Reconstruction Theorems for Flux Compactifications

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Setting and notations

- (M, g) = oriented, connected and paracompact pseudo-Riemannian smooth manifold of dimension d.
- (p, q)=the signature type of the metric g on M $(p + q = d)$; we say that (M, g) is of *simple normal type* if $p - q \equiv_8 0, 2$. Most of the talk will be limited to this case, although appropriate generalizations of the results exist for all other cases.
- If V is a smooth vector bundle on M, recall that the *support* of a global section $s \in \Gamma(M, V)$ is the open subset of M defined through:

$$\mathrm{supp}(s) \stackrel{\text{def.}}{=} \{p \in M | s(p) \neq 0_p\} \quad ,$$

where $0_p \in V_p$ denotes the zero vector of the fiber V_p of V at p. In general, $\mathrm{supp}(s)$ is disconnected.
- If A is any subset of M, we let $\chi_A : M \to \mathbb{R}$ denote its characteristic function:

$$\chi_A(p) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{if } p \in M \setminus A \end{cases} \quad .$$

Locally constant signs

Definition. A *locally constant sign (l.c.s.) function* is a function $\mu : A \to \{-1, 1\}$ which is locally constant on $A \subset M$, where A is endowed with the topology induced from M.

Such a function is constant on each connected component of A, where it takes value $+1$ or -1.

Locally constant sign functions defined on A form a commutative group with respect to pointwise multiplication and every element of this group squares to the identity; this group is isomorphic with the direct product of $\mathrm{Card}[\pi_0(A)]$ copies of $\mathbb{Z}_2$.

Definition. We say that two sections $s, s' \in \Gamma(M, V)$ *differ by locally constant signs* (and write $s \sim s'$) if $\mathrm{supp}(s') = \mathrm{supp}(s)$ and there exists a locally constant sign function $\mu : \mathrm{supp}(s) \to \{-1, 1\}$ such that $s' = \mu s$. This defines an equivalence relation on $\Gamma(M, V)$; we denote the equivalence class of s under this relation through $[s]$ and say that $[s]$ is a *global section of V considered up to locally constant signs*.

Reconstructing systems of global sections of Euclidean vector bundles from systems of endomorphisms

Let S be a vector bundle over M endowed with a fiberwise Euclidean scalar product $\mathscr{B}$. For any $\xi, \xi' \in \Gamma(M, S)$, define $E_{\xi,\xi'} \in \Gamma(M, \mathrm{End}(S))$ through:

$$\boxed{E_{\xi,\xi'}(\xi'') \stackrel{\text{def.}}{=} \mathscr{B}(\xi'', \xi')\xi \quad , \quad \forall \xi'' \in \Gamma(M, S)} \quad .$$

We have:

$$\mathrm{tr} E_{\xi,\xi'} = \mathscr{B}(\xi, \xi') \quad , \quad (E_{\xi,\xi'})^t = E_{\xi',\xi} \quad ,$$

where ^{t} denotes the $\mathscr{B}$-transpose as well as:

$$E_{\xi_1,\xi_2} \circ E_{\xi_3,\xi_4} = \mathscr{B}(\xi_2, \xi_3) E_{\xi_1,\xi_4} \quad .$$

Reconstructing unconstrained sections

Theorem. Giving l.c.s. classes of smooth global sections $\xi_i \in \Gamma(M, S)$ $(i = 1, \ldots, s)$ is equivalent with giving l.c.s. classes of global endomorphisms $E_i \in \Gamma(M, \mathrm{End}(S))$ $(i = 1, \ldots, s)$ which satisfy the conditions:

$$\boxed{\begin{array}{l} E_i^2 = \mathrm{tr}(E_i) E_i \\ E_i^t = E_i \quad . \end{array}}$$

Moreover, E_i have the form:

$$\boxed{E_i = \mu_i E_{\xi_i, \xi_i} \quad \text{i.e.} \quad E_i(\xi) = \mu_i \mathscr{B}(\xi, \xi_i) \xi_i}$$

where $\mu_i : \mathrm{supp} Q_i \to \{-1, 1\}$ are l.c.s. functions and they determine ξ_i up to l.c.s. through these relations. The following relations hold:

$$\begin{aligned} \mathrm{supp}(E_i) &= \mathrm{supp}(\xi_i) = \mathrm{supp}(\mu_i) \\ \mathrm{tr}(E_i) &= \mu_i \quad , \quad \forall i = 1 \ldots s \quad . \end{aligned}$$

Finally, ξ_i have unit $\mathscr{B}$-norm everywhere iff. $|\mathrm{tr}(E_i)| = 1_M$ for all i, in which case μ_i are constant on M and equal to ± 1.

Reconstructing everywhere orthonormal sections

Proposition. Giving an everywhere orthonormal system of global smooth sections $\xi_i \in \Gamma(M, S)$ is equivalent to giving a system of global endomorphisms $E_i \in \Gamma(M, \mathrm{End}(S))$ which satisfy the following conditions:

$$\boxed{\begin{array}{l} E_i^2 = E_i \\ E_i^t = E_i \\ \mathrm{tr}(E_i) = 1_M \\ \mathrm{tr}(E_i \circ E_j) = 0_M \ \ \mathrm{for} \ \ i < j \ \ . \end{array}} \ .$$

Furthermore, a solution $(E_i)_{i=1\ldots s}$ of this system determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the conditions:

$$E_i = E_{\xi_i, \xi_i} \ \ ,$$

up to *independent* ambiguities of the form:

$$\xi_i \to -\xi_i \ \ .$$

Flat constrained sections

Let $\mathcal{Q} \in \Gamma(M, \mathrm{End}(S))$ be a smooth global endomorphism of S and $D : \Gamma(M, S) \to \Omega^1(M, \mathrm{End}(S))$ be a connection on S which is compatible with $\mathscr{B}$ in the sense that $\mathscr{B}$ is D-flat:

$$\partial_m \mathscr{B}(\xi', \xi'') = \mathscr{B}(D_m\xi', \xi'') + \mathscr{B}(\xi', D_m\xi'') \ , \ \ \forall \xi', \xi'' \in \Gamma(M, S) \ .$$

We have the identities:

$$\begin{aligned} D_m^{\mathrm{ad}}(E_{\xi,\xi'}) &= E_{D_m\xi,\xi'} + E_{\xi,D_m\xi'} \ , \\ \mathcal{Q} \circ E_{\xi,\xi'} &= E_{\mathcal{Q}\xi,\xi'} \ , \ \ E_{\xi,\xi'} \circ \mathcal{Q}^t = E_{\xi,\mathcal{Q}\xi'} \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ , \end{aligned}$$

where D^{ad} is the connection induced by D on the bundle $\mathrm{End}(S)$.

Definition. A *$\mathcal{Q}$-constrained and D-flat section* of S is a smooth global section $\xi \in \Gamma(M, S)$ which satisfies:

$$\boxed{D_m\xi = 0 \ , \ \ \mathcal{Q}\xi = 0} \ .$$

Flat constrained sections

The $\mathscr{B}$-pairing of any two D_m-flat pinors is constant on M and in particular that any two solutions ξ_1, ξ_2 of the equations above have constant $\mathscr{B}$-pairing. Since the parallel transport of D_m preserves $\mathscr{B}$ by, it follows that any two solutions which are linearly independent at a point are linearly independent everywhere and can be replaced via the Gram-Schmidt algorithm by two solutions which are $\mathscr{B}$-orthogonal everywhere and whose local values at any point span the same subspace of the fiber of S at that point as the two original solutions. This implies that we can always find a basis of solutions which consists of sections of S that are everywhere $\mathscr{B}$-orthogonal.

Characterizing flat constrained sections

Theorem. Giving s flat constrained sections $\xi_1, \ldots, \xi_s$ which are $\mathscr{B}$-orthonormal everywhere is equivalent to giving s globally-defined endomorphisms $E_1, \ldots, E_s \in \Gamma(M, \mathrm{End}(S))$ which satisfy (6) as well as the conditions:

$$D_m^{\mathrm{ad}}(E_i) = \mathcal{Q} \circ E_i = 0 \quad , \quad \forall i = 1 \ldots s \quad .$$

Furthermore, a solution $(E_i)_{i=1\ldots s}$ of (6) determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the conditions:

$$E_i = E_{\xi_i, \xi_i} \quad ,$$

up to *independent* ambiguities of the form:

$$\xi_i \to -\xi_i \quad .$$

Since by the argument recalled above any system of independent solutions can be replaced (upon making linear combinations with coefficients from $\mathcal{C}^\infty(M, \mathbb{R})$) with a system of solutions which are $\mathscr{B}$-orthonormal everywhere, this result gives a complete characterization.

Basics

Let S be a pin bundle on M, with underlying pin representation $\gamma : (\wedge T^* M, \diamond) \to (\mathrm{End}(S), \circ)$. Let $\mathscr{B}$ be an admissible bilinear form on S (in the sense of Alekseevski and Cortes). We assume that the signature (p, q) of g is such that we are in the normal case, i.e. the Schur algebra of γ equals the base field $\mathbb{R}$. This occurs iff $p - q \equiv_8 0, 1, 2$ and has two subcases, namely the normal simple case ($p - q \equiv_8 0, 2$) and the normal non-simple case ($p - q \equiv_8 1$). The restriction of γ gives an isomorphism of bundles of algebras from a certain sub-bundle $\wedge^\gamma T^* M$ of the exterior bundle to $\mathrm{End}(S)$, whose inverse we call *vertical dequantization morphism* and we denote through γ^{-1}. Let:

$$\check{T} \stackrel{\mathrm{def.}}{=} \gamma^{-1}(T) \in \Omega^\gamma(M) \stackrel{\mathrm{def.}}{=} \Gamma(M, \wedge^\gamma T^* M)$$

for any globally-defined endomorphism $T \in \Gamma(M, \mathrm{End}(S))$. Recall the description of $\Omega^\gamma(M)$ in the normal cases:

$$\Omega^\gamma(M) = \begin{cases} \Omega(M) & \text{if we are in the normal simple case} \\ \Omega^{\epsilon_\gamma}(M) & \text{if we are in the normal non-simple case} \end{cases},$$

where $\Omega^\pm(M)$=$\mathcal{C}^\infty(M, \mathbb{R})$-modules of twisted (anti-)self-dual forms.

Symmetric admissible pairings and scalar products

- In the normal simple case the bundle S admits (up to rescaling by everywhere non-vanishing smooth functions) two admissible pairings which have opposite types and which are both symmetric when $d \equiv_8 0$, both antisymmetric when $d \equiv_8 4$ while having opposite symmetries when $d \equiv_8 2, 6$. Symmetric choices of admissible pairing exist iff. $d \equiv_8 0, 2, 6$ (the fist of these cases allowing for two essentially distinct choices). In this case, d is even and we have $\Omega^\gamma(M) = \Omega(M)$.
- In the normal non-simple case the bundle S admits a single admissible pairing $\mathscr{B}$ (up to multiplication by a nowhere vanishing smooth function), which is symmetric when $d =_8 1, 7$ and antisymmetric otherwise. In the symmetric case, the type $\epsilon_{\mathscr{B}}$ of the pairing equals $+1$ when $d \equiv_8 1$ and -1 when $d \equiv_8 7$, i.e. we have $d \equiv_8 \epsilon_{\mathscr{B}}$. In this case, d is odd and we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$ and $\Omega^\gamma(M) = \Omega^{\epsilon_\gamma}(M)$ where $\epsilon_\gamma \in \{-1, 1\}$ is the signature of γ.

In those cases when a symmetric choice of admissible pairing is possible, we can pick an admissible form which is in fact a scalar product.

Reversion, trace and generalized products

We shall need the *modified reversion* $\tau_{\mathscr{B}}$ defined through:

$$\tau_{\mathscr{B}} \stackrel{\text{def.}}{=} \tau \circ \pi^{\frac{1-\epsilon_{\mathscr{B}}}{2}} = \begin{cases} \tau\ , & \text{if } \epsilon_{\mathscr{B}} = +1 \\ \tau \circ \pi\ , & \text{if } \epsilon_{\mathscr{B}} = -1 \end{cases} ,$$

(where $\epsilon_{\mathscr{B}}$ is the *type* of $\mathscr{B}$) and the trace $\mathcal{S} : \Omega^{\gamma}(M) \to \mathcal{C}^{\infty}(M, \mathbb{R})$:

$$\mathcal{S}(\omega) \stackrel{\text{def.}}{=} \omega^{(0)} N_{p,q} \mathrm{rk} S \ ,$$

where $\omega^{(0)} \in \mathcal{C}^{\infty}(M, \mathbb{R})$ denotes the rank zero component of ω and $N_{p,q}$ equals 1 or 2 according to whether we are in the simple or non-simple case. We have:

$$\mathcal{S}(\check{T}) = \mathrm{tr}(T) \ , \ \forall T \in \Gamma(M, \mathrm{End}(S)) \ .$$

For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. we have the expansion:

$$\omega \diamond \eta = \sum_{m=0}^{\min(k,l)} (-1)^{\left[\frac{m+1}{2}\right]} \pi^m(\omega) \triangle_m \eta = \sum_{m=0}^{\min(k,l)} (-1)^{km+\left[\frac{m+1}{2}\right]} \omega \triangle_m \eta \ ,$$

where $\triangle_m$ are the so-called generalized products. The symbol $\langle\ ,\ \rangle$ denotes the $\mathcal{C}^{\infty}(M, \mathbb{R})$-valued pairing induced by g on $\Omega(M)$.

Characterizing systems of everywhere orthonormal pinors

Proposition. Giving s smooth global sections $\xi_1, \ldots, \xi_s \in \Gamma(M, S)$ which are everywhere $\mathscr{B}$-orthonormal amounts to giving s globally-defined smooth forms $\check{E}_1, \ldots, \check{E}_s \in \Omega^\gamma(M)$ specified up to independent signs (i.e. up to independent ambiguities of the form $\check{E}_i \to -\check{E}_i$) such that the following system of relations is satisfied, where $i, j = 1 \ldots s$:

$$\boxed{\begin{aligned} &\check{E}_i \diamond \check{E}_i = \check{E}_i \\ &\tau_{\mathscr{B}}(\check{E}_i) = \check{E}_i \\ &\mathcal{S}(\check{E}_i) = 1_M \\ &\mathcal{S}(\check{E}_i \diamond \check{E}_j) = 0_M \quad \text{for} \quad i < j \end{aligned}}$$

Furthermore, a solution of this system determines the corresponding sections ξ_i through the relations:

$$\check{E}_i = \check{E}_{\xi_i, \xi_i}$$

up to *independent* ambiguities of the type:

$$\xi_i \to -\xi_i \quad .$$

Constrained Killing spinors

Consider the situation in which S is endowed with a $\mathscr{B}$-compatible connection D_m and with a global endomorphism $\mathcal{Q} \in \Gamma(M, \mathrm{End}(S))$. Then D-flat and $\mathcal{Q}$-constrained sections of S are called *constrained generalized Killing (s)pinors* (CGKS). These arise whens studying the problem of characterizing supersymmetric backgrounds of supergravity theories in the presence of fluxes.

The connection D_m induces a derivation $\check{D}_m$ of the reduced Kähler-Atiyah algebra $\Omega^{\gamma}(M)$ while $\mathcal{Q}$ induces an element $\check{\mathcal{Q}} \stackrel{\mathrm{def.}}{=} \gamma^{-1}(\mathcal{Q})$ of this algebra. Furthermore, we have $\check{D_m(T)} = \check{D}_m(\check{T})$ and $\check{\mathcal{Q} \circ T} = \check{\mathcal{Q}} \circ \check{T}$ for all $T \in \Gamma(M, \mathrm{End}(S))$.

Characterizing systems of constrained generalized Killing (s)pinors

Theorem. Giving s globally-defined smooth (s)pinors $\xi_1, \ldots, \xi_s$ which satisfy (7) and which are $\mathscr{B}$-orthonormal everywhere is equivalent to giving s globally-defined forms $\check{E}_1, \ldots, \check{E}_s \in \Omega^\gamma(M)$ which satisfy the conditions of the previous Proposition as well as the conditions:

$$\boxed{\check{D}_m^{\mathrm{ad}}(\check{E}_i) = \check{\mathcal{Q}} \diamond \check{E}_i = 0_M} \quad , \quad \forall i = 1 \ldots s \quad .$$

Furthermore, a solution $(\check{E}_i)_{i=1\ldots s}$ determines the corresponding everywhere $\mathscr{B}$-orthonormal system of sections of S via the conditions:

$$\check{E}_i = \check{E}_{\xi_i, \xi_i} \quad ,$$

up to *independent* ambiguities of the form:

$$\xi_i \to -\xi_i \quad .$$

In the normal non-simple case, one has a variant of this result formulated using the so called truncated model of the reduced Kähler-Atiyah algebra.

Basics

Consider supergravity on an 11-dimensional spin manifold $\tilde{M}$ with Lorentzian metric $\tilde{g}$ (of 'mostly plus' signature). Besides the metric, the action of the theory contains the three-form potential with four-form field strength $\tilde{G} \in \Omega^4(\tilde{M})$ and the gravitino $\tilde{\Psi}_M$, which is a real pinor of spin 3/2. For supersymmetric bosonic classical backgrounds, both the gravitino and its supersymmetry variation must vanish, which requires that there exists at least one solution $\tilde{\eta}$ to the equation:

$$\delta_{\tilde{\eta}} \tilde{\Psi}_M \stackrel{\text{def.}}{=} \tilde{\mathcal{D}}_M \tilde{\eta} = 0 \quad ,$$

where uppercase indices run from 0 to 10 and $\tilde{\mathcal{D}}_M$ is the supercovariant connection. The eleven-dimensional supersymmetry generator $\tilde{\eta}$ (a Majorana pinor field of spin 1/2) is a smooth section of the pin bundle $\tilde{S}$ of M, which is a rank 32 real vector bundle defined over $\tilde{M}$.

Compactifications down to AdS_3

Consider compactification down to an AdS_3 space of cosmological constant $\Lambda = -8\kappa^2$, where κ is a positive real parameter. Thus $\tilde{M} = N \times M$, where N is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the AdS_3 metric while M is an oriented Riemannian eight-manifold whose metric we denote by g. The metric on $\tilde{M}$ is a warped product:

$$\mathrm{d}\tilde{s}^2_{11} \quad = \quad e^{2\Delta}\mathrm{d}s^2_{11} \quad \text{where} \quad \mathrm{d}s^2_{11} = \mathrm{d}s^2_3 + g_{mn}\mathrm{d}x^m\mathrm{d}x^n \quad .$$

The warp factor Δ is a smooth function defined on M while $\mathrm{d}s^2_3$ is the squared length element on N. For the field strength $\tilde{G}$, we use the ansatz:

$$\tilde{G} = e^{3\Delta}G \quad \text{with} \quad G = \mathrm{vol}_3 \wedge f + F \quad ,$$

where $f \in \Omega^1(M)$, $F \in \Omega^4(M)$ and vol_3 is the volume form of N. For $\tilde{\eta}$, we use the ansatz:

$$\tilde{\eta} = e^{\frac{\Delta}{2}}\eta \quad \text{with} \quad \eta = \psi \otimes \xi \quad ,$$

where ξ, ψ are real pinors on M and N. We *do not require that ξ has definite chirality*.

Reduction to a CGK problem

Assuming that ψ is a Killing pinor on the AdS_3 space, the supersymmetry condition reduces to a CGK condition for ξ, where D_m and $\mathcal{Q} \in \Gamma(M, \mathrm{End}(S))$ can be computed directly.

The space of solutions to the CGKS equations is a finite-dimensional $\mathbb{R}$-linear subspace $\mathcal{K}(D, \mathcal{Q})$ of the space $\Gamma(M, S)$ of smooth sections of S. This subspace is trivial for generic metrics g and fluxes F and f on M, since the generic compactification of the type we consider breaks all supersymmetry. The interesting problem is to find those metrics and fluxes on M for which some fixed amount of supersymmetry is preserved in three dimensions, i.e. for which the space $\mathcal{K}(D, \mathcal{Q})$ has some given non-vanishing dimension, which we denote by s; this is the number of supersymmetries preserved in three dimensions. The pairing $\mathscr{B}$ on S is D_m-flat (see (7)), so that D_m is an $O(16)$-connection. Hence requiring that our background preserves s independent supersymmetry generators (i.e. requiring that $\dim_{\mathbb{R}} \mathcal{K}(D, \mathcal{Q}) = s$) is *equivalent* to requiring that there are s CGK spinors which are $\mathscr{B}$-orthonormal at every point of M.

The pairing and reversion in this case

Since $p - q \equiv_8 0$, we are in the normal simple case, so $\Omega^\gamma(M) = \Omega(M)$. For these values of p and q (namely $p = 8$ and $q = 0$), one has (up to rescalings by smooth nowhere vanishing real-valued functions defined on M) two admissible pairings $\mathscr{B}_\pm$ on S, both of which are symmetric and having the types $\epsilon_{\mathscr{B}_\pm} = \pm 1$. Since any choice of admissible pairing leads to the same result, we choose to work with $\mathscr{B} \stackrel{\mathrm{def.}}{=} \mathscr{B}_+$ without loss of generality. Then $\tau_{\mathscr{B}}$ coincides with the canonical reversion τ of the Kähler-Atiyah algebra of (M, g), which is defined through:

$$\tau(\omega) \stackrel{\mathrm{def.}}{=} (-1)^{\frac{k(k-1)}{2}} \omega \quad , \quad \forall \omega \in \Omega^k(M) \quad .$$

Upon rescaling by a smooth function, we can in fact take $\mathscr{B}$ to be a fiberwise scalar product on S; we hence denote the corresponding norm through $|| \ ||$.

The case $s = 1$

This case was studied by Martelli & Sparks and Tsimpis and more completely in our previous work using Kähler-Atiyah algebra techniques. Since $N_{8,0} = 1$ and $\mathrm{rk} S = 16$, the general definition of the trace on the Kähler-Atiyah algebra becomes:

$$\mathcal{S}(\omega) = 16\omega^{(0)} \quad , \quad \forall \omega \in \Omega(M) \quad .$$

The non-quadratic relations in the reconstruction theorem applied to this case amount to the statement that $\check{E} \in \Omega(M)$ expands as

$$\boxed{\check{Q} = \frac{1}{16}(1 + V + \Phi + Z + b\nu)} \quad ,$$

where $b \in \mathcal{C}^\infty(M, \mathbb{R})$, $V \in \Omega^1(M), \Phi \in \Omega^4(M)$, $Z \in \Omega^5(M)$ and ν is the canonical volume form of (M, g). The reconstruction theorem tells us that imposing $\check{Q} \diamond \check{Q} = \check{Q}$ *guarantees* that $Q = E_{\xi,\xi}$ and hence that $\check{Q} = \check{E}_{\xi,\xi}$ for some globally-defined normalized pinor $\xi \in \Gamma(M, S)$ which is determined up to sign by this condition. This recovers the Fierz identities of Martelli-Sparks purely from Kähler-Atiyah algebra relations. The GKS equations in Kähler-Atiyah form recover their remaining results.

The case $s > 2$ and generalization

Applying the reconstruction theorem to this case shows that the full system of Fierz relations plus the GKS constraints (differential and algebraic) is equivalent with two copies of the system found the $s = 1$ case plus a single quadratic constraint which couples these two systems. This gives a drastic simplification of the system of relations that we extracted in previous work and provides a way to describe the geometry in terms of certain foliations.

Using the reconstruction theorem, one can similarly characterize supersymmetric backgrounds of this type which preserve more than two supersymmetries, since the reconstruction theorem allows one to reduce the case of s GKS spinors to s copies of a single GKS spinor plus a set of $\frac{s(s-1)}{2}$ quadratic algebraic constraints, thus allowing a systematic study for $s > 1$.

One can generalize these results to the non-normal case and to pairings $\mathscr{B}$ which are antisymmetric, using a certain 'non-degeneracy condition'.