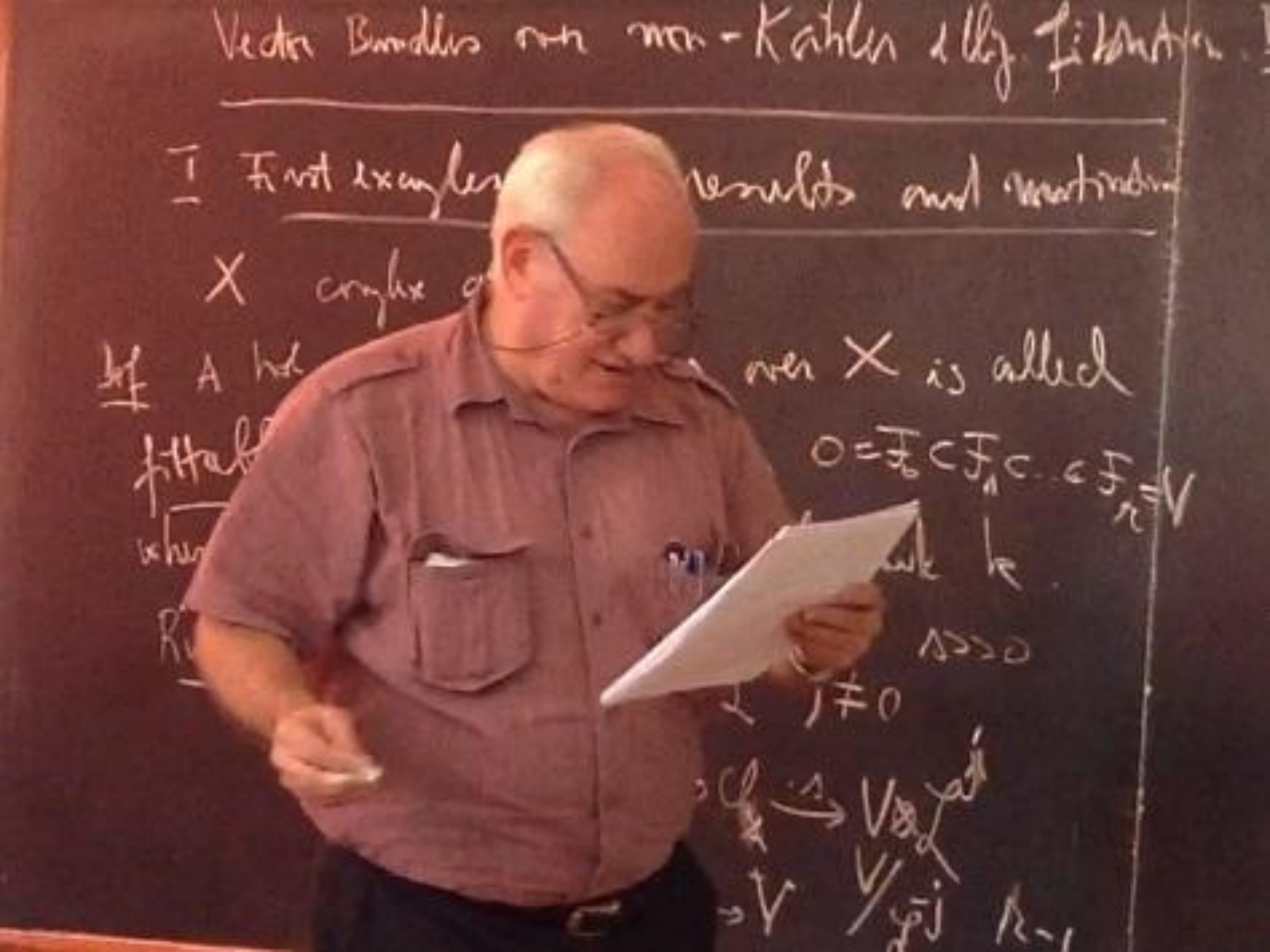
Vector Bundles on non-Kähler elliptic fibrations
I First examples results and motivation
X complex
Def A hol over X is called
filtrable
0 = F_0 ⊂ F_1 ⊂ ... ⊂ F_r = V
≠ 0

Vector Bundles over non-Kähler ell. fibrations

I First examples, some results and motivation

X complex cpt. mfd.

Def A hol. v.b. V of rk $= r$ over X is called *filtrable* if $\exists$ a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset .. \subset \mathcal{F}_r = V$ where $\mathcal{F}_k$ is a coh. subsheaf of rank k.

Rmk X proj mfd $H^0(V \otimes \mathcal{L}^j) \neq 0$ $j >> 0$

$\mathcal{L}$ v ample

$$0 \to \mathcal{O} \xrightarrow{s} V \otimes \mathcal{L}^j$$

$$\mathcal{L}^{-j} \hookrightarrow V \qquad V/\mathcal{L}^{-j} \quad r-1$$

Def. A hol. v. b. V over X is called **reducible** if $\exists$ coh. analytic subsheaf $\mathcal{F}$ such that $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(V)$

$\overset{\shortparallel}{n}$

irreducible $\qquad$ **$n=2$** $\qquad$ $\boxed{\text{non-filtrable} = \text{irred.}}$

Exp (1992) E–F $\qquad$ non-filtrable rk 2 v. b

$\overset{\shortparallel}{\text{irreducible}}$

V on X 2-torus $\quad NS(X) = 0 \qquad X = \mathbb{C}^2/\Lambda$

versal def of T_X $\quad$ dim(versal def) $>$ dim(space parametrizing all filtrable-bundles

Exp 2 $\quad X$ K3-surface $\quad$ simply connected $\quad \omega_X \cong \mathcal{O}_X \qquad \mathrm{Pic}(X) = 0 = \{\mathcal{O}_X\}$.

$H^0(X, T_X) = 0 \quad H^0(X, T_X^*) = 0 \qquad a(X) = 0$

T_X is non-filtrable $=$ irred.

Exp. 3. X K3-surface $\mathrm{Pic}(X)=0$

non-vanishing H-field $\overset{!}{\Longrightarrow}$ X non-Kähler

$$0 \to \underset{1}{\mathcal{O}_X} \to \underset{4}{T_X \otimes T_X} \to \underset{3}{\overset{\swarrow V}{S^2 T_X}} \to 0$$

V is irreducible.

Def. X surface, $\mathcal{F}$ coh. sheaf of rank $r>0$.

$c_1(\mathcal{F})$, $c_2(\mathcal{F})$ $\quad \Delta(\mathcal{F}) := \frac{1}{r}\left(c_2(\mathcal{F}) - \frac{r-1}{2r} c_1^2(\mathcal{F})\right)$. discriminant of $\mathcal{F}$

Def $a \in NS(X)$, $r>0$, X non-algebraic

$$m(r,a) := -\frac{1}{2r} \max_{\mu_i} \left\{ \sum_{i=1}^{r} \left(\frac{a}{r} - \mu_i\right)^2 \;\middle|\; \mu_i \in NS(X),\ \mu_1+\mu_2+\dots+\mu_r = a \right\}$$

1985 B, Flenner. $r=2$

Thm (Bănică - Le Potier, 87) X non-alg. surface.

A) $\Delta(\mathcal{F}) \geq 0$

B) If $r \geq 2$ a top. v.b. V over X has a filtrable hol. structure iff $c_1(V) \in NS(X)$, $\Delta(V) \geq m(r, c_1(V))$ with one exception: X K3-surface, $a(X)=0$, $c_1(V) \in r\,NS(X)$, $\Delta(V) = \frac{1}{r}$.

Vector Bundles over non-Kähler ellip. fibrations

X CY 3-fold

II. fibr. over non-Kähler elliptic surfaces

(B, Moraru).

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ B \end{tikzcd}$$ curve

E fibre elliptic

$X \xrightarrow{\pi} B$ principal elliptic bundles, 2-fold.

Th (B, JB) $NS(X)/\mathrm{Tors}\,NS(X) \cong \mathrm{Hom}(J_B, \mathrm{Pic}^0(E))$

$\hat{q}(X)$

$\mathrm{Pic}^0(E) = E^* \simeq E$

section

$B \longrightarrow B \times E^*$

$\parallel$

$J(X)$

affine over

in E... fibration

X CY 3-fold

V v.b of rank 2

$b_1, b_2, \dots, b_k \in B$

$(J_B, \mathrm{Pic}^0(E))$

$\boxed{\overline{C} \cdot E^* = 2}$

section

$B \longrightarrow \underset{\parallel}{B \times E^*}$

over

in the Jacobian $J(X) \supset \overline{C}$

$r = 2$ twisted — F-M transform

FM

$$S_V = \sum_1^k (\{b_i\} \times E^*) + \overline{C}$$

$$\delta = \det(V)$$

$$J(X) = B \times E^* \xrightarrow{i_\delta} B \times E^*$$

$$(b, \lambda) \longrightarrow (b, \delta_b \otimes \lambda^{-1})$$

$$J(X) \xrightarrow{\xi} J(X)/_{i_\delta} = \mathbb{F}_\delta$$

ruled surface

$$\overline{C} \longrightarrow A = \underset{\text{a curve}}{\xi}(\overline{C})$$

$$G_V = \sum_1^k f_i + A$$

fibres in $\mathbb{F}_\delta$

graph of V

$$G : \mathcal{M}_{\delta, c_2} \longrightarrow \mathrm{Div}(\mathbb{F}_\delta)$$

$$V \longmapsto G_V$$

1) smooth part of $\mathcal{M}_{\delta, c_2}$

2) Neces. cond. for $\mathcal{M}_{\delta, c_2} \neq \phi$

3) General fibre of G = Proj. variety

4) dim of singular fibres

Stable 2-v.b.

$\det(V) = \delta$

$c_2 = c_2(V)$

Gieseker–Maruyama

$$\overline{C} \xrightarrow{2:1} B$$

Vector Bundles on non-Kähler elliptic Fibrations

~~E elliptic curve~~

II dim $X \geq 3$ $X \xrightarrow{\pi} S$ proj. elliptic fibration

dim $X \geq 3$ dim $S \geq 2$ non-Kähler

Halanay, Trautmann

$\mathrm{Pic}^0_S(X)$ ⇙

$J(X) = B \times E^*$ "coarse moduli space" (never fine)

$\omega_S = \mathcal{O}_S$

⇓

$\omega_X = \mathcal{O}_X$

twisted universal sheaf

$D^b(X) \xrightarrow{\text{F-M}} D^b(J(X), \alpha)$ $\alpha \in H^2(J, \mathcal{O}_J^*)$ not torsion

$\mathcal{M}_{\text{totally semi-stable}}(X) \xrightarrow{\sim}$ <u>Drinfeld space</u>

$$\begin{tikzcd}
X \arrow[d, "\pi"] \\
S \arrow[d] \\
B
\end{tikzcd}$$