressed via coefficients of an almost Hermitian d–metric $\aleph$.
  Multiplying $\aleph$ by a positive constant, we do not change the associated almost Kähler d–metric.
- The constructions can be inverted for prequantized Hodge nonholonomc manifold $(\mathbf{Y}, \mathbf{L}, {}^{\shortmid}_{\natural}\theta, \aleph)$, see similar constructions for holonomic configurations in Section 1 of (Lazaroiu et all).
  In our approach, we work with equivalence of $(\mathbf{L}, {}^{\shortmid}_{\natural}\mathbf{N}, \aleph)$ and $(\mathbf{L}', {}^{\shortmid}_{\natural}\mathbf{N}', \aleph')$ if there is an N-adapted isomorphism $\psi : \mathbf{L} \to \mathbf{L}'$ of holomorphic line bundles such that $\psi^*(\aleph') = \aleph$ and $\psi^*({}^{\shortmid}_{\natural}\mathbf{N}') = {}^{\shortmid}_{\natural}\mathbf{N}$.
  Equivalence classes of N–anholonomic quantum line bundles for $(\mathbf{Y}, \mathbf{L})$ for a distinguished $\mathbf{H}om(\pi_1(\mathbf{Y}), S^1)$–torsor splitting into respective h- and v–torsors, $Hom(\pi_1(h\mathbf{Y}), S^1)$–torsor and $Hom(\pi_1(v\mathbf{Y}), S^1)$–torsor.

□

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

For a nonholonomc quantum line bundle $(\mathbf{L}, {}^{\shortmid}_{\natural}\mathbf{N}, \aleph)$, consider on ${}^{s}\mathbf{L}$ an induced d–metric ${}^{s}\aleph := \aleph^{\otimes s}$; corresponding Chern d–connection ${}^{\mathbf{sc}}_{\natural}\mathbf{D} := ({}^{\mathbf{c}}_{\natural}\mathbf{D})^{\otimes s}$, or any distorted preferred connections; the almost Kähler - Cartan ${}^{s\theta}_{\natural}\mathbf{D} := ({}^{\theta}_{\natural}\mathbf{D})^{\otimes s}$.

We can introduce the almost symplectic fundamental form ${}^{s\shortmid}_{\natural}\theta$ on ${}^{s}\mathbf{L}$, which can be identified (up to respective coefficients) to the Chern d–connection, ${}^{\mathbf{s}\shortmid\mathbf{c}}_{\natural}\mathcal{R} := ({}^{\shortmid\mathbf{c}}_{\natural}\mathcal{R})^{\otimes s}$. We obtain ${}^{s\shortmid}_{\natural}\theta = \frac{i}{2\pi}\,{}^{\mathbf{sc}}_{\natural}\mathcal{R}$, or ${}^{s\shortmid}_{\natural}\theta = \frac{i}{2\pi}\,{}^{\mathbf{s}\shortmid\mathbf{c}}_{\natural}\mathcal{R}$,

Fixing a positive measure μ on $\mathbf{Y}$, define an induced almost Hermitian scalar product on the space of sooth sections $Sec({}^{s}\mathbf{L})$, ${}^{\mu,\aleph}_{s}\langle s_1, s_2\rangle := \int d\mu\, {}^{s}\aleph(\mathbf{s_1}, \mathbf{s_2})$, $\forall \mathbf{s_1}, \mathbf{s_2} \subset Sec({}^{s}\mathbf{L})$.

This way we can perform a L^2–completion of $Sec({}^{s}\mathbf{L})$ to ${}^{s}\mathbf{L}(\mathbf{L}, \aleph, \mu)$ with such a scalar product which admits further *h*- and *v*–decompositions because a nontrivial N–connection structure. Here we note that the finite–dimensional subspace of holomorphic nonholonomic sections, $H^0({}^{s}\mathbf{L}) \subset Sec({}^{s}\mathbf{L})$, also contains an induced scalar product (the same symbol).

Theorem: $\exists\mu$ *standard identified with the Liouville measure determined by the canonical volume form* $\frac{({}^{\shortmid}_{\natural}\theta)^s}{s!}$ *of* $(\mathbf{Y}, {}^{\shortmid}_{\natural}\theta)$ *when the N–adapted scalar product* ${}^{\aleph}_{s}\langle s_1, s_2\rangle := \int \frac{({}^{\shortmid}_{\natural}\theta)^s}{s!}\,{}^{s}\aleph(\mathbf{s_1}, \mathbf{s_2})$, *with splitting* $\mathbf{s_1} = hs_1 + vs_1$ *and* $\mathbf{s_2} = hs_2 + vs_2$,

$$
\begin{aligned}
{}^{\aleph}_{s}\langle s_1, s_2\rangle &= {}^{\aleph}_{s}\langle hs_1, hs_2\rangle + {}^{\aleph}_{s}\langle vs_1, vs_2\rangle,\\
{}^{\aleph}_{s}\langle hs_1, hs_2\rangle &= \int \frac{(h\,{}^{\shortmid}_{\natural}\theta)^s}{s!}\,{}^{s}\aleph(hs_1, hs_2) \text{ and } {}^{\aleph}_{s}\langle vs_1, vs_2\rangle = \int \frac{(v\,{}^{\shortmid}_{\natural}\theta)^s}{s!}\,{}^{s}\aleph(vs_1, vs_2),
\end{aligned}
$$

encode the information from possible solutions of the (nonholonomic) Ricci soliton and (modified) Einstein equations.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

Canonical authomorphis of a prequantized Hodge d–manifold and smooth N–adapted scalar products

The geometry of nonholonomic almost Kähler – Cartan and Ricci solitons has a rich structure which is characterized by a series of new properties.

Definition *A N–adapted automorphism of a prequantized Hodge nonhlolonomic manifold* $(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph)$ *is a pair* $\gamma := (\gamma_0 = h\gamma_0 + v\gamma_0, \gamma_1 = h\gamma_1 + v\gamma_1)$ *with h- and v–splitting when* γ_0 *is a N–adapted holomorphic ismotery of* $(\mathbf{Y}, {}_{\natural}^{\shortmid}\theta)$ *and* γ *is a holonmorphic bundle isometry of* $(\mathbf{L}, \aleph)$ *above* γ_0. *Such N–adapted automorphisms form a d–group denoted* $\mathbf{A}ut(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph) := h\mathbf{A}ut \oplus v\mathbf{A}ut$ *with Whitney sum induced by the N–connection structure.*

In particular, we can consider that in above Definition $\gamma_1(u)$ is a N-adapted isometry from $(\mathbf{L}_u, \aleph(u))$ to $(\mathbf{L}_{\gamma_0(u)}, \aleph(\gamma_0(u)))$ $\forall u \in \mathbf{Y}$. An automorphism γ is trivial if $\gamma_0 = id_{\mathbf{Y}}$ and $\gamma_1(u) = (e^{i\alpha})\cdot$ for a real constant α $\forall u \in \mathbf{Y}$, i.e. $U(1)$ is contained as a subgroup in $\mathbf{A}ut(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph)$. In general, this group acts linearly on the space of sections $H^0({}^s\mathbf{L})$. The actions $\rho_k : \mathbf{A}ut(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph) \to End(H^0({}^s\mathbf{L}))$ are unitary with respect to the L^2–scalar product ${}_s^{\aleph}\langle s_1, s_2\rangle$ from above Theorem.

Using the quotients resulting in a subgroup of respective d–group,

$$\mathbf{A}ut_{\mathbf{L},\aleph}(\mathbf{Y}, {}_{\natural}^{\shortmid}\theta) := \mathbf{A}ut(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph)/U(1) \subset \mathbf{A}ut(\mathbf{Y}, {}_{\natural}^{\shortmid}\theta),$$

we select those holomorphic N–adapted isometries γ_0 of the almost Kähler – Cartan manifold $(\mathbf{Y}, {}_{\natural}^{\shortmid}\theta)$ which admit a N–adapted lift $\gamma_1 : \mathbf{L} \to \mathbf{L}$ such that the pair (γ_0, γ_1) is an automorphism of $(\mathbf{Y}, \mathbf{L}, {}_{\natural}^{\shortmid}\theta, \aleph)$.

Putting together such considerations, we prove

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

Theorem: *There is an N–adapted sequence of d–groups*

$$\begin{array}{rcl} 1 \to U(1) & \to & \mathbf{A}ut(\mathbf{Y}, \mathbf{L}, \natural\theta, \aleph) \to \mathbf{A}ut_{\mathbf{L},\aleph}(\mathbf{Y}, \natural\theta) \to 1 \\ \text{with h- and v–splitting} \quad 1 & \to & U(1) \to h\mathbf{A}ut(\mathbf{Y}, \mathbf{L}, h\natural\theta, h\aleph) \to h\mathbf{A}ut_{\mathbf{L},h\aleph}(\mathbf{Y}, h\natural\theta) \to 1 \text{ and} \\ 1 & \to & U(1) \to v\mathbf{A}ut(\mathbf{Y}, \mathbf{L}, v\natural\theta, v\aleph) \to v\mathbf{A}ut_{\mathbf{L},h\aleph}(\mathbf{Y}, v\natural\theta) \to 1. \end{array}$$

The constructions related to above Theorem are standard ones if the de Rham and Dolbeaut operators, respectively, $d = \partial + \overline{\partial}$ and ∂, are used in definition of the standard Chern connection $\natural^{\mathbf{c}} D$ for the data $(\mathbf{L}, \aleph)$. Using local complex coordinate frames on an open set $U_\sigma := \{u \in \mathbf{Y} | \sigma(u) \neq 0\}$, we construct standard relations for the Kähler geometry with local potential of type.

We have $\natural\theta := i\partial\overline{\partial}q$ for $q = -\log \natural\aleph(\sigma, \sigma)$, when $\natural^{\mathbf{c}} D \equiv d + \partial q$, $\natural\theta = \frac{i}{2\pi} \natural^{\mathbf{c}}\mathcal{R}$ with $\natural^{\mathbf{c}}\mathcal{R} = -\partial\overline{\partial}q = -2\pi i \natural\theta$.

Any section $S \in Sec({}^{s}\mathbf{L})$ considered above U_σ can be written in the form $s = f\sigma^{\otimes s}$ for a smooth complex–valued function f on U_σ. We consider both S and f to be holomorphic and use a measure $\mu(\mathbf{Y} \setminus U_\sigma) = 0$. This provides a proof:

Corollary: *There are N–adapted isometries of $Sec({}^{s}\mathbf{L})$ and $H^0({}^{s}\mathbf{L})$ with the spaces of smooth, respectively holomoprhic functions on U_σ endowed with scalar product*

$${}^{\sigma}_{s}\langle f_1, f_2\rangle = \int_{U_\sigma} d\mu e^{-sq_\sigma} \overline{f}_1, f_2.$$

This identifies ${}^{s}\mathbf{L}^2(\mathbf{L}, \aleph, \mu)$ with the space ${}^{s}\mathbf{L}^2(U_\sigma, e^{-sq_\sigma}\mu)$.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

Almost Hermitian bundle d–metrics & polarized almost Kähler – Cartan forms

Let us fix a polarized complex $(\mathbf{Y}, \mathbf{L})$ and denote by $\widetilde{\mathbf{L}}$ the total space of $\mathbf{L}$ and by $\widetilde{\mathbf{L}}_0$ be the total space with the graph *0* of the zero section excluded. Consider square norm functions $\widetilde{\aleph} \in \mathcal{C}^{\infty}(\widetilde{\mathbf{L}}_0, \mathbb{R}_+)$ of type $\widetilde{\aleph}(\widetilde{u}) := \aleph(\widetilde{u}, \widetilde{u})$, for $\widetilde{u} \in \widetilde{\mathbf{L}}$, when these smooth non–negative functions on $\widetilde{\mathbf{L}}$ are strictly positive on $\widetilde{\mathbf{L}}_0$ and have the property $\widetilde{\aleph}(c\widetilde{u}) = |c|\widetilde{\aleph}(\widetilde{u})\ \forall \widetilde{u} \in \widetilde{\mathbf{L}}$ and $\forall\ c \in \mathbb{C}$, for $\widetilde{\aleph}_{|0} = 0$. This way, the Hermitian metrics $\aleph$ on $\mathbf{L}$ are uniquely determined by $\widetilde{\aleph}$, i.e. the set $Met\{\mathbf{L}, \aleph\}$ of Hermitan metrics on $\mathbf{L}$ can be identified with the set of functions $\{\widetilde{\aleph}(\widetilde{u})\}$ which form an infinite–dimensional convex cone in $\mathcal{C}^{\infty}(\widetilde{\mathbf{L}}_0, \mathbb{R})$.

Fix a polarized complex $(\mathbf{Y}, \mathbf{L})$ and denote by $\widetilde{\mathbf{L}}$ the total space of $\mathbf{L}$ and by $\widetilde{\mathbf{L}}_0$ be the total space with the graph *0* of the zero section excluded. Consider square norm functions $\widetilde{\aleph} \in \mathcal{C}^{\infty}(\widetilde{\mathbf{L}}_0, \mathbb{R}_+)$ of type $\widetilde{\aleph}(\widetilde{u}) := \aleph(\widetilde{u}, \widetilde{u})$, for $\widetilde{u} \in \widetilde{\mathbf{L}}$, when these smooth non–negative functions on $\widetilde{\mathbf{L}}$ are strictly positive on $\widetilde{\mathbf{L}}_0$ and property $\widetilde{\aleph}(c\widetilde{u}) = |c|\widetilde{\aleph}(\widetilde{u})\ \forall \widetilde{u} \in \widetilde{\mathbf{L}}$ and $\forall\ c \in \mathbb{C}$, for $\widetilde{\aleph}_{|0} = 0$.

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We use a parametrization for the case when $\mathbf{L}$ is very ample (see details and references in Lazaroiu et all which have a straightforward extension to nonholonomic configurations).
It is used the so–called evaluation functional $\widetilde{u} : H^0(\mathbf{L}) \rightarrow \mathbb{C}$ constructed as a N–adapted linear functional

$$\zeta(\pi(\widetilde{u})) = \widehat{u}(\zeta)\widetilde{u},\ \zeta \in H^0(\mathbf{L})\ \forall \widetilde{u} \in \widetilde{\mathbf{L}}_0,$$

where $\pi : \widetilde{\mathbf{L}} \rightarrow \mathbf{Y}$ is the N–adapted bundle projection. For non-vanishing c, $\widehat{cu} = c^{-1}\widehat{u}$ and the condition of very ampleness implies $\widehat{u} \neq 0\ \forall \widetilde{u} \in \widetilde{\mathbf{L}}_0$.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

The N–adapted Bergman d–metric

An almost Hermitian d–metric determines a N–adapted scalar product $(\ ,\ ) = {}^{h}(\ ,\ ) + {}^{v}(\ ,\ )$; on the finite–dimensional nonholonomic space $H^0(\mathbf{L})$ induces a h- and v–scalar product on the dual spaces $H^0(\mathbf{L})^* = Hom_{\mathbb{C}}\left(H^0(\mathbf{L}), \mathbb{C}\right)$.

Theorem–Definition: *There is a N–adapted version of the Bergman metric, i.e. a Bergman d–metric on* $\mathbf{L}$ *defined by the - and v–scalar product* $(\ ,\ ) = {}^{h}(\ ,\ ) + {}^{v}(\ ,\ )$ *which allows to considder a reference Hermitian d–metric for any* $\widetilde{u} \in \widetilde{\mathbf{L}}_0, \widetilde{\aleph}_B(\widetilde{u}) = \parallel \widehat{u} \parallel^{-2}$ *and* $\widetilde{\aleph}_{B|0} = 0.$

This Theorem–Definition generalizes for nonholonomic configurations the results related to Bergman metrics. Because this is a d–metric, we have h- and v–components $h\widetilde{\aleph}_B(\widetilde{x}) = \parallel \widehat{x} \parallel^{-2}$ and $v\widetilde{\aleph}_B(\widetilde{y}) = \parallel \widehat{y} \parallel^{-2}$, when $u = (x, y)$.

Corollary: *Having a reference d–metric* $\widetilde{\aleph}_B$*, we can describe any other almost Hermitian d–metric* $\aleph$ *via the postive epsilon function of* $\aleph$, $\epsilon := \frac{\widetilde{\aleph}(\widetilde{u})}{\widetilde{\aleph}_B(\widetilde{u})} \in \mathcal{C}^{\infty}(\mathbf{Y}, \mathbb{R}_+^*)$ *relative to* $(\ ,\ ) = {}^{h}(\ ,\ ) + {}^{v}(\ ,\ )$*, when* $\aleph(\widetilde{u}, \widetilde{u}) = \epsilon(\pi(\widetilde{u}))\aleph_B(\widetilde{u}, \widetilde{u})$.

The function ϵ splits into h- and v–components, respectively, ${}^{h}\epsilon := \frac{h\widetilde{\aleph}(\widetilde{x})}{h\widetilde{\aleph}_B(\widetilde{x})}$ and ${}^{v}\epsilon := \frac{v\widetilde{\aleph}(\widetilde{y})}{v\widetilde{\aleph}_B(\widetilde{y})}$.

Definition: *The* $\mathbf{L}$*–polarized almost Kähler – Cartan metric on* $\mathbf{Y}$ *determined by* $\widetilde{\aleph}_B$ *is called the Bergman d–metric on* $\mathbf{Y}$ *induced by a distinguished scalar product* $(\ ,\ ) = {}^{h}(\ ,\ ) + {}^{v}(\ ,\ )$*. Its almost Kähler – Cartan form is denoted* ${}^{\shortmid}_{\natural}\theta_B$.

Remark: *The almost Kähler – Cartan form determined by the Hermitian nonholonomic bundle d–metric is* ${}^{\shortmid}_{\natural}\theta = {}^{\shortmid}_{\natural}\theta_B - \frac{i}{2\pi}\partial\overline{\partial}\log\epsilon$ *and* ${}_{\natural}\theta = {}^{\shortmid}_{\natural}\theta_B - \frac{i}{2\pi}\, {}^{\shortmid}_{\natural}\mathbf{e}\, {}^{\shortmid}_{\natural}\overline{\mathbf{e}}\log\epsilon$.

The Bergman d–metric provides a framework for geometric quantization.

Nonholonomic Quantum Line Bundles and Generalized Bergman Metrics

Nonholonomically induced scalar products and Bergman d–metric

For a N–anholonomic **Y**, $\dim_{\mathbb{C}} \mathbf{Y} = n$, we associate $n+1 := \dim_{\mathbb{C}} H^0(\mathbf{L})$ when $\zeta_{\underline{\alpha}} \in H^0(\mathbf{L})$ define a basis.
Parameterize the square norm $\forall \widetilde{u} \in \widetilde{\mathbf{L}}_0$ as$\| \ \widehat{u} \ \|^2 = \Xi^{\underline{\alpha}\underline{\beta}} \overline{\widehat{u}(\zeta_{\underline{\alpha}})} \widehat{u}(\zeta_{\underline{\beta}})$, where $\Xi^{\underline{\alpha}\underline{\beta}}$ is the inverse matrix to $\Xi_{\underline{\alpha}\underline{\beta}} := (\zeta_{\underline{\alpha}}, \zeta_{\underline{\beta}})$.
This is a d–metric with splitting in N–adapted frames as $\Xi_{\underline{\alpha}\underline{\beta}} = \{\Xi_{\underline{00}}, \Xi_{\underline{ij}}, \Xi_{\underline{ab}}\}$, when $\zeta_{\underline{\alpha}} = (\zeta_{\underline{0}}, \zeta_{\underline{j}}, \zeta_{\underline{b}})$.
The respective Bergman d–metric takes such a form

$$\aleph_B(\widetilde{u}, \widetilde{u}) = [\Xi^{\underline{\alpha}\underline{\beta}} \overline{\widehat{u}(\zeta_{\underline{\alpha}})} \widehat{u}(\zeta_{\underline{\beta}})]^{-1}.$$

We can compute the epsilon function of arbitrary Hermitian d–metric $\aleph$ on **L** following formula $\epsilon(\widetilde{u}) = \Xi^{\underline{\alpha}\underline{\beta}} \aleph(\widetilde{u}) \left(\zeta_{\underline{\alpha}}(\widetilde{u}) \zeta_{\underline{\beta}}(\widetilde{u}) \right)$.
Inversely, we can express an almost Hermitian d–metric $\aleph$ in terms of its relative epsilon function $\aleph(\widetilde{u}, \widetilde{u}) = \epsilon(\widetilde{u}) \aleph_B(\widetilde{u}, \widetilde{u})$.
The L^2–scalar distinguished product on $H^0(\mathbf{L})$ is defined by $\aleph_B$ and the volume form of ${}_{\natural}^{\shortmid}\theta_B$ when

$$\langle {}^1\zeta, {}^2\zeta \rangle = \int_{\mathbf{Y}} \frac{({}_{\natural}^{\shortmid}\theta_B)^n}{n!} \aleph_B({}^1\zeta, {}^2\zeta)$$

for ${}^1\zeta, {}^2\zeta \in H^0(\mathbf{L})$. In general, $\langle \ , \ \rangle$ and do not coincide with the distinguished scalar product $(\ , \ ) = {}^h(\ , \ ) + {}^v(\ , \ )$.
We note that algebraically we can work in a similar fashion both with holonomic and nonholonomic structures even in the last case there is a specific *h*- and *v*-dubbing.

Further developments are possible by considering projective spaces with Kodaira embedding like in section 2.3 of Lazaroiu et all,

see also corresponding references therein. We omit such considerations with N–splitting in this work.

Ricci Solitons & the Karabegov–Schlichenmaier DQ

Aim: Perform DQ using N–adapted frames (for Fedosov operators), the Cartan d–connection and distortions with Neijenhuis tensor, $\rightarrow$ star product.

$$\check{\Gamma}^{\alpha'}_{\ \beta'\gamma'} = \check{\mathbf{e}}^{\alpha'}_{\ \alpha}\check{\mathbf{e}}^{\ \beta}_{\beta'}\check{\mathbf{e}}^{\ \gamma}_{\gamma'}\Gamma^{\alpha}_{\ \beta\gamma} + \check{\mathbf{e}}^{\alpha'}_{\ \alpha}\mathbf{e}_{\gamma}(\check{\mathbf{e}}^{\ \alpha}_{\beta'}),\ \check{\Gamma}' = \Gamma + \check{Z}$$

$\check{\mathbf{e}}_{\nu'} = \check{\mathbf{e}}^{\ v}_{v'}(u)\mathbf{e}_{\nu}$ $\check{\mathbf{e}}^{\nu'} = \check{\mathbf{e}}^{v'}_{\ v}(u)\mathbf{e}^{v}$, new sets $\check{\mathbf{N}} = \{\check{N}^{a'}_{j}\}$ when $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma} = (1/4)\check{\Omega}^{\alpha}_{\ \beta\gamma}$.

"Formal power" series and Wick product

$C^{\infty}(\mathbf{V})[[\ell]]$ of "formal series" on ℓ with coefficients from $C^{\infty}(\mathbf{V})$ on a Poisson $(\mathbf{V}, \{\cdot,\cdot\})$, where the bracket $\{\cdot,\cdot\}$. Operator

$$^{1}f * {}^{2}f = \sum_{r=0}^{\infty} {}_{r}C(\,^{1}f,\,^{2}f)\,\ell^{r},$$

${}_{r}C, r \geq 0$, are bilinear operators with ${}_{0}C(\,^{1}f,\,^{2}f) = \,^{1}f\,^{2}f$ and ${}_{1}C(\,^{1}f,\,^{2}f) - {}_{1}C(\,^{2}f,\,^{1}f) = i\{\,^{1}f,\,^{2}f\}$; $i^{2} = -1$; an associative algebra structure on $C^{\infty}(\mathbf{V})[[\ell]]$ with a ℓ–linear and ℓ–addical continuous star product.

Local coordinates $(u,z) = (u^\alpha, z^\beta)$, on $T\mathbf{V}$; elements as series

$$a(v,z) = \sum_{r\geq 0, |\{\alpha\}|\geq 0} a_{r,\{\alpha\}}(u) z^{\{\alpha\}}\ \ell^r, \quad \text{is a multi–index}\{\alpha\}$$

On $T_u\mathbf{V}$, a formal Wick product with $\check{\Lambda}^{\alpha\beta} := \check{\theta}^{\alpha\beta} - i\ \check{\mathbf{g}}^{\alpha\beta}$,

$$a \circ b\,(z) := \exp\left(i\frac{\ell}{2}\ \check{\Lambda}^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z^\beta_{[1]}}\right) a(z) b(z_{[1]})\ |_{z=z_{[1]}}$$

The d–connection extended on space $\check{\mathcal{W}} \otimes \check{\Lambda}$ to operator

$$\check{\mathbf{D}}\,(a \otimes \xi) := \left(\check{\mathbf{e}}_\alpha(a) - u^\beta\ \check{\Gamma}^\gamma_{\alpha\beta}\ {}^z\check{\mathbf{e}}_\alpha(a)\right) \otimes (\check{\mathbf{e}}^\alpha \wedge \xi) + a \otimes d\xi,$$

where ${}^z\check{\mathbf{e}}_\alpha$ is a similar to $\check{\mathbf{e}}_\alpha$ but depend on z–variables. This operator is a N–adapted $\deg_a$–graded derivation of the d–algebra $(\check{\mathcal{W}} \otimes \check{\Lambda}, \circ)$.

Fedosov N–adapted operators

Definition: The Fedosov N–adapted operators are

$$\check{\delta}(a) = \check{\mathbf{e}}^{\alpha} \wedge {}^{z}\check{\mathbf{e}}_{\alpha}(a) \text{ and } \check{\delta}^{-1}(a) = \begin{cases} \frac{i}{p+q} z^{\alpha}\, \check{\mathbf{e}}_{\alpha}(a), \text{ if } p+q>0, \\ 0, \text{ if } p=q=0, \end{cases}$$

where $a \in \check{\mathcal{W}} \otimes \Lambda$ is homogeneous w.r.t. the grading deg_s and deg_a with $deg_s(a) = p$ and $deg_a(a) = q$.

Theorem: Any d-metric/ equivalent symplectic structure, $\check{\theta}(\cdot,\cdot) := \mathbf{g}(\check{\mathbf{J}}\cdot,\cdot)$, define a flat canonical Fedosov d–connection $\check{\mathcal{D}} : -\check{\delta} + \check{\mathbf{D}} - \frac{i}{\ell} ad_{Wick}(r)$; $\check{\mathcal{D}}^2 = 0$; $\exists$ a unique element $r \in \check{\mathcal{W}} \otimes \check{\Lambda}$, $deg_a(r) = 1$, $\check{\delta}^{-1} r = 0$, solving $\check{\delta} r = \check{\mathcal{T}} + \check{\mathcal{R}} + \check{\mathbf{D}} r - \frac{i}{\ell} r \circ r$. This element is computed recursively,

$$r^{(0)} = r^{(1)} = 0,\ r^{(2)} = \check{\delta}^{-1} \check{\mathcal{T}},\ r^{(3)} = \check{\delta}^{-1}(\check{\mathcal{R}} + \check{\mathbf{D}} r^{(2)} - \frac{i}{\ell} r^{(2)} \circ r^{(2)}),$$

$$r^{(k+3)} = \check{\delta}^{-1}(\check{\mathbf{D}} r^{(k+2)} - \frac{i}{\ell} \sum_{l=0}^{k} r^{(l+2)} \circ r^{(l+2)}),\ k \geq 1,$$

$a^{(k)}$ is the *Deg*–homogeneous component of degree k of $a \in \check{\mathcal{W}} \otimes \check{\Lambda}$.

Main theorems for Fedosov–Ricci solitons

Analogs of torsion and curvature operators of $\check{\mathbf{D}}$ on $\check{\mathcal{W}} \otimes \check{\Lambda}$,

$$\check{\mathcal{T}} := \frac{z^{\gamma}}{2}\, \check{\theta}_{\gamma\tau}\, \check{\mathbf{T}}^{\tau}_{\alpha\beta}(u)\, \check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}, \qquad \check{\mathcal{R}} := \frac{z^{\gamma} z^{\varphi}}{4} \check{\theta}_{\gamma\tau} \check{\mathbf{R}}^{\tau}{}_{\varphi\alpha\beta}(u)\, \check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}$$

Properties: $\left[\check{\mathbf{D}}, \check{\delta}\right] = \frac{i}{\ell} ad_{Wick}(\check{\mathcal{T}})$ and $\check{\mathbf{D}}^2 = -\frac{i}{\ell} ad_{Wick}(\check{\mathcal{R}})$.
The bracket $[\cdot,\cdot]$ is the $\deg_a$–graded commutator of endomorphisms of $\check{\mathcal{W}} \otimes \check{\Lambda}$ and ad_{Wick} is defined via the $\deg_a$–graded commutator in $\left(\check{\mathcal{W}} \otimes \check{\Lambda}, \circ\right)$.

Theorem 1: A star–product on the almost Kähler model of a nonholonomic Ricci soliton is defined on $C^{\infty}(\mathbf{V})[[\ell]]$ by ${}^{1}f * {}^{2}f \doteq \sigma(\tau({}^{1}f)) \circ \sigma(\tau({}^{2}f))$, where the projection $\sigma: \check{\mathcal{W}}_{\mathcal{K}_{\mathcal{D}}} \to C^{\infty}(\mathbf{V})[[\ell]]$ onto the part of $\deg_s$–degree zero is a bijection and the inverse map $\tau: C^{\infty}(\mathbf{V})[[\ell]] \to \check{\mathcal{W}}_{\check{\mathcal{D}}}$ can be calculated recursively w.r..t the total degree Deg, $\tau(f)^{(0)} = f$,

$$\tau(f)^{(k+1)} = \check{\delta}^{-1}\left(\check{\mathbf{D}}\tau(f)^{(k)} - \frac{i}{v}\sum_{l=0}^{k} ad_{Wick}(r^{(l+2)})(\tau(f)^{(k-l)})\right), \text{ for } k \geq 0.$$

Main theorems for Fedosov–Ricci solitons

${}^{f}\xi$ is the Hamiltonian vector field for a function $f \in C^{\infty}(\mathbf{V})$ on $(\mathbf{V}, \check{\theta})$. Antisymmetric ${}^{-}C({}^{1}f, {}^{2}f) := \frac{1}{2}\left(C({}^{1}f, {}^{2}f) - C({}^{2}f, {}^{1}f)\right)$ of bilinear $C({}^{1}f, {}^{2}f)$.
A star–product is normalized if ${}_{1}C({}^{1}f, {}^{2}f) = \frac{i}{2}\{{}^{1}f, {}^{2}f\}$, $\{\cdot,\cdot\}$ is the Poisson bracket defined by $\check{\theta}$. For a normalized $*$, the bilinear ${}_{2}^{-}C$ is a de Rham–Chevalley 2–cocycle $\exists$ a unique closed 2–form $\check{\varkappa}$, ${}_{2}C({}^{1}f, {}^{2}f) = \frac{1}{2}\check{\varkappa}({}^{f_1}\xi, {}^{f_2}\xi) \forall {}^{1}f, {}^{2}f \in C^{\infty}(\mathbf{V})$.
Consider the class c_0 of a normalized star–product $*$ as the equivalence class $c_0(*) \doteqdot [\check{\varkappa}]$, computed as a unique 2–form,

$$\check{\varkappa} = -\frac{i}{8}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\check{\mathbf{R}}^{\tau}{}_{\alpha'\alpha\beta}\,\check{\mathbf{e}}^{\alpha}\wedge\check{\mathbf{e}}^{\beta} - i\,\check{\lambda},\ \text{for } \check{\lambda} = d\,\check{\mu},\ \check{\mu} = \frac{1}{6}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\,\check{\mathbf{T}}^{\tau}{}_{\alpha'\beta}\,\check{\mathbf{e}}^{\beta}.$$

The h- and v–projections $h\Pi = \frac{1}{2}(Id_h - iJ_h)$ and $v\Pi = \frac{1}{2}(Id_v - iJ_v)$. The final step is to compute the closed Chern–Weyl form

$$\check{\gamma} = -iTr\left[(h\Pi, v\Pi)\,\check{\mathbf{R}}\,(h\Pi, v\Pi)^{T}\right] = -iTr\left[(h\Pi, v\Pi)\,\check{\mathbf{R}}\right] = -\frac{1}{4}\check{\mathbf{J}}_{\tau}^{\ \alpha'}\check{\mathbf{R}}^{\tau}{}_{\alpha'\alpha\beta}\,\check{\mathbf{e}}^{\alpha}\wedge\check{\mathbf{e}}^{\beta}.$$

The canonical class is $\check{\varepsilon} := [\check{\gamma}] \rightarrow$ proof of

Theorem 2: The zero–degree cohomology coefficient $c_0(*)$ for the almost Kä hler

model of a nonholonomic Ricci soliton is $c_0(*) = -(1/2i)\,\check{\varepsilon}$.

Noncommutative Ricci Solitons: Dirac almost sympl.?

Data for (non) holonomic Ricci solitons and Einstein spaces encoded into almost Kähler data $(\tilde{\theta}, \tilde{\mathbf{J}}, {}_{\theta}\tilde{\mathbf{D}})$, $(\theta, \mathbf{J}, \mathbf{D})$, when $\mathbf{D}\theta = 0$ and $\mathbf{DJ} = 0$.
The almost symplectic structure $\theta \rightarrow$ non–degenerate Poisson structure $\rightarrow$ N–adapted and covariant product for noncommutative geometry (NC).

Definition: The canonical (Cartan) covariant star product

$$\alpha \tilde{\star} \beta := \sum_k \frac{\ell^k}{k!} \theta^{\mu_1 \nu_1} ... \theta^{\mu_k \nu_k} (\mathbf{D}_{\mu_1} \ldots \mathbf{D}_{\mu_k}) \cdot (\mathbf{D}_{\nu_1} \ldots \mathbf{D}_{\nu_k}).$$

$\tilde{\star}$ is adapted to N–connection, maps d–tensors into d–tensors. For $\mathbf{D} \rightarrow \nabla$, similar NC generalizations of Riemann geometry if θ is fixed for a symplectic manifold, $\tilde{\star} \rightarrow \star$, h- and v–splitting, $\tilde{\star} = ({}^h\tilde{\star}, {}^v\tilde{\star})$ if $\mathbf{D}_{\mu_1} = (\mathbf{D}_{i_1}, \mathbf{D}_{a_1})$,

$$\alpha \; ({}^h\tilde{\star})\beta = \sum_k \frac{\ell^k}{k!} \theta^{i_1 j_1} ... \theta^{i_k j_k} (\mathbf{D}_{i_1} \ldots \mathbf{D}_{i_k}) \cdot (\mathbf{D}_{j_1} \ldots \mathbf{D}_{j_k})$$

similar for $\alpha \; ({}^v\tilde{\star})\beta$ written for abstract v–indices.

Properties of canonical and Cartan star products

Theorem: The product $\alpha\tilde{\star}\beta := \alpha\beta + \sum_k^{\infty} \ell^k \mathbf{C}_k(\alpha,\beta)$ has such properties

1. associativity, $\alpha\tilde{\star}(\beta\tilde{\star}\gamma) = (\alpha\tilde{\star}\beta)\tilde{\star}\gamma$;
2. Poisson bracket, $\mathbf{C}_1(\alpha,\beta) = \{\alpha,\beta\} = \theta^{\mu\nu}\mathbf{D}_\mu\alpha \cdot \mathbf{D}_\nu\beta$, antisymmetry, $\{\alpha,\beta\} = -\{\beta,\alpha\}$, and the Jacoby identity, $\{\alpha,\{\beta,\gamma\}\} + \{\gamma,\{\alpha,\beta\}\} + \{\beta,\{\alpha,\gamma\}\} = 0$;
3. N–adapted stability of type $\alpha\tilde{\star}\beta = \alpha\cdot\beta$ if $\mathbf{D}\alpha = 0$ or $\mathbf{D}\beta = 0$;
4. the Moyal symmetry, $\mathbf{C}_k(\alpha,\beta) = (-1)^k\mathbf{C}_k(\beta,\alpha)$;
5. N-adapted derivation with Leibniz rule,

$$\mathbf{D}(\alpha\tilde{\star}\beta) = (\mathbf{D}\alpha)\tilde{\star}\beta + \alpha\tilde{\star}(\mathbf{D}\beta) = ((h\mathbf{D}+v\mathbf{D})\alpha)\,\tilde{\star}\beta + \alpha\tilde{\star}((h\mathbf{D}+v\mathbf{D})\beta).$$

Hermitian $\overline{\alpha\tilde{\star}\beta} = \overline{\beta}\tilde{\star}\overline{\alpha}$; $(\tilde{\star},\mathbf{D})$ similar $(\star,\nabla)$. $\mathbf{D}_\mu\mathbf{g}_{\alpha\beta} = 0,\ \mathbf{g}_{\alpha\beta} = \frac{1}{2}\left(\overline{\mathbf{e}}_\alpha\tilde{\star}\mathbf{e}_\beta + \overline{\mathbf{e}}_\beta\tilde{\star}\mathbf{e}_\alpha\right)$, is not very restrictive as for symplectic geometries. ${}_\theta\tilde{\mathbf{D}}\tilde{\theta} = 0 \rightarrow \theta^{\mu\nu}\tilde{\star}\alpha = \theta^{\mu\nu}\cdot\alpha$.

Data $(\tilde{\star},\ {}_\theta\tilde{\mathbf{D}})$, elaborate an associative star product calculus completely defined by the metric structure in N–adapted form and keeps the covariant property.

Generating (non) commutative Ricci solitons

Noncommutativity via "generalized uncertainty" relations $\widehat{u}^{\alpha}\widehat{u}^{\beta} - \widehat{u}^{\beta}\widehat{u}^{\alpha} = i\theta^{\alpha\beta}(u)$
$\widehat{u}^{\alpha}$ are quantum analogs of coordinates, $\theta^{\alpha\beta}$ is an anti–symmetric tensor, $\theta \sim \hbar$).
Constant valued matrix for $u^{\alpha}u^{\beta} - u^{\beta}u^{\alpha} = i\theta^{\alpha\beta}$, with $\widehat{u}^{\alpha} \sim u^{\alpha}$, $\theta^{\alpha\beta}$
$= diag[\begin{pmatrix} 0 & {}^{h}\theta = \theta \sim \hbar \\ -\,{}^{h}\theta = -\theta \sim -\hbar & 0 \end{pmatrix}, \begin{pmatrix} 0 & {}^{v}\theta = \theta \sim \hbar \\ -\,{}^{v}\theta = -\theta \sim -\hbar & 0 \end{pmatrix}]$
We begin with conventional 2+2 splitting, ${}_{\shortmid}\mathbf{e}_{\alpha} = {}_{\shortmid}\mathbf{e}_{\alpha}^{\underline{\alpha}}(u,\theta)\partial_{\underline{\alpha}}$ such formal series

$$\begin{aligned}
{}_{\shortmid}\mathbf{e}_{\alpha}^{\underline{\alpha}} &= \mathbf{e}_{\alpha}^{\underline{\alpha}} + i\theta^{\alpha_1\beta_1}\mathbf{e}_{\alpha\ \alpha_1\beta_1}^{\underline{\alpha}} + \theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\mathbf{e}_{\alpha\ \alpha_1\beta_1\alpha_2\beta_2}^{\underline{\alpha}} + \mathcal{O}(\theta^3), \\
{}_{\shortmid}\mathbf{e}_{\star\underline{\alpha}}^{\alpha} &= \mathbf{e}_{\underline{\alpha}}^{\alpha} + i\theta^{\alpha_1\beta_1}\mathbf{e}_{\underline{\alpha}\alpha_1\beta_1}^{\alpha} + \theta^{\alpha_1\beta_1}\theta^{\alpha_2\beta_2}\mathbf{e}_{\underline{\alpha}\alpha_1\beta_1\alpha_2\beta_2}^{\alpha} + \mathcal{O}(\theta^3).
\end{aligned}$$

Generate noncommutative ${}_{\shortmid}\mathbf{g}_{\alpha\beta} = ({}_{\shortmid}^{h}\mathbf{g}, {}_{\shortmid}^{v}\mathbf{g}) = \frac{1}{2}\eta_{\underline{\alpha}\underline{\beta}}[{}_{\shortmid}\mathbf{e}_{\alpha}^{\underline{\alpha}}\tilde{\star}({}_{\shortmid}\mathbf{e}_{\beta}^{\underline{\beta}})^{+} + {}_{\shortmid}\mathbf{e}_{\beta}^{\underline{\beta}}\tilde{\star}({}_{\shortmid}\mathbf{e}_{\alpha}^{\underline{\alpha}})^{+}]$,
$(\ldots)^{+}$ is the Hermitian conjugation and $\eta_{\underline{\alpha}\underline{\beta}}$ is the flat Minkowski spacetime metric.

$D = \{\Gamma^{\beta}_{\ \alpha\gamma}\} \rightarrow {}_{\shortmid}D = \{{}_{\shortmid}\Gamma^{\beta}_{\ \alpha\gamma}\}$, $\quad {}_{\shortmid}D_{\alpha}\tilde{\star}X^{\beta} = \partial X^{\beta}/\partial u^{\alpha} + X^{\gamma}\tilde{\star}\,{}_{\shortmid}\Gamma^{\beta}_{\ \alpha\gamma}$.
The geometric rule: take the partial N–derivatives as for commutative spaces but twist the products via $\tilde{\star}$ when the product results in series, with

$$X^{\gamma}\tilde{\star}\,{}_{\shortmid}\Gamma^{\beta}_{\ \alpha\gamma} := X^{\gamma}\Gamma^{\beta}_{\ \alpha\gamma} + \sum_{k}^{\infty}\ell^{k}\mathbf{C}_{k}(X,\Gamma).$$

Noncommutative Ricci soliton and Einstein eqs

Using the principle of general commutative covariance,

$${}_{\shortmid}\mathbf{g}_{\alpha\beta} = {}_{\shortmid}\mathbf{g}_{\alpha\beta}(u,\theta) = \mathbf{g}_{\alpha\beta}(u) + \sum_{k}^{\infty} \theta^k \mathbf{C}_{k\alpha\beta}({}_{\shortmid}\mathbf{e}_{\gamma};\mathbf{g})$$

$${}_{\shortmid}\widehat{\mathbf{R}}_{\alpha\beta} \equiv \widehat{\mathbf{R}}_{\alpha\beta}(u,\theta) = \widehat{\mathbf{R}}_{\alpha\beta}(u) + \sum_{k}^{\infty} \theta^k \widehat{\mathbf{C}}_{k\alpha\beta}({}_{\shortmid}\mathbf{e}_{\gamma};\mathbf{g})$$

The Ricci solitonic/field equations

$$\begin{aligned} {}_{\shortmid}R_{ij} &= {}_{\shortmid}\Lambda(x^i,y^a)\,{}_{\shortmid}g_{ij} \\ {}_{\shortmid}R_{ab} &= {}_{\shortmid}\Lambda(x^i,y^a)\,{}_{\shortmid}g_{ab} \\ {}_{\shortmid}R_{ai} &= 0, \quad {}_{\shortmid}R_{ia} = 0, \end{aligned}$$

noncommutatively modified cosmological constant ${}_{\shortmid}\Lambda = \frac{\lambda + {}_{\shortmid}\mathbf{D}_{\gamma}\,{}_{\shortmid}\mathbf{D}^{\gamma}\,{}_{\shortmid}f_R - {}_{\shortmid}f/2}{1 - {}_{\shortmid}f_R}$. Chose nonholonomic distributions & noncommutative deforms $\mathbf{g} \rightarrow {}_{\shortmid}\mathbf{g}$,

$$\begin{aligned} {}_{\shortmid}g_{ij} &= g_{ij}(u) + \mathring{g}_{ij}(u)\theta^2 + \mathcal{O}(\theta^4), \; {}_{\shortmid}h_{ab} = h_{ab}(u) + \mathring{h}_{ab}(u)\theta^2 + \mathcal{O}(\theta^4), \\ {}_{\shortmid}N_i^3 &= w_i(u) + \mathring{w}_i(u)\theta^2 + \mathcal{O}(\theta^4), \; {}_{\shortmid}N_i^4 = n_i(u) + \mathring{n}_i(u)\theta^2 + \mathcal{O}(\theta^4). \end{aligned}$$

Decoupling and integrability of Ricci solitonic eqs

Gravitational eqs (for ${}_{\shortmid}\widehat{\mathbf{D}}$ can be integrated in very general off–diagonal forms for

$$\begin{aligned}
{}_{\shortmid}\mathbf{g}_{\alpha\beta}(x^k, y^3, \theta) &= diag\{ {}_{\shortmid}g_i(x^k,\theta) = \epsilon_i e^{\,{}_{\shortmid}\psi(x^k,\theta)} = g_i(x^k) + \mathring{g}_i(x^k)\theta^2 + \mathcal{O}(\theta^4), \\
{}_{\shortmid}h_a(x^k, y^3, \theta) &= h_a(x^k, y^3) + \mathring{h}_a(x^k, y^3)\theta^2 + \mathcal{O}(\theta^4)\}, \\
{}_{\shortmid}N_i^3(x^k, y^3, \theta) &= {}_{\shortmid}w_i(x^k, y^3, \theta) = w_i(x^k, y^3) + \mathring{w}_i(x^k, y^3)\theta^2 + \mathcal{O}(\theta^4), \\
{}_{\shortmid}N_i^4(x^k, y^3, \theta) &= {}_{\shortmid}n_i(x^k, y^3, \theta) = n_i(u) + \mathring{n}_i(u)\theta^2 + \mathcal{O}(\theta^4)
\end{aligned}$$

and ${}_{\shortmid}\Lambda \approx \Lambda(x^k, \theta)$; $\epsilon_i = \pm 1$ depend on chosen signature of metric for $\theta \to 0$.

Decoupling with respect to N–adapted frames; computation of $\widehat{\mathbf{R}}_{\alpha\beta}(u, \theta)$,

$a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial y^3$,

$$\begin{aligned}
\epsilon_1\, {}_{\shortmid}\psi^{\bullet\bullet} + \epsilon_2\, {}_{\shortmid}\psi'' &= \Lambda, \\
{}_{\shortmid}\phi^*(\ln|\,{}_{\shortmid}h_4|)^* &= \Lambda\, {}_{\shortmid}h_3, \\
{}_{\shortmid}\beta\, {}_{\shortmid}w_i + {}_{\shortmid}\alpha_i &= 0, \quad {}_{\shortmid}n_i^{**} + {}_{\shortmid}\gamma\, {}_{\shortmid}n_i^* = 0,
\end{aligned}$$

${}_{\shortmid}\gamma = (\ln|\,{}_{\shortmid}h_4|^{3/2} - \ln|\,{}_{\shortmid}h_3|)^*$, ${}_{\shortmid}\alpha_i = {}_{\shortmid}h_4^* \partial_i\, {}_{\shortmid}\phi$, ${}_{\shortmid}\beta = {}_{\shortmid}h_4^*\, {}_{\shortmid}\phi^*$, ${}_{\shortmid}\phi$ is given by ${}_{\shortmid}h_3$ and ${}_{\shortmid}h_4$ via ${}_{\shortmid}\phi = \ln|2(\ln\sqrt{|\,{}_{\shortmid}h_4|})^*| - \ln\sqrt{|\,{}_{\shortmid}h_3|}$.

Constructing integral varieties

Generic off–diagonal metrics with 6 independent coefficients

"New" generating, ${}_{\shortmid}\Phi(x^k, y^3, \theta) := e^{\,{}_{\shortmid}\phi}$ and ${}_{\shortmid}\psi[\Lambda(x^k, \theta)]$; integration, ${}^0h_a = {}^0h_a(x^k, \theta)$, ${}^1n_k(x^k)$, ${}^2n_k(x^k)$ functions

$$
\begin{aligned}
{}_{\shortmid}g_i &= \epsilon_i e^{\,{}_{\shortmid}\psi}, \\
{}_{\shortmid}h_3 &= {}^0h_3[1 + {}_{\shortmid}\Phi^*/2\Lambda\sqrt{|\,{}^0h_3|}]^2, \\
{}_{\shortmid}h_4 &= {}^0h_4 \exp[\,{}_{\shortmid}\Phi^2/8\Lambda], \\
{}_{\shortmid}w_i &= -\partial_i\,{}_{\shortmid}\phi/\,{}_{\shortmid}\phi^* = -\partial_i(\,{}_{\shortmid}\Phi)/(\,{}_{\shortmid}\Phi)^*, \\
{}_{\shortmid}n_k &= {}^1n_k + {}^2n_k \int dy^3\,{}_{\shortmid}h_3/(\sqrt{|\,{}_{\shortmid}h_4|})^3 \\
&= {}^1n_k + {}^2n_k \frac{{}^0h_3}{|\,{}^0h_4|^{3/2}} \int dy^3 [1 + \frac{{}_{\shortmid}\Phi^*}{2\Lambda\sqrt{|\,{}^0h_3|}}]^2 \exp[-\frac{3\,{}_{\shortmid}\Phi^2}{16\Lambda}],
\end{aligned}
$$

LC–conditions: ${}_{\shortmid}w_i^* = {}_{\shortmid}\mathbf{e}_i \tilde{\star} \ln|\,{}_{\shortmid}h_4|, \partial_i\,{}_{\shortmid}w_j = \partial_j\,{}_{\shortmid}w_i,\ {}_{\shortmid}n_i^* = 0.$

Black ellipsoids and solitonic waves as Ricci solitons

Spherical $u^{\alpha} = (x^1, x^2 = \vartheta, y^3 = \varphi, y^4 = t)$, when $x^1 = \xi = \int dr/\sqrt{|\underline{q}(r)|}$

Noncommutative Ricci solitonic black ellipsoids

Generating function for rotoid configuration $e^{2\,{}^{\shortmid}\phi} = 8\Lambda \ln |1 - \theta^2 \zeta(\xi, \tilde{\vartheta}, \varphi)/\underline{q}(\xi)|$.

$$
\begin{aligned}
{}^{rot}_{\lambda}\mathbf{g} &= e^{\psi(\xi,\tilde{\vartheta})}\,(d\xi^2 + d\tilde{\vartheta}^2) + r^2(\xi)\sin^2\vartheta(\xi,\tilde{\vartheta})\,(1 + \frac{({}^{\shortmid}e^{\phi})^*}{2\Lambda\sqrt{|{}^{\circ}h_3|}})^2 \\
& \quad {}^{\shortmid}\mathbf{e}_{\varphi} \otimes {}^{\shortmid}\mathbf{e}_{\varphi} - \left[\underline{q}(\xi) + \theta^2 \zeta(\xi,\tilde{\vartheta},\varphi)\right] \; {}^{\shortmid}\mathbf{e}_t \otimes {}^{\shortmid}\mathbf{e}_t, \\
{}^{\shortmid}\mathbf{e}_{\varphi} &= d\varphi - \theta^2 \left(\frac{\partial_{\xi}\phi}{\partial_{\varphi}\phi} d\xi + \frac{\partial_{\vartheta}\phi}{\partial_{\varphi}\phi} d\vartheta \right), \; {}^{\shortmid}\mathbf{e}_t = dt + \theta^2 \left[n_1 d\xi + n_2 d\vartheta\right],
\end{aligned}
$$

Prescribing $\zeta = \underline{\zeta}(\xi,\tilde{\vartheta}) \sin(\omega_0 \varphi + \varphi_0)$, constant parameters ω_0 and φ_0, $\underline{\zeta}(\xi,\tilde{\vartheta}) \simeq \underline{\zeta} = const$. The smaller horizon (when the term before ${}^{\shortmid}\mathbf{e}_t \otimes {}^{\shortmid}\mathbf{e}_t$ became ${}^{\shortmid}h_4 = 0$) is described by formula $r_+ \simeq 2\, m_0 / \left(1 + \theta^2 \underline{\zeta} \sin(\omega_0 \varphi + \varphi_0)\right)$.

Black ellipsoids and solitonic waves as Ricci solitons

Spherical $u^{\alpha} = (x^1, x^2 = \vartheta, y^3 = \varphi, y^4 = t)$, when $x^1 = \xi = \int dr/\sqrt{|\underline{q}(r)|}$

$^{\circ}\mathbf{g} = d\xi \otimes d\xi + r^2(\xi)\, d\vartheta \otimes d\vartheta + r^2(\xi) \sin^2\theta\, d\varphi \otimes d\varphi - \underline{q}(\xi)\, dt \otimes dt,$

an empty de Sitter space if $\underline{q}(r) = 1 - 2\frac{m(r)}{r} - \lambda \frac{r^2}{3}$; the total mass–energy within the radius r is $m(r)$; $m(r) = 0 \rightarrow$ cosmological horizon at $r = r_c = \sqrt{3/\lambda}$.

Noncommutative Ricci solitonic black holes and "non-Ricci" solitonic backgrounds

${}_{\shortmid}\phi = \eta(\xi, \tilde{\vartheta}, t, \theta) : \pm\eta'' + (\partial_t \eta + \eta\, \eta^{\bullet} + \epsilon \eta^{\bullet\bullet\bullet})^{\bullet} = 0,$

In the dispersionless limit $\epsilon \rightarrow 0$ the solutions transforms in those for the Burgers' equation $\partial_t \eta + \eta\, \eta^{\bullet} = 0$.

$$\begin{aligned} ds^2 \quad = \quad & e^{\,{}_{\shortmid}\psi}[d\xi^2 + d\tilde{\vartheta}^2] - \underline{q}(1 + \frac{\partial_t e^{\,{}_{\shortmid}\phi}}{2\Lambda\sqrt{|\underline{q}(\xi)|}})^2[dt - \frac{\partial_\xi\, {}_{\shortmid}\phi}{\partial_t\, {}_{\shortmid}\phi}d\xi - \frac{\partial_\vartheta\, {}_{\shortmid}\phi}{\partial_t\, {}_{\shortmid}\phi}d\tilde{\vartheta}]^2 + \\ & r^2(\xi)\sin^2\tilde{\vartheta}\exp[\frac{e^{2\,{}_{\shortmid}\phi}}{8\Lambda}][d\varphi + ({}^1n_1 + {}^2n_1 \int dt \frac{h_3}{(\sqrt{|h_4})^3})d\xi \\ & + ({}^1n_2 + {}^2n_2 \int dt \frac{h_3}{(\sqrt{|h_4|})^3})d\tilde{\vartheta}]^2, \ x^1 = \xi, x^2 = \tilde{\vartheta}, y^3 = t, y^4 = \varphi, \end{aligned}$$

Conclusions

- Nonholonomic (pseudo) Riemannian manifolds $\rightarrow$ an unified almost symplectic formalism for (supersymmetric/ noncommutative) Ricci solitons.
- We can develop nonholonomic versions of Berezin and Bergman–Toeplitz geometric quantization.
- DQ of commutative almost Kähler structures following Karabegov–Schlichenmaier constructions working with a special Cartan distinguished connection; the idea taken from Finsler-Lagrange geometry but defined on (pseudo) Riemannian manifolds.
- NC extensions of constructions are possible due to D. Vassilevich proposal to define associative star products as for Fedosov quantization but working with almost symplectic structures uniquely determined by metrics and nonlinear connections.
- The anholonomic deformation method $\rightarrow$ construct generic off–diagonal exact solutions for noncommutative Ricci solitons & (modified) gravity.