Singularities and integrability of birational dynamical systems on the projective plane

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February 6, 2014
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Tropical dynamical systems
From nonlinear discrete equations (mappings) to surface theory

The main motivation! Finding an INTEGRABILITY DETECTORS for discrete equations.
Example:

\[ x_{n+1} + x_n + x_{n-1} = \frac{a}{x_n} \]

Integrability ≡ internal symmetry, existence of invariants, computing general solution etc.
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**Question!** Is there any invariant \( K_n \equiv K(x_n, x_{n-1}) \) (conservation law, i.e. \( K_{n-1} = K_n = K_{n+1} = \ldots \)) of the above equation? If yes can it be computed algorithmically?
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Iterating:

\[ x_{n+1} = -x_n - x_{n-1} + \frac{z}{x_n} + c = -f - \epsilon - \frac{z}{\epsilon} + c = \infty \]
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Singularity pattern \((f, 0, \infty, \infty, 0, -f)\). So after a **finite** number of steps the singularities are confined and initial information is recovered — singularity confinement.
Our example can be written as:

\[ \phi : \begin{cases} 
  x_{n+1} = y_n \\
  y_{n+1} = -x_n - y_n + a/y_n 
\end{cases} \quad (1) \]

seen as a chain of birational mappings \( \ldots \rightarrow (x, y) \rightarrow (x, y) \rightarrow (\bar{x}, \bar{y}) \rightarrow \ldots \) where \( x = x_{n-1}, \bar{x} = x_n, \bar{x} = x_{n+1} \) and so on.

Each step is an automorphism of the field of rational functions \( \mathbb{C}(x, y) \).
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Singularity confinement:

\[
(f, 0) \rightarrow (0, \infty) \rightarrow (\infty, \infty) \rightarrow (\infty, 0) \rightarrow (0, f)
\]

and the secret is the following:

If \( (x_0, y_0) = (f, \varepsilon) \) then the following products are finite

\[
x_1 y_1 = a + O(\varepsilon), \quad \frac{x_2}{y_2} = -1 + O(\varepsilon), \quad x_3 y_3 = -a + O(\varepsilon)
\]
So let's construct a surface by glueing

\[ \mathbb{C}^2 \cup \mathbb{C}^2 = \left( x_1, \frac{1}{x_1 y_1} \right) \cup \left( x_1 y_1, \frac{1}{y_1} \right) \]
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But this is nothing but blow up of the affine space \( \text{Spec} \mathbb{C}[x, Y] \) with the center \((x, Y) = (0, 0)\) which gives the surface \((Y = 1/y)\):

\[
X_1 = \{(x, Y, [z_0 : z_1]) \in \text{Spec} \mathbb{C}[x, Y] \times \mathbb{P}^1 | xz_0 = Yz_1 \} = \\
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So by blowing up \( \mathbb{C}^2 \) in the points \((x_1, y_1) = (0, \infty), (x_2, y_2) = (\infty, \infty), (x_3, y_3) = (\infty, 0)\) the equation then make sense on this new surface.

Accordingly we do analyze any discrete order two nonlinear equation by identifying the singularities and blow them up.

From now on we shall replace \( \mathbb{C}^2 \) with \( \mathbb{P}^1 \times \mathbb{P}^1 \) and any nonlinear equation will be a birational mapping \( \varphi : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) After blowing up the singular points we get a surface \( X \) and our mapping is lifted to a regular mapping:

\[ \varphi : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \]
Algorithm for analysing mappings

- check if $\varphi : X \rightarrow X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface $S$ and the final mapping $\varphi : S \rightarrow S$ without any singularity.
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- from the nonlinear mapping we go to the induced bundle mapping $\varphi_* : \text{Pic}(S) \to \text{Pic}(S)$ whose action on the Picard group is linear.
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- back to the nonlinear world, by computing the real invariants as proper transforms of the those found above.
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- integrability \( \equiv \) Weyl group of affine type (and \( S \) is a rational elliptic surface).

- linearisability \( \equiv \) infinite number of blow ups, analytical stability, ruled surface \( S \).
Rational elliptic surface:
A complex surface $X$ is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
- $\pi$ is not birational to the projection: $E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve $E$
- no fibers contains exceptional curves of first kind.
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**Halphen surface of index $m$:** A rational surface $X$ is called a *Halphen surface of index $m$* if the anticanonical divisor class $-K_X$ is decomposed into prime divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Halphen surfaces can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by successive 8 blow-ups. In addition the dimension of the linear system $|-kK_X|$ is zero for $k = 1, \ldots, m - 1$ and 1 for $k = m$. Here, the linear system $|-mK_X|$ is the set of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2m, 2m)$ passing through each point of blow-up with multiplicity $m$. 

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If the fibers contain exceptional curves of first kind the elliptic surface is called relatively non-minimal. To make it minimal one has to blow down that curves.

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Analytical stability and blowing-down structure
Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational automorphism with iterates growing quadratically with $n$. For any such automorphism we can blow up $\mathbb{P}^1 \times \mathbb{P}^1$ and construct a rational surface $X$ such that: $\tilde{\phi} : X \to X$ and $\tilde{\phi}$ is \textit{analytically stable} which means:

$$(\tilde{\phi}^*)^n = (\tilde{\phi}^n)^* : \text{Pic}(X) \to \text{Pic}(X)$$

Analytical stability is equivalent with the following: There is no divisor $D$ such that exist $k > 0$ and $\tilde{\phi}(D) =$point, $\tilde{\phi}^k(D) =$ indeterminate

$$D \to \bullet \to \bullet \to \ldots \bullet \to D'$$

$$\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\mu \downarrow & & \mu \downarrow \\
X & \xrightarrow{\tilde{\phi}} & X
\end{array}$$
compute the surface $X$ where $\tilde{\phi} : X \to X$ is analytically stable
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there is a singularity pattern $\bullet \to D_1 \to D_2 \to \ldots \to D_k \to \bullet$ having $(-1)$ curves in the components of some $D_i$ and this set of $(-1)$ curves is preserved by the action of $\tilde{\phi} : X \to X$. 
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Blow down the $(-1)$ curves in the following way: Let $C$ be the $(-1)$ divisor class and $F_1, F_2$ two divisor classes such that

\[ F_1 \cdot F_1 = F_2 \cdot F_2 = 0, \quad F_1 \cdot F_2 = 1, \quad C \cdot F_1 = C \cdot F_2 = 0 \]
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all the above procedure is allowed by the Castelnuovo theorem, and if $\dim |F_1| = \dim |F_2| = 1$ we can put $|F_1| = \alpha_1 x' + \beta_1 y', |F_2| = \alpha_2 x'' + \beta_2 y''$
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then we have a new coordinate system where $X$ is minimal given by the following transformation:

$$\mathbb{C}^2 \ni (x, y) \longrightarrow \left(\frac{y'}{x'}, \frac{y''}{x''}\right) \in \mathbb{P}^1 \times \mathbb{P}^1$$
Singularities and surfaces

Basic example

\[ x_{n+1} = -x_n \frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \] (2)

\[ \bar{x} = y \]

\[ \bar{y} = -x \frac{(y - a)(y - 1/a)}{(y + a)(y + 1/a)} \] (3)
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Indeterminate points for \( \phi \) and \( \phi^{-1} \):

\[ P_1 : (x, y) = (0, -a), \quad P_2 : (x, y) = (0, -1/a), \]
\[ P_3 : (X, y) = (0, a), \quad P_4 : (X, y) = (0, 1/a), \]
\[ P_5 : (x, y) = (a, 0), \quad P_6 : (x, y) = (1/a, 0), \]
\[ P_7 : (x, Y) = (-a, 0), \quad P_8 : (x, Y) = (-1/a, 0). \]
**Figure:** Space of initial conditions and orthogonal complement.
The Picard group of $X$ is a $\mathbb{Z}$-module

$$\text{Pic}(X) = \mathbb{Z}H_x \oplus \mathbb{Z}H_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i,$$

$H_x, H_y$ are the total transforms of the lines $x = \text{const.}, y = \text{const}$. $E_i$ are the total transforms of the eight blowing up points. The intersection form:

$$H_z \cdot H_w = 1 - \delta_{zw}, \quad E_i \cdot E_j = -\delta_{ij}, \quad H_z \cdot E_k = 0$$

for $z, w = x, y$. Anti-canonical divisor of $X$:

$$-K_X = 2H_x + 2H_y - \sum_{i=1}^{8} E_i.$$
If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice $(h_i, e_j \in \mathbb{Z})$ the induced bundle mapping is acting on it as
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$$\phi^* (h_0, h_1, e_1, ..., e_8) = (h_0, h_1, e_1, ..., e_8) \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
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$$\phi_*(h_0, h_1, e_1, ..., e_8)$$

$$=(h_0, h_1, e_1, ..., e_8)\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
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0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}.$$ .

It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$: 

$$D_0 = H_x - E_1 - E_2, \quad D_1 = H_y - E_5 - E_6,$$

$$D_2 = H_x - E_3 - E_4, \quad D_3 = H_y - E_7 - E_8.$$
there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all $E_i$ for any $k$).

\[
F \equiv \alpha xy - \beta((x^2 + 1)(y^2 + 1) + (a + 1/a)(y - x)(xy + 1)) = 0 \\
\Leftrightarrow kxy - ((x^2 + 1)(y^2 + 1) + (a + 1/a)(y - x)(xy + 1)) = 0.
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  fibration)

So the conservation law will be:

$$I = \left(\frac{(x^2 + 1)(y^2 + 1) + (a + 1/a)(y - x)(xy + 1)}{xy}\right)^2$$
Symmetries
Related to orthogonal complement of the space of initial condition $A_3^{(1)}$

$$\text{rankPic}(X) = \text{rank} \langle H_0, H_1, E_1, ... E_8 \rangle_{\mathbb{Z}} = 10$$

Define:

$$\langle D \rangle = \sum_{i=0}^{3} \mathbb{Z} D_i$$

$$\langle D \rangle^\perp = \{ \alpha \in \text{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3 \}$$
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which have 6-generators:

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$$\alpha_0 = E_4 - E_3, \alpha_1 = E_1 - E_2, \alpha_2 = H_1 - E_1 - E_5$$

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Elementary reflections:

$$w_i : \text{Pic}(x) \rightarrow \text{Pic}(X), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

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Permutation of roots:

$$\sigma_{10}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$\sigma_{tot}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$$

Hence the group generated by reflections and permutations becomes an extended Weyl group.
This extended Weyl group becomes the group of Cremona isometries for the space of initial conditions $X$ since:

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Accordingly our mapping lives in a Weyl group and has the following decomposition in elementary reflections:

$$\phi_\ast = \sigma_{\text{tot}} \circ w_3 \circ w_5 \circ w_4 \circ w_3$$

All elements $\omega \in \widetilde{W}(D_5^{(1)})$ which commutes with $\phi_\ast$, namely ($\omega \circ \phi_\ast = \phi_\ast \circ \omega$) form the symmetries of the mapping.

The equation is related to the translations in this affine Weyl group. In general for an affine Weyl group with null vector $\delta$ the translation of an element $D$ with respect to the root $\alpha_i$ is given by

$$t_{\alpha_i} : D \rightarrow D - (D, \delta)\alpha_i + (D, \alpha_i + \delta)\delta$$

and our mapping is "the fourth root" of a translation:

$$\phi_\ast^4 \equiv t_{\alpha_3} \circ t_{\alpha_3} \circ t_{\alpha_4} \circ t_{\alpha_5} = t_{2\alpha_3+\alpha_4+\alpha_5}$$
Differential Nahm equations are nonlinear ODE order two describing symmetric monopoles associated to some rotational symmetry groups. The solutions are expressed through rational expressions of Weierstrass elliptic functions and their derivatives (Hitchin, Manton, Murray - ’95)
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- **Icosahedral symmetry**: 
  \[
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Hirota-Kimura discretisation

It applies to some class of ODE (quadratic) and has close relation with Hirota bilinear method. More precisely start with:

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\dot{x}_i = \sum_{j=1}^{N} a_{ij}x_j^2 + \sum_{j<k} b_{ijk}x_jx_k + c_i
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\[ D_t G_i \cdot F = \sum_{j=1}^{N} a_{ij} G_j^2 + \sum_{j<k} b_{ijk} G_j G_k + c_i F^2 \]

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$$\bar{G}_i F - G_i \bar{F} = \epsilon (\sum_{j=1}^{N} a_{ij} G_j \bar{G}_j + \sum_{j<k} b_{ijk} (\alpha \bar{G}_j G_k + (1 - \alpha) G_j \bar{G}_k) + c_i F \bar{F})$$

or in the nonlinear form (Kahan '93, Hirota-Kimura, '00)

$$\bar{x}_i - x_i = \epsilon (\sum_{j=1}^{N} a_{ij} x_j \bar{x}_j + \sum_{j<k} b_{ijk} (\alpha \bar{x}_j x_k + (1 - \alpha) x_j \bar{x}_k) + c_i)$$
Discrete Nahm equations

Using the above Kahan-Hirota-Kimura procedure one can easily discretize the above mentioned Nahm equations (Petrera, Pfadler, Suris ’12)
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K(\epsilon) = \frac{y(3x - y)^2(4x + y)^3}{1 + \epsilon^2c_2 + \epsilon^4c_4 + \epsilon^6c_6}
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\[
c_2 = -35x^2 + 7y^2
\]
\[
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\]
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Minimization and invariants

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The tetrahedral symmetry (simple can be brought to QRT):

\[ \bar{x} - x = \epsilon(x\bar{x} - y\bar{y}) \]
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use the substitution \( u = (1 - \epsilon x)/y, \nu = (1 + \epsilon x)/y \) and we get QRT-mapping \((\bar{u} = \nu)\) and

\[ 3\bar{u}u - u(\bar{u} + u) - u^2 + 4\epsilon^2 = 0 \]
with the invariant

\[ K = \frac{-3(u - \bar{u})^2 + 4\epsilon^2}{2\epsilon^2 (u + \bar{u})(u\bar{u} - \epsilon^2)} \equiv \frac{3x^2 y - y^3}{1 - \epsilon^2 (x^2 + y^2)} \]

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The red substitution looks like curves corresponding to divisor classes of some blow-down structure.
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What we learn:
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The cases of octahedral and icosahedral symmetry cannot be transformed to QRT forms by these type of substitutions.
So we need to analyse carefully the singularity structure. What is seen is that we have more singularities and apparently some of them are useless making the corresponding rational elliptic surface to be more complicated.
The case of octahedral symmetry:

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We simplify by the following:

\[
x = \frac{1}{3}(\chi - 2y), \quad \bar{x} = \frac{1}{3}(\bar{\chi} - 2\bar{y}), \quad u = (1 - \epsilon \chi)/y, \quad v = (1 + \epsilon \chi)/y
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to the non-QRT type system:
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to the non-QRT type system:

\[
\begin{align*}
\bar{u} &= v \\
\bar{v} &= \frac{(u + 2v - 20\epsilon)(v + 10\epsilon)}{4u - v + 10\epsilon}
\end{align*}
\]
The case of octahedral symmetry:

\[ \bar{x} - x = \epsilon(2x\bar{x} - 12y\bar{y}) \]

\[ \bar{y} - y = -\epsilon(3y\bar{x} + 3x\bar{y} + 4y\bar{y}) \]

We simplify by the following:

\[ x = \frac{1}{3}(\chi - 2y), \; \bar{x} = \frac{1}{3}(\bar{\chi} - 2\bar{y}), \; u = (1 - \epsilon\chi)/y, \; v = (1 + \epsilon\chi)/y \]

to the non-QRT type system:

\[ \begin{cases} 
\bar{u} & = v \\
\bar{v} & = \frac{(u + 2v - 20\epsilon)(v + 10\epsilon)}{4u - v + 10\epsilon} \end{cases} \] \hspace{1cm} (4)

The space of initial conditions is given by the \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up at the following nine points:

\[ E_1 : (u, v) = (-10\epsilon, 0), \; E_2 (0, 10\epsilon), \; E_3 (10\epsilon, 5\epsilon), \]
\[ E_4 (5\epsilon, 0), \; E_5 (0, -5\epsilon), \; E_6 (-5\epsilon, -10\epsilon) \]
\[ E_7 (\infty, \infty), \; E_8 : (1/u, u/v) = (0, -1/2), \; E_9 : (1/u, u/v) = (0, -2). \]
The action on the Picard group:

\[
\bar{H}_u = 2H_u + H_v - E_1 - E_3 - E_7 - E_8, \quad \bar{H}_v = H_u \\
\bar{E}_1 = E_2, \quad \bar{E}_2 = H_u - E_3, \quad \bar{E}_3 = E_4, \quad \bar{E}_4 = E_5, \quad \bar{E}_5 = E_6, \\
\bar{E}_6 = H_u - E_1, \quad \bar{E}_7 = H_u - E_8, \quad \bar{E}_8 = E_9, \quad \bar{E}_9 = H_u - E_7.
\]
The action on the Picard group:

\[ \tilde{H}_u = 2H_u + H_v - E_1 - E_3 - E_7 - E_8, \quad \tilde{H}_v = H_u \]
\[ \tilde{E}_1 = E_2, \quad \tilde{E}_2 = H_u - E_3, \quad \tilde{E}_3 = E_4, \quad \tilde{E}_4 = E_5, \quad \tilde{E}_5 = E_6, \]
\[ \tilde{E}_6 = H_u - E_1, \quad \tilde{E}_7 = H_u - E_8, \quad \tilde{E}_8 = E_9, \quad \tilde{E}_9 = H_u - E_7. \]

Three invariant divisor classes:

\[ \alpha_0 = H_u + H_v - E_1 - E_2 - E_7, \quad \alpha_1 = H_u + H_v - E_1 - E_2 - E_8 - E_9, \]
\[ \alpha_2 = E_7 - E_8 - E_9, \quad \alpha_3 = H_u + H_v - E_3 - E_4 - E_5 - E_6 - E_7. \]
The action on the Picard group:

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\[
\bar{E}_1 = E_2, \quad \bar{E}_2 = H_u - E_3, \quad \bar{E}_3 = E_4, \quad \bar{E}_4 = E_5, \quad \bar{E}_5 = E_6,
\]

\[
\bar{E}_6 = H_u - E_1, \quad \bar{E}_7 = H_u - E_8, \quad \bar{E}_8 = E_9, \quad \bar{E}_9 = H_u - E_7.
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\[
\alpha_2 = E_7 - E_8 - E_9, \quad \alpha_3 = H_u + H_v - E_3 - E_4 - E_5 - E_6 - E_7.
\]

The curve corresponding to \(\alpha_0\) is a (-1) curve which must be blown down.

\(E_1 \to H_a = H_u + H_v - E_2 - E_7\) and \(E_2 \to H_b = H_u + H_v - E_1 - E_7\), 0-curves intersecting each other: The corresponding curves are given by:

\[
a_1 u + a_2 (\nu - 10 \epsilon) = 0, \quad b_1 (u + 10 \epsilon) + b_2 \nu = 0
\]
The action on the Picard group:

\[
\begin{align*}
\tilde{H}_u &= 2H_u + H_v - E_1 - E_3 - E_7 - E_8, \quad \tilde{H}_v = H_u \\
\tilde{E}_1 &= E_2, \quad \tilde{E}_2 = H_u - E_3, \quad \tilde{E}_3 = E_4, \quad \tilde{E}_4 = E_5, \quad \tilde{E}_5 = E_6, \\
\tilde{E}_6 &= H_u - E_1, \quad \tilde{E}_7 = H_u - E_8, \quad \tilde{E}_8 = E_9, \quad \tilde{E}_9 = H_u - E_7.
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\[
a_1 u + a_2 (v - 10\epsilon) = 0, \quad b_1 (u + 10\epsilon) + b_2 v = 0
\]

So if we set \( a = (v - 10\epsilon)/u \quad b = (u + 10\epsilon)/v \) our dynamical system becomes

\[
\begin{align*}
\bar{a} &= \frac{3ab - 2a + 2}{a - 4} \\
\bar{b} &= \frac{4 - a}{2a + 1}
\end{align*}
\]
This system has the following space of initial conditions which define a minimal rational elliptic surface:

\[ F_1 : (a, b) = (0, \infty), \quad F_2 : (a, b) = (\infty, 0), \]
\[ F_3 : (a, b) = (-1/2, 4), \quad F_4 : (a, b) = (-2, \infty) \]
\[ F_5 : (a, b) = (\infty, -2), \quad F_6 : (a, b) = (4, -1/2), \]
\[ F_7 : (a, b) = (-2, -1/2), \quad F_8 : (a, b) = (-1/2, -2). \]
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The invariant is nothing but the proper transform of the anti-canonical divisor:

\[ K_X = 2H_a + 2H_b - \bigoplus_{i=1}^{8} F_i \]

namely

\[ K = \frac{(ab - 1)(ab + 2a + 2b - 5)}{4ab + 2a + 2b + 1} \]
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F_5 : (a, b) = (\infty, -2), \quad F_6 : (a, b) = (4, -1/2), \\
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\]

which is the same as the one given at the beginning [Suris et al. 2012]

\[
K(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)}
\]
The case of icosahedral symmetry:

\[ \bar{x} - x = \epsilon(2x\bar{x} - y\bar{y}) \]

\[ \bar{y} - y = -\epsilon(5y\bar{x} + 5x\bar{y} - y\bar{y}) \]
The case of icosahedral symmetry:

\[ \bar{x} - x = \epsilon (2x\bar{x} - y\bar{y}) \]

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The space of initial condition is given by the \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up at the following 12 points:

- \( E_1 : (x, y) = (\infty, \infty) \)
- \( E_2(-1/7\epsilon, -3/7\epsilon) \)
- \( E_3(-1/7\epsilon, 4/7\epsilon) \)
- \( E_4(1/7\epsilon, 3/7\epsilon) \)
- \( E_5(1/7\epsilon, -4/7\epsilon) \)
- \( E_6(1/5\epsilon, 0) \)
- \( E_7(1/3\epsilon, 0) \)
- \( E_8(1/\epsilon, 0) \)
- \( E_9(-1/\epsilon, 0) \)
- \( E_{10}(-1/3\epsilon, 0) \)
- \( E_{11}(-1/5\epsilon, 0) \)
- \( E_{12} : (1/x, x/y) = (0, 1/3) \)
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\[ E_4 (1/7\epsilon, 3/7\epsilon), \quad E_5 (1/7\epsilon, -4/7\epsilon), \quad E_6 (1/5\epsilon, 0), \]
\[ E_7 (1/3\epsilon, 0), \quad E_8 (1/\epsilon, 0), \quad E_9 (-1/\epsilon, 0), \]
\[ E_{10} (-1/3\epsilon, 0), \quad E_{11} (-1/5\epsilon, 0), \quad E_{12} : (1/x, x/y) = (0, 1/3) \]

Singularity confinement gives the following pattern:

\[ H_y - E_1 (y = \infty) \rightarrow \text{point} \rightarrow \cdots \rightarrow \text{point} \rightarrow H_y - E_1 \]
\[ \cdots \rightarrow \text{point} \rightarrow \text{point} \rightarrow H_x - E_1 (x = \infty) \rightarrow \text{point} \rightarrow \text{point} \rightarrow \cdots . \]
The case of icosahedral symmetry:

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The space of initial condition is given by the \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up at the following 12 points:

\[ E_1 : (x, y) = (\infty, \infty), \ E_2(-1/7\epsilon, -3/7\epsilon), \ E_3(-1/7\epsilon, 4/7\epsilon), \]
\[ E_4(1/7\epsilon, 3/7\epsilon), \ E_5(1/7\epsilon, -4/7\epsilon) \ E_6(1/5\epsilon, 0), \]
\[ E_7(1/3\epsilon, 0), \ E_8(1/\epsilon, 0), \ E_9(-1/\epsilon, 0), \]
\[ E_{10}(-1/3\epsilon, 0), \ E_{11}(-1/5\epsilon, 0). E_{12} : (1/x, x/y) = (0, 1/3) \]

Singularity confinement gives the following pattern:

\[ H_y - E_1 (y = \infty) \rightarrow \text{point} \rightarrow \cdots (4 \text{ points}) \cdots \rightarrow \text{point} \rightarrow H_y - E_1 \]
\[ \cdots \rightarrow \text{point} \rightarrow \text{point} \rightarrow H_x - E_1 (x = \infty) \rightarrow \text{point} \rightarrow \text{point} \rightarrow \cdots . \]

The curve \( 4x + y = 0 : H_x + H_y - E_1 - E_3 - E_5 \) is invariant and we blow it down
So $E_3 \rightarrow H_v = H_x + H_y - E_1 - E_5$ and $E_5 \rightarrow H_u = H_x + H_y - E_1 - E_3$ with

$$H_u \cdot H_u = H_v \cdot H_v = 0, H_u \cdot H_v = 1$$

where the linear systems of $H_u$ and $H_v$ are given by

$$|H_u| : u_0(1 + 7\epsilon x) + u_1(4x + y)$$
$$|H_v| : v_0(1 - 7\epsilon x) + v_1(4x + y).$$
So $E_3 \rightarrow H_v = H_x + H_y - E_1 - E_5$ and $E_5 \rightarrow H_u = H_x + H_y - E_1 - E_3$ with

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If we take the new variables $u$ and $v$ as

$$u = \frac{2(1 + 7\varepsilon x)}{\varepsilon(4x + y)}, \quad v = \frac{2(1 - 7\varepsilon x)}{\varepsilon(4x + y)},$$
So $E_3 \rightarrow H_v = H_x + H_y - E_1 - E_5$ and $E_5 \rightarrow H_u = H_x + H_y - E_1 - E_3$ with

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$$u = \frac{2(1 + 7\varepsilon x)}{\varepsilon(4x + y)}, \quad v = \frac{2(1 - 7\varepsilon x)}{\varepsilon(4x + y)},$$

then we have a new space for initial conditions given by nine blow up points:

$$F_1 : (u, v) = (2, -2), F_2 : (0, -4), F_3 : (4, 0), F_4 : (6, -1), F_5 : (5, -2),$$
$$F_6 : (4, -3), F_7 : (3, -4), F_8 : (2, -5), F_9 : (1, -6).$$
The dynamical system becomes an automorphism having the following topological singularity patterns

\[ H_v - F_9 \rightarrow F_2 \rightarrow F_1 \rightarrow F_3 \rightarrow H_u - F_4 \]
\[ H_v - F_3 \rightarrow F_4 \rightarrow F_5 \rightarrow F_6 \rightarrow F_7 \rightarrow F_8 \rightarrow F_9 \rightarrow H_u - F_2 \]

and \( H_u \rightarrow H_u + H_v - F_2 - F_4 \).
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and \( H_u \to H_u + H_v - F_2 - F_4 \).

The invariant \((-1)\) curve \( H_u + H_v - F_1 - F_2 - F_3 \), which should be blown down.

\[ F_3 \to H_s = H_u + H_v - F_1 - F_2, \quad F_2 \to H_t = H_u + H_v - F_1 - F_3 \]

where the linear systems of \( H_s \) and \( H_t \) are given by

\[ |H_s| : s_0 u(v + 2) + s_1 (u - v - 4) \]
\[ |H_t| : t_0 v(u - 2) + t_1 (u - v - 4) \]
The dynamical system becomes an automorphism having the following topological singularity patterns

\[ H_v - F_9 \to F_2 \to F_1 \to F_3 \to H_u - F_4 \]
\[ H_v - F_3 \to F_4 \to F_5 \to F_6 \to F_7 \to F_8 \to F_9 \to H_u - F_2 \]

and \( H_u \to H_u + H_v - F_2 - F_4 \).
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\[ |H_s| : s_0 u(v + 2) + s_1 (u - v - 4) \]
\[ |H_t| : t_0 v(u - 2) + t_1 (u - v - 4) \]

and hence we take the new variables \( s \) and \( t \) as

\[ s = -\frac{3u(v + 2)}{2(u - v - 4)}, \quad t = -\frac{3v(u - 2)}{2(u - v - 4)} \]
Minimization and invariants

\[
\begin{align*}
\bar{s} &= \frac{2st - 3s - 3t + 9}{s + t - 3} \\
\bar{t} &= \frac{2(s - 3)(t + 3)}{3s - t - 9}
\end{align*}
\]

with the blow-up points
\begin{align*}
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with the blow-up points

\begin{align*}
F_1' : (s, t) &= (3, 0), \quad F_2'(0, 3), \quad F_3'(-3, 2), \quad F_4' : \left(\frac{s}{t - 3}, t - 3\right) = (5, 0), \\
F_5'(2, 3), \quad F_6'(3, 2), \quad F_7' : (s - 3, \frac{t}{s - 3}) = (0, 5), \quad F_8'(2, -3)
\end{align*}
Minimization and invariants

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\begin{cases}
\bar{s} = \frac{2st - 3s - 3t + 9}{s + t - 3} \\
\bar{t} = \frac{2(s - 3)(t + 3)}{3s - t - 9}
\end{cases}
\]

with the blow-up points

\[
F_1^\prime: (s, t) = (3, 0), \quad F_2^\prime(0, 3), \quad F_3^\prime(-3, 2), \quad F_4^\prime: \left(\frac{s}{t - 3}, t - 3\right) = (5, 0),
\]

\[
F_5^\prime(2, 3), \quad F_6^\prime(3, 2), \quad F_7^\prime: (s - 3, \frac{t}{s - 3}) = (0, 5), \quad F_8^\prime(2, -3)
\]

The invariants can be computed by using the the anticanonical divisor:

\[
K = \frac{(s - t)^2 + 4(s + t) - 21}{(s - 2)(t - 2)(2st - 5s - 5t + 15)} = \frac{-56\epsilon^6 y(-3x + y)^2(4x + y)^3}{d_1 d_2 d_3}
\]  \hspace{1cm} (6)

where

\[
\begin{align*}
d_1 &= -3 - 12\epsilon x + 15\epsilon^2 x^2 - 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2 \\
d_2 &= -3 + 12\epsilon x + 15\epsilon^2 x^2 + 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2 \\
d_3 &= -3 + 27\epsilon^2 x^2 + 10\epsilon^2 xy + 10\epsilon^2 y^2.
\end{align*}
\]
Tropical dynamical systems

Motivation: How simple can a nonlinearity be? How do behave the discrete equations with the simplest nonlinearity?
Tropical dynamical systems

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Answer: The simplest nonlinear function is \( f(x) = |x| = 2 \max(0, x) - x \) and the procedure of reducing a nonlinear discrete equation to one having only max-nonlinearities and addition in an algorithmical way is called *ultradiscretisation* or *tropicalisation*.
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Mathematically the tropicalisation has been introduced as follows: Calling \( \mathbb{R}_{\max} = \mathbb{R} \cup \{ -\infty \} \) we introduce the semiring \( \{ \mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e \} \) through the following definitions:

- \( a \oplus b := \max(a, b) \), \( a \otimes b := a + b \)
- \( \varepsilon := -\infty \), \( e := 0 \)

The main news is that there is no additive inverse and the addition is idempotent, making all calculation extremely hard.
Minimization and invariants

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A nonlinear discrete equation (ordinary or partial) with *positive definite dependent variable* $x_n$ can be ultradiscretised or tropicalised using the following substitution and formula:

$$x_n = e^{X_n/\varepsilon} \lim_{\varepsilon \to 0^+} \varepsilon \ln(1 + x_n) = \max(0, X_n)$$
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\[
x_n = e^{X_n/\epsilon} \quad \lim_{\epsilon \to 0^+} \varepsilon \ln(1 + x_n) = \max(0, X_n)
\]

Example:

\[
x_{n+1}x_{n-1} = a \frac{1 + x_n}{x_n^2}, \quad I_n = \frac{a(1 + x_n + x_{n+1}) + x_n^2 x_{n+1}^2}{x_n x_{n+1}}
\]

If \( x_n = \exp(X_n/\epsilon) \), \( a = \exp A/\epsilon \) then we get the tropical equation and the invariant:
\[ X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n, \quad \mathcal{I} = \max(2X_n + 2X_{n+1}, A, A + X_n, A + X_{n+1}) - X_{n+1} - X_n \]
\[ X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n, \quad I = \max(2X_n + 2X_{n+1}, A, A + X_n, A + X_{n+1}) - X_{n+1} - X_n \]

Question: What is singularity here? Can one compute the invariant?
\[ X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n, \quad \mathcal{I} = \max(2X_n + 2X_{n+1}, A, A + X_n, A + X_{n+1}) - X_{n+1} - X_n \]

Question: What is singularity here? Can one compute the invariant? The only \textbf{visible} singularity is the discontinuity point. Can we imagine a "singularity confinement" here?
\[ X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n, \quad \mathcal{I} = \max(2X_n + 2X_{n+1}, A, A + X_n, A + X_{n+1}) - X_{n+1} - X_n \]

Question: What is singularity here? Can one compute the invariant?
The only visible singularity is the discontinuity point. Can we imagine a "singularity confinement" here?
YES! We shall thus examine the behaviour of a singularity appearing at, say, \( n = 1 \) where \( X_1 = \epsilon \), while \( X_0 \) is regular and look at the propagation of this singularity both forwards and backwards. Introducing the notation \( \mu \equiv \max(\epsilon, 0) \), the presence of \( \mu \) indicates that the value of \( X \) is singular. We get (for \( X_0 > A \)): 
\[ X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n, \quad I = \max(2X_n + 2X_{n+1}, A, A + X_n, A + X_{n+1}) - X_{n+1} - X_n \]

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\[
\begin{align*}
X_{-3} &= A - \epsilon \\
X_{-2} &= X_0 - A + 2\epsilon \\
X_{-1} &= -X_0 + A - \epsilon \\
X_0 &= X_0 \\
X_1 &= \epsilon \\
X_2 &= A - X_0 - 2\epsilon + \mu \\
X_3 &= 2X_0 - A + 3\epsilon - 2\mu \\
X_4 &= A - X_0 - \epsilon + \mu \\
X_5 &= -\epsilon \\
X_6 &= X_0 + 2\epsilon
\end{align*}
\]
Conclusions

- Singularities are essential in analysing discrete dynamical systems.
- The singularity structure may give a non-minimal elliptic surface. In order to make it minimal one has to blow down some -1 divisor classes (one has to prove the existence of the blow-down structure).
- After minimization the mapping can be "solved".
- We expect to find analogies in the case of tropical dynamical systems using tropical algebraic geometry.

All the results are published here: