On minimization of rational elliptic surfaces obtained from birational dynamical systems

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- Discrete Nahm equations
- Minimization of elliptic surfaces and invariants

The systems under consideration have the rational reversible form:

$$(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (\overline{x}, \overline{y}) \in \mathbb{P}^1 \times \mathbb{P}^1$$

$$\overline{x} = F(x, y)$$

$$\overline{y} = G(x, y)$$

and also the inverse (F, G, Φ, Γ are rational functions of x, y)

$$\underline{x} = \Phi(x, y)$$

$$\underline{y} = \Gamma(x, y)$$

The projective space $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by the following coordinate systems ($X = 1/x, Y = 1/y$):

$$\mathbb{P}^1 \times \mathbb{P}^1 = (x, y) \cup (X, y) \cup (x, Y) \cup (X, Y)$$

Analytical stability and blowing-down structure

Analytical stability and blowing-down structure

Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a birational automorphism with iterates growing quadratically with n.
For any such automorphism we can blow up $\mathbb{P}^1 \times \mathbb{P}^1$ and construct a rational surface X such that: $\tilde{\phi} : X \to X$ with $\phi = \tilde{\phi}$ in general and $\tilde{\phi}$ is analytically stable which means: $(\tilde{\phi^*})^n = (\tilde{\phi^n})^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$
Analitical stability is equivalent with the following: There is no divisor D such that exist $k > 0$ and $\tilde{\phi}(D) =$point, $\tilde{\phi}^k(D) =$ indeterminate

$$D \to \bullet \to \bullet \to ...\bullet \to D'$$

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\phi}} & X \\ \mu \downarrow & & \downarrow \mu \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

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- Blow down the (-1) curves in the following way: Let C be the (-1) divisor class and F_1, F_2 two divisor classes such that

$$F_1 \cdot F_1 = F_2 \cdot F_2 = 0, \quad F_1 \cdot F_2 = 1, \quad C \cdot F_1 = C \cdot F_2 = 0$$

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- the genus formula is helping here $g = 1 + \frac{1}{2}(F^2 + F \cdot K_X)$ which must be zero
- then we have a new coordinate system where X is minimal given by the following transformation:

$$\mathbb{C}^2 \ni (x, y) \longrightarrow \left(\frac{y'}{x'}, \frac{y''}{x''}\right) \in \mathbb{P}^1 \times \mathbb{P}^1$$

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A complex surface X is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
- π is not birational to the projection : $E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve E
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Generalized Halphen surface: A rational surface X is called a *generalized Halphen surface* if the anticanonical divisor class $-K_X$ is decomposed into effective divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Generalized Halphen surfaces can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by succesive 8 blow-ups. They can be classified by the topology of D as follows ($D_{\mathrm{red}} = \cup D_i$):

- $\mathrm{rank} \quad H_1(D_{\mathrm{red}}, \mathbb{Z}) = 2$, surface is elliptic,
- $\mathrm{rank} \quad H_1(D_{\mathrm{red}}, \mathbb{Z}) = 1$, surface is multiplicative
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If the fibers contain exceptional curves of first kind the elliptic surface is called non-minimal. To make it minimal one has to blow down that curves.

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Icosahedral symmetry:

$$\dot{x} = 2x^2 - y^2$$

$$\dot{y} = -10xy + y^2$$

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It applies to some class of ODE (quadratic) and has close relation with Hirota bilinear method. More precisely start with:

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij} x_j^2 + \sum_{j<k} b_{ijk} x_j x_k + c_i$$

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In order to find the time discretisation first we bilinearize it by using projective substitution $x_i = G_i/F$ and we get:

$$D_t G_i \cdot F = \sum_{j=1}^{N} a_{ij} G_j^2 + \sum_{j<k} b_{ijk} G_j G_k + c_i F^2$$

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$$\bar{G}_i F - G_i \bar{F} = \epsilon(\sum_{j=1}^{N} a_{ij} G_j \bar{G}_j + \sum_{j<k} b_{ijk}(\alpha \bar{G}_j G_k + (1-\alpha) G_j \bar{G}_k) + c_i F \bar{F})$$

or in the nonlinear form (Kahan '93, Hirota-Kimura, '00)

$$\bar{x}_i - x_i = \epsilon(\sum_{j=1}^{N} a_{ij} x_j \bar{x}_j + \sum_{j<k} b_{ijk}(\alpha \bar{x}_j x_k + (1-\alpha) x_j \bar{x}_k) + c_i)$$

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$$\bar{y} - y = -\epsilon(y\bar{x} + x\bar{y})$$

with the integral of motion:

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$$K(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)}$$

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- Icosahedral symmetry

$$\bar{x} - x = \epsilon(2x\bar{x} - y\bar{y})$$

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with the integral of motion:

$$K(\epsilon) = \frac{y(3x-y)^2(4x+y)^3}{1+\epsilon^2 c_2 + \epsilon^4 c_4 + \epsilon^6 c_6}$$

with

$$c_2 = -35x^2 + 7y^2$$

$$c_4 = 7(37x^4 + 22x^2y^2 - 2xy^3 + 2y^4)$$

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The tetrahedral symmetry (simple can be brought to QRT):

$$\bar{x} - x = \epsilon(x\bar{x} - y\bar{y})$$

$$\bar{y} - y = -\epsilon(y\bar{x} + x\bar{y})$$

use the substitution $u = (1-\epsilon x)/y, v = (1+\epsilon x)/y$ and we get QRT-mapping $(\bar{u} = v)$ and

$$3\bar{u}\underline{u} - u(\bar{u} + \underline{u}) - u^2 + 4\epsilon^2 = 0$$

with the invariant

$$K = \frac{-3(u-\bar{u})^2 + 4\epsilon^2}{2\epsilon^2(u+\bar{u})(u\bar{u}-\epsilon^2)} \equiv \frac{3x^2y - y^3}{1-\epsilon^2(x^2+y^2)}$$

What we learn:

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The cases of octahedral and icosahedral symmetry cannot be transformed to QRT forms by these type of substitutions.
So we need to analyse carefully the singularity structure. What is seen is that we have more singularities and apparently some of them are useless making the corresponding rational elliptic surface to be more complicated.

Warming up exercise

$$x_{n+1} = -x_{n-1}\frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \tag{1}$$

$$\begin{aligned} \overline{x} &= y \\ \overline{y} &= -x\frac{(y - a)(y - 1/a)}{(y + a)(y + 1/a)} \end{aligned} \tag{2}$$

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Indeterminate points for ϕ and ϕ^{-1}:

$$\begin{aligned} P_1 : (x, y) = (0, -a), &\quad P_2 : (x, y) = (0, -1/a), \\ P_3 : (X, y) = (0, a), &\quad P_4 : (X, y) = (0, 1/a), \\ P_5 : (x, y) = (a, 0), &\quad P_6 : (x, y) = (1/a, 0), \\ P_7 : (x, Y) = (-a, 0), &\quad P_8 : (x, Y) = (-1/a, 0). \end{aligned}$$

The Picard group of X is a **Z**-module

$$\mathrm{Pic}(X) = \mathbb{Z}H_x \oplus \mathbb{Z}H_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}E_i,$$

H_x, H_y are the total transforms of the lines $x = \mathrm{const.}$, $y = \mathrm{const.}$
E_i are the total transforms of the eight blowing up points.
The intersection form:

$$H_z \cdot H_w = 1 - \delta_{zw}, \quad E_i \cdot E_j = -\delta_{ij}, \quad H_z \cdot E_k = 0$$

for $z, w = x, y$. Anti-canonical divisor of X:

$$-K_X = 2H_x + 2H_y - \sum_{i=1}^{8} E_i.$$

If $A = h_0 H_x + h_1 H_y + \sum_{i=1}^{8} e_i E_i$ is an element of the Picard lattice ($h_i, e_j \in \mathbf{Z}$) the induced bundle mapping is acting on it as

$$
\begin{aligned}
&\phi_*(h_0, h_1, e_1, ..., e_8) \\
=&(h_0, h_1, e_1, ..., e_8) \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
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It preserves the decomposition of $-K_X = \sum_{i=0}^{3} D_i$:

$$D_0 = H_x - E_1 - E_2,\ D_1 = H_y - E_5 - E_6$$
$$D_2 = H_x - E_3 - E_4,\ D_3 = H_y - E_7 - E_8$$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any k).

$$
\begin{aligned}
F \equiv & \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0 \\
& \Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.
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$$\Leftrightarrow \; kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.$$

- this family of curves defines a rational elliptic surface.

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So the conservation law will be:

$$I = \left(\frac{(x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)}{xy} \right)^2$$

The case of octahedral symmetry:

$$\bar{x} - x = \epsilon(2x\bar{x} - 12y\bar{y})$$

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$x = \frac{1}{3}(\chi - 2y), \quad \bar{x} = \frac{1}{3}(\bar{\chi} - 2\bar{y}), u = (1 - \epsilon\chi)/y, v = (1 + \epsilon\chi)/y$ to the non-QRT type system:

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$$\begin{cases} \bar{u} & = v \\ \bar{v} & = \dfrac{(u + 2v - 20\epsilon)(v + 10\epsilon)}{4u - v + 10\epsilon} \end{cases} . \tag{3}$$

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The space of initial conditions is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following nine points:

$$E_1 : (u, v) = (-10\epsilon, 0),\ E_2(0, 10\epsilon),\ E_3(10\epsilon, 5\epsilon),$$
$$E_4(5\epsilon, 0),\ E_5(0, -5\epsilon),\ E_6(-5\epsilon, -10\epsilon)$$
$$E_7(\infty, \infty),\ E_8 : (1/u, u/v) = (0, -1/2),\ E_9 : (1/u, u/v) = (0, -2).$$

The action on the Picard group:

$$\bar{H}_u = 2H_u + H_v - E_1 - E_3 - E_7 - E_8,\ \bar{H}_v = H_u$$
$$\bar{E}_1 = E_2,\ \bar{E}_2 = H_u - E_3,\ \bar{E}_3 = E_4,\ \bar{E}_4 = E_5,\ \bar{E}_5 = E_6,$$
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Three invariant divisor classes:

$$\alpha_0 = H_u + H_v - E_1 - E_2 - E_7,\ \alpha_1 = H_u + H_v - E_1 - E_2 - E_8 - E_9,$$
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The curve corresponding to α_0 is a (-1) curve which must be blown down. $E_1 \to H_a = H_u + H_v - E_2 - E_7$ and $E_2 \to H_b = H_u + H_v - E_1 - E_7$, 0-curves intersecting each other: The corresponding curves are given by:

$$a_1 u + a_2(v - 10\epsilon) = 0, \quad b_1(u + 10\epsilon) + b_2 v = 0$$

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$$a_1 u + a_2(v - 10\epsilon) = 0, \quad b_1(u + 10\epsilon) + b_2 v = 0$$

So if we set $a = (v - 10\epsilon)/u \quad b = (u + 10\epsilon)/v$ our dynamical system becomes

$$\begin{cases} \bar{a} &= \dfrac{3ab - 2a + 2}{a - 4} \\ \bar{b} &= \dfrac{4 - a}{2a + 1} \end{cases} . \tag{4}$$

This system has the following space of initial conditions which define a minimal rational elliptic surface:

$$\begin{aligned}
&F_1 : (a, b) = (0, \infty),\ \ F_2 : (a, b) = (\infty, 0),\\
&F_3 : (a, b) = (-1/2, 4),\ \ F_4 : (a, b) = (-2, \infty)\\
&F_5 : (a, b) = (\infty, -2),\ \ F_6 : (a, b) = (4, -1/2),\\
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The invariant is nothing but the proper transform of the anti-canonical divisor:

$$K_X = 2H_a + 2H_b - \oplus_{i=1}^{8} F_i$$

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which is the same as the one found by Suris et al.

$$K(\epsilon) = \frac{y(2x+3y)(x-y)^2}{1-10\epsilon^2(x^2+4y^2)+\epsilon^4(9x^4+272x^3y-352xy^3+696y^4)}$$

The case of icosahedral symmetry:

$$\bar{x} - x = \epsilon(2x\bar{x} - y\bar{y})$$

$$\bar{y} - y = -\epsilon(5y\bar{x} + 5x\bar{y} - y\bar{y})$$

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The space of initial condition is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following 12 points:

$$\begin{aligned}
&E_1 : (x, y) = (\infty, \infty),\ E_2(-1/7\epsilon, -3/7\epsilon),\ E_3(-1/7\epsilon, 4/7\epsilon),\\
&E_4(1/7\epsilon, 3/7\epsilon),\ E_5(1/7\epsilon, -4/7\epsilon)\, E_6(1/5\epsilon, 0),\\
&E_7(1/3\epsilon, 0),\ E_8(1/\epsilon, 0),\ E_9(-1/\epsilon, 0),\\
&E_{10}(-1/3\epsilon, 0),\ E_{11}(-1/5\epsilon, 0). E_{12} : (1/x, x/y) = (0, 1/3)
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Singularity confinement gives the following pattern:

$$H_y - E_1\ (y = \infty) \to \text{point} \to \cdots (4 \text{ points}) \cdots \to \text{point} \to H_y - E_1$$
$$\cdots \to \text{point} \to \text{point} \to H_x - E_1\ (x = \infty) \to \text{point} \to \text{point} \to \cdots .$$

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$$\cdots \to \text{point} \to \text{point} \to H_x - E_1\ (x = \infty) \to \text{point} \to \text{point} \to \cdots .$$

The curve $4x + y = 0 : H_x + H_y - E_1 - E_3 - E_5$ is invariant and we blow it down

So $E_3 \to H_v = H_x + H_y - E_1 - E_5$ and $E_5 \to H_u = H_x + H_y - E_1 - E_3$ with

$$H_u \cdot H_u = H_v \cdot H_v = 0, H_u \cdot H_v = 1$$

where the linear systems of H_u and H_v are given by

$$\begin{aligned} |\mathcal{H}_{\sqcap}| :& u_0(1 + 7\epsilon x) + u_1(4x + y) \\ |\mathcal{H}_{\sqsubseteq}| :& v_0(1 - 7\epsilon x) + v_1(4x + y). \end{aligned}$$

So $E_3 \to H_v = H_x + H_y - E_1 - E_5$ and $E_5 \to H_u = H_x + H_y - E_1 - E_3$ with

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If we take the new variables u and v as

$$u = \frac{2(1 + 7\epsilon x)}{\epsilon(4x + y)}, \ v = \frac{2(1 - 7\epsilon x)}{\epsilon(4x + y)},$$

So $E_3 \to H_v = H_x + H_y - E_1 - E_5$ and $E_5 \to H_u = H_x + H_y - E_1 - E_3$ with

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$$u = \frac{2(1 + 7\epsilon x)}{\epsilon(4x + y)},\ v = \frac{2(1 - 7\epsilon x)}{\epsilon(4x + y)},$$

then we have a new space for initial conditions given by nine blow up points:

$$F_1 : (u, v) = (2, -2), F_2 : (0, -4), F_3 : (4, 0), F_4 : (6, -1), F_5 : (5, -2),$$
$$F_6 : (4, -3), F_7 : (3, -4), F_8 : (2, -5), F_9 : (1, -6).$$

The dynamical system becomes an automorphism having the following topological singularity patterns

$$H_v - F_9 \to F_2 \to F_1 \to F_3 \to H_u - F_4$$
$$H_v - F_3 \to F_4 \to F_5 \to F_6 \to F_7 \to F_8 \to F_9 \to H_u - F_2$$

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and $H_u \to H_u + H_v - F_2 - F_4$.
The invariant (-1) curve $H_u + H_v - F_1 - F_2 - F_3$, which should be blown down.

$$F_3 \to H_s = H_u + H_v - F_1 - F_2, \quad F_2 \to H_t = H_u + H_v - F_1 - F_3$$

where the linear systems of H_s and H_t are given by

$$|\mathcal{H}_{\int}| : s_0 u(v+2) + s_1(u - v - 4)$$
$$|\mathcal{H}_{\sqcup}| : t_0 v(u-2) + t_1(u - v - 4)$$

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and hence we take the new variables s and t as

$$s = -\frac{3u(v+2)}{2(u-v-4)}, t = -\frac{3v(u-2)}{2(u-v-4)}$$

$$\begin{cases} \bar{s} &= \dfrac{2st - 3s - 3t + 9}{s + t - 3} \\ \bar{t} &= \dfrac{2(s-3)(t+3)}{3s - t - 9} \end{cases} .$$

with the blow-up points

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$$F_1' : (s,t) = (3,0),\ F_2'(0,3),\ F_3'(-3,2),\ F_4' : (\frac{s}{t-3}, t-3) = (5,0),$$
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The invariants can be computed by using the the anticanonical divisor:

$$K = \frac{(s-t)^2 + 4(s+t) - 21}{(s-2)(t-2)(2st-5s-5t+15)} = \frac{-56\epsilon^6 y(-3x+y)^2(4x+y)^3}{d_1 d_2 d_3} \quad (5)$$

where

$$d_1 = -3 - 12\epsilon x + 15\epsilon^2 x^2 - 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2$$
$$d_2 = -3 + 12\epsilon x + 15\epsilon^2 x^2 + 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2$$
$$d_3 = -3 + 27\epsilon^2 x^2 + 10\epsilon^2 xy + 10\epsilon^2 y^2.$$

Conclusions

- The singularity structure may give a non-minimal elliptic surface. In order to make it minimal one has to blow down some -1 divisor classes (one has to prove the existence of the blow-down structure)
- after minimization the mapping can be "solved"
- the procedure applies not only to confining mappings but also to linearisable mappings and is quite effective since we do not have to compute the action on the Picard group (which is more complicated in linearisable cases)

Main reference:
A. S. Carstea, T. Takenawa, arXiv:1211.5393 (to appear in JNMP vol 20)