Stratified geometry of representation varieties

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1. Representation varieties and jump loci
2. Flat connections and Aomoto complexes
3. Abelian covers
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Representation varieties

- X reasonable space (connected CW-complex/algebraic variety)
- $\pi = \pi_1 X$ finitely generated group
- G linear Lie group (over $\mathbb{C}$, $\mathbb{R}$), $\mathfrak{g}$ Lie algebra

representation variety $1 \in \mathrm{Hom}_{\mathrm{gr}}(\pi, G)$ affine variety

- topology (local systems on X of type G)
- differential geometry ($\mathfrak{g}$–valued flat connections on X)
- algebraic geometry (locally constant sheaves)
- differential equations (completely integrable systems)

rank 1 case $\mathbb{T}(\pi) = \mathrm{Hom}_{\mathrm{gr}}(\pi, \mathbb{C}^\times) \supseteq \mathbb{T}^0(\pi) = (\mathbb{C}^\times)^n$ character torus

Artin groups

- Γ finite simple graph, vertices V, edges E
- coloured by $\ell : \mathsf{E} \to \mathbb{Z}_{\geq 2}$

- Artin group $\pi_{\Gamma,\ell}$ generated by $v \in \mathsf{V}$
- for $\{u, v\} \in \mathsf{E}$, relation $uvu \cdots = vuv \cdots$ words of length $\ell(\{u, v\})$
- RAAG (right–angled) π_Γ when $\ell \equiv 2$

- Γ complete: $\pi_\Gamma = \mathbb{Z}^n$ free abelian
- Γ discrete: $\pi_\Gamma = F_n$ free

Universality Theorem

Theorem (Kapovich–Millson, IHES 1998)

For any point $x \in \mathcal{X}$ on an affine variety defined over $\mathbb{Q}$, there is a coloured graph (Γ, ℓ) *and* $1 \neq \rho \in \mathrm{Hom}_{\mathrm{gr}}(\pi_{\Gamma,\ell}, PSL_2(\mathbb{C}))$ *such that* $\mathrm{Hom}_{\mathrm{gr}}(\pi_{\Gamma,\ell}, PSL_2(\mathbb{C}))_{(\rho)} = \mathcal{X}_{(x)} \times \mathbb{C}^3_{(0)}$, *as analytic germs.*

Theorem (with A. Măcinic, R. Popescu, A. Suciu, arXiv 2013)

For an arbitrary coloured graph (Γ, ℓ), $\mathrm{Hom}_{\mathrm{gr}}(\pi_{\Gamma,\ell}, PSL_2(\mathbb{C}))_{(1)}$ *is very special and admits explicit combinatorial description.*

Theorem (with A. Dimca, to appear Commun. Contemp. Math.)

In general, $\mathrm{Hom}_{\mathrm{gr}}(\pi, G)_{(1)}$ *depends only on the* Malcev–Lie algebra *of* π *[Quillen, Annals 1969] and* $\mathfrak{g}$.

Moduli space of smooth projective curves ($g \geq 2$)

- mapping class group M_g: orientation–preserving homeo's of genus g closed Riemann surface, up to isotopy
- $M_g = \mathrm{Out}^+(\pi_g)$, by Dehn–Nielsen
- Fuchsian (i.e., discrete) subgroups $\mathrm{Hom}^0_{\mathrm{gr}}(\pi_g, G)$, $G = PSL_2(\mathbb{R})$

Theorem (Goldman, BAMS 1982)

$\mathrm{Hom}^0_{\mathrm{gr}}(\pi_g, G) = \{\rho \in \mathrm{Hom}_{\mathrm{gr}}(\pi_g, G) \mid e(\rho) = 2g - 2\}$ *(connected component).*

- uniformization: $PSL_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$ and $\mathbb{H}/\rho$ is a smooth curve

Theorem (A. Weil, Annals 1960's)

The biquotient $M_g \backslash \mathrm{Hom}^0_{\mathrm{gr}}(\pi_g, G)/G$ *gives moduli space of genus* g *smooth projective curves, of* $\dim_{\mathbb{C}} = 3g - 3$ *(Riemann).*

Jump loci

- $\iota : G \to \mathrm{GL}(V)$ rational representation, $\theta : \mathfrak{g} \to \mathfrak{gl}(V)$ tangent map
- fix $q \geq 1$ (computational complexity)
- assume from now on X *q–finite* (finite q–skeleton)

- characteristic varieties give natural stratification for $i \leq q$

$$\mathcal{V}^i_r(X, \iota) = \{\rho \in \mathrm{Hom}_{\mathrm{gr}}(\pi, G) \mid \dim H^i(X, {}_{\iota\rho}V) \geq r\}$$

- for $r = 0$, $\mathcal{V}^i_0(X, \iota) = \mathrm{Hom}_{\mathrm{gr}}(\pi, G)$

Question

Compute at 1 and extract topological/geometric information on X

Pencils

- topological *Green–Lazarsfeld sets* $\mathcal{V}^i_r(X, \mathrm{id}_{\mathbb{C}^\times}) =: \mathcal{V}^i_r(X)$
- for a projective manifold X, see Green–Lazarsfeld, Invent. 1987
- pencil on a quasi–projective manifold $f : X \to S$ regular onto a curve, with connected generic fiber
- pencil *of general type* iff $\chi_S < 0$

Theorem (Arapura, JAG 1997)

Positive–dimensional components through 1 *of* $\mathcal{V}^1_1(X)$ *are in bijection with pencils of general type on* X.

Commutative Differential Graded Algebras

- aim: extend computation for germs of representation varieties to jump loci
- strategy: replace X by CDGA model $A^\bullet$
- assume from now on $A^\bullet$ is *q–finite* ($A^0 = \mathbb{C} \cdot 1$, $\dim A^{\leq q} < \infty$)

- $A^\bullet \otimes \mathfrak{g}$ becomes DGLA: $d(a \otimes g) = da \otimes g$, $[a \otimes g, a' \otimes g'] = aa' \otimes [g, g']$

- variety of flat connections $0 \in \mathcal{F}(A, \mathfrak{g}) \subseteq A^1 \otimes \mathfrak{g}$, defined by the *Maurer–Cartan equation*

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

- rank 1 case: $\mathcal{F}(A, \mathbb{C}) := \mathcal{F}(A) = H^1 A$

Covariant derivative

- semidirect Lie product $V \rtimes_\theta \mathfrak{g}$
- covariant derivative $d_\omega := d + \mathrm{ad}_\omega : A^\bullet \otimes V \to A^{\bullet+1} \otimes V$
- Aomoto complex $(A^\bullet \otimes V, d_\omega)$ for $\omega \in \mathcal{F}(A, \mathfrak{g})$

Definition

The resonance varieties

$$\mathcal{R}^i_r(A, \theta) = \{\omega \in \mathcal{F}(A, \mathfrak{g}) \mid \dim H^i(A^\bullet \otimes V, d_\omega) \geq r\};$$

natural stratification by subvarieties for $i \leq q$, and $\mathcal{R}^i_0(A, \theta) = \mathcal{F}(A, \mathfrak{g})$.

Example

When $\theta = \mathrm{id}_\mathbb{C}$ and $d = 0$, usual Aomoto complex $(A^\bullet, \omega \cdot)$, for $\omega \in A^1$.

From topology to algebra

- via D. Sullivan's De Rham CDGA of a space $\Omega^\bullet(X)$ (IHES 1977)
- $A \simeq_q B$ (same *q–homotopy type* in CDGA) iff same Sullivan q–minimal model
- localization of CDGA w.r. to weak q–equivalences, i.e., maps inducing homology iso's in degrees $\leq q$ and homology mono in degree $q+1$

Theorem (with A. Dimca, to appear CCM)

If $\Omega^\bullet(X) \simeq_q A^\bullet$, there is a stratified reduced local analytic iso, for $i \leq q$,

$$e : \mathcal{R}^i_r(A,\theta)_{(0)} \xrightarrow{\sim} \mathcal{V}^i_r(X,\iota)_{(1)}$$

• extended away from 1 by Budur–Wang/arXiv 2013, for X compact Kahler, $\iota = \mathrm{id}_{\mathrm{GL}_n}$ and ρ semisimple
• extension fails for $n=1$ and $\rho = 1$, if X *not compact* Kahler [CCM]

Examples

- when X is *q–formal* (i.e., $\Omega^\bullet(X) \simeq_q (H^\bullet X, d = 0)$), $A^\bullet = (H^\bullet X, d = 0)$
- X compact Kahler, $X = K(\pi_\Gamma, 1)$, $X = M(\mathcal{A})$ complement of complex hyperplane arrangement: ∞–formal
- X quasi–projective manifold with $W_1 H^1 X = 0$: 1–formal

- X quasi–projective manifold has good compactifications $\overline{X}$
- $\overline{X}$ gives *Gysin model* $A^\bullet(\overline{X})$, cf. Morgan, IHES 1978
- may use $A^\bullet = A^\bullet(\overline{X})$, for $q = \infty$

Remark

In both examples, $\Omega^\bullet(X) \simeq_q A^\bullet$ is defined over $\mathbb{Q}$.

Positive weight

Definition

$A^\bullet$ is a CDGA with positive weights if $A^\bullet = \bigoplus_{j\in\mathbb{Z}} A^\bullet_j$, $d(A^\bullet_j) \subseteq A^\bullet_j$, $A^\bullet_j \cdot A^\bullet_k \subseteq A^\bullet_{j+k}$, and $A^1_j = 0$ for $j \leq 0$.

Theorem (with A. Dimca, to appear CCM)

If $\Omega^\bullet(X) \simeq_q A^\bullet$ is defined over $\mathbb{Q}$ and $A^\bullet$ has positive weights, then, for all $i \leq q$ and $r \geq 0$

(1) *$\mathcal{R}^i_r(A, \mathrm{id}_{\mathbb{C}})$ is a finite union of linear subspaces in $H^1 A$, that are defined over $\mathbb{Q}$.*

(2) *All irreducible components through 1 of $\mathcal{V}^i_r(X)$ are subtori in $\mathbb{T}(\pi)$.*

• obstructions to q–formality/quasi–projectivity

• used and extended by Budur–Wang/arXiv 2012 away from 1, when X is a quasi–projective manifold

Universality, again

- The preceding obstructions are in marked contrast with the general case

Theorem (Simpson, Proc. Symp. Pure Math. 1997)

Given any $i \geq 1$ and an arbitrary subvariety W in $(\mathbb{C}^\times)^n$ defined over $\mathbb{Z}$, there is an ∞–finite space X such that $\mathbb{T}(\pi) = (\mathbb{C}^\times)^n$ and $W \cup \{1\} = \mathcal{V}^i_1(X) \cup \{1\}$.

Remark

Rationality obstruction (1) used in Duke 2009 (with A. Dimca and A. Suciu) to construct a finite–dimensional CGA defined over $\mathbb{Q}$ that cannot be the cohomology ring of a formal space. By Quillen and Sullivan [loc. cit.], this cannot happen in the 1–connected case.

Finiteness properties of abelian covers

- group epimorphism $\nu : \pi \twoheadrightarrow Q$
- Galois cover $X^{\nu} \to X$ with group Q

Theorem (with A. Suciu, J. Topol. 2012)

If Q is abelian, the following are equivalent.

(1) $\dim H_{\leq q} X^{\nu} < \infty$

(2) $\nu^{*}\mathbb{T}(Q) \cap \left(\bigcup_{i \leq q} \mathcal{V}_1^i(X) \right)$ *is finite*

• i.e., rank 1 jump loci control homological finiteness properties of abelian covers

Moduli spaces for Riemann surfaces

- mapping class group M_g: orientation–preserving homeo's of genus g closed Riemann surface, up to isotopy
- closely related to the moduli space of genus g projective curves
- Torelli group T_g: $h \in M_g$ inducing identity on H_1
- Torelli space $\mathcal{T}_g = K(T_g, 1)$: moduli space of genus g projective curves with symplectic marking on H_1
- Johnson subgroup $K_g \subseteq T_g$: generated by Dehn twists about simple separating curves on the Riemann surface

Theorem (D. Johnson, Annals 1983)

For $g \geq 3$, T_g is finitely generated.

Theorem (D. Johnson, Topology 1985)

Universal torsion–free abelian cover of $\mathcal{T}_g$ is given by K_g.

Question and answers

Question (early 1980s)

Is $\dim H_1 K_g < \infty$?

Answer uses jump loci criterion (2) above.

Theorem (with A. Dimca and R. Hain, Annals 2013 and JEMS, to appear; R. Hain, arXiv 2013)

If $g \geq 4$, then

- $\mathcal{V}_1^1(T_g) \cap \mathbb{T}^0(T_g) = \{1\}$
- *$H_1 K_g$ is finite–dimensional, computable by symplectic representation theory.*

Question (Serre, early 1950s)

Characterize and construct (quasi)projective groups.

Theorem (with A. Dimca and A. Suciu, Duke 2009)

RAAG π_Γ *is quasi–projective iff* $\pi_\Gamma = \prod_{i=1}^{m} F_{n_i}$.

Question (Kollár, 1995)

Let π be projective. Up to commensurability (i.e., localizing w.r. to group morphisms with finite kernel and cofinite image), does it have an algebraic $K(\pi, 1)$?

Theorem (with A. Dimca and A. Suciu, Crelle 2009)

NO. More precisely, for any $q \geq 3$, *there is a projective* π *without* q*–finite* $K(\pi, 1)$ *(even up to commensurability).*

Milnor fibers of line arrangements

- $\mathcal{A} = \{L_1, \dots, L_n\}$ line arrangement in $\mathbb{CP}^2$, $\cap_{i=1}^n L_i = \emptyset$
- given by $P(x, y, z) = 0$ that splits into n distinct linear factors
- combinatorics given by the multiple points and their position on lines
- Milnor fiber $F_{\mathcal{A}}$: $P(x, y, z) = 1$
- monodromy $h : F_{\mathcal{A}} \to F_{\mathcal{A}}$ induced by $\exp(\frac{2\pi\sqrt{-1}}{n})$
- monodromy action: $H_\bullet(F_{\mathcal{A}}, \mathbb{Q})$ becomes $\mathbb{Q}[t]$–module (t acts by $H_\bullet h$)
- nontrivial part: $H_\bullet^{\neq 1}(F_{\mathcal{A}}, \mathbb{Q})$

Question (open)

Compute $H_1^{\neq 1}(F_{\mathcal{A}}, \mathbb{Q})$. Is it determined by combinatorics?

Milnor fibers and modular resonance

Theorem (Cogolludo–Libgober, to appear Crelle; Libgober, Adv. Stud. Pure Math. 2012)

Property $H_1^{\neq 1}(F_{\mathcal{A}}, \mathbb{Q}) \neq 0$ is combinatorial, when $\mathcal{A}$ has only double and triple points.

- natural $\mathbb{F}_p$–basis for $H^1(M(\mathcal{A}), \mathbb{F}_p)$ indexed by $L \in \mathcal{A}$
- canonical element $\omega := \sum_{L \in \mathcal{A}} L \in H^1(M(\mathcal{A}), \mathbb{F}_p)$
- modular Aomoto complex: $(H^\bullet(M(\mathcal{A}), \mathbb{F}_p), \omega \cdot)$
- modular Aomoto–Betti number: $\beta_p(\mathcal{A}) := \dim_{\mathbb{F}_p} H^1(H^\bullet(M(\mathcal{A}), \mathbb{F}_p), \omega \cdot)$
- $\beta_p(\mathcal{A})$ is combinatorial, cf. Orlik–Solomon, Invent. 1980

Milnor fibers via $\mathfrak{sl}_2$–flat connections

Theorem (with A. Suciu, arXiv 2014)

Assume $\mathcal{A}$ has only double and triple points.

(1) $\beta_3(\mathcal{A}) \leq 2$

(2) $H_1^{\neq 1}(F_{\mathcal{A}}, \mathbb{Q}) = \left(\frac{\mathbb{Q}[t]}{t^2+t+1}\right)^{\beta_3(\mathcal{A})}$

- graphic arrangement $\mathcal{A}_\Gamma$: hyperplanes $z_i = z_j$, for $\{i, j\} \in$ E
- generic section of $\mathcal{A}_\Gamma$ has only double and triple points

Theorem (with A. Măcinic, Topology Appl. 2009)

Formula (2) above holds for $\mathcal{A}_\Gamma$.