Courant algebroids in bosonic string theory

Ilija Ivanišević

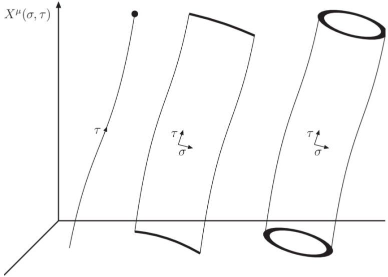
Group for Gravitation, Particles and Fields
Institute of Physics Belgrade

"Modern Methods in Quantum Gravity and Cosmology", Bucharest 2025

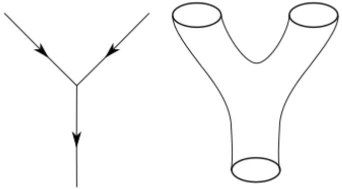
arXiv:**1903.04832**, EPJC 80 (2020)
arXiv:**2010.10662**, JHEP 03 (2021)

String theory

- In string theory, the basic elements of nature are one-dimensional strings, with different particles arising from different vibrations of these strings.
- String theory predicts the existence of a spin-2 boson, which is the **graviton**. For one-loop corrections, gravity obtained this way turns out to be **renormalizable**.



world line, world sheet for open, and world sheet for closed string

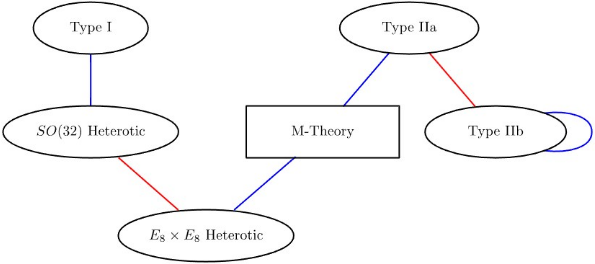


Feynman diagrams for particle vs string interaction

- Bosonic string theory was first developed.

Superstrings and M-theory

- Fermions in string theory are introduced with the help of supersymmetry. At first glance, it appeared that there are five different anomaly-free superstring theories.
- Superstring theories are not different, they are connected via web of dualities with one 11-dimensional theory - M-theory.



Dualities: red lines correspond to the T-duality, while blue lines correspond to the S-duality.

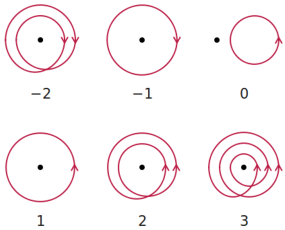
- **T-duality:** duality between theories formulated in different geometries
- **S-duality:** duality betwewen theories with strong and weak coupling constants

T-duality - example: bosonic string in $\mathbb{R}^{1,24} \times S^1$

- Translation along a compactified dimension by a is generated by $e^{ip_{25}a}$. Translation for $a = 2\pi R$ has to correspond to the identity operator, hence we can introduce **momenta numbers** n:

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}.$$

- Algebraic number of times that string winds around the compact dimension is characterized by **winding number** m.



- Mass spectrum

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}, \quad R \leftrightarrow \frac{l_s^2}{R}, \quad m \leftrightarrow n$$

Lie algebroid

- **Lie algebroid** consists of a vector bundle E, bracket on the smooth section of E and *anchor* $\rho : E \to T\mathcal{M}$, such that they satisfy the following conditions:

$$\rho[\xi_1, \xi_2] = [\rho(\xi_1), \rho(\xi_2)]_L \,,$$
$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (\mathcal{L}_{\rho(\xi_1)} f) \; \xi_2 \,,$$
$$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0 \,.$$

- It is possible to introduce various geometric concepts to different vector bundles.
- Lie derivative:

$$\hat{\mathcal{L}}_\xi f = \mathcal{L}_{\rho(\xi)} f \,, \quad \hat{\mathcal{L}}_{\xi_1} \xi_2 = [\xi_1, \xi_2] \,.$$

- exterior derivative:

$$\begin{aligned} \hat{d}\lambda(\xi_0, ..., \xi_p) \quad = & \sum_{i=0}^{p} (-1)^i \mathcal{L}_{\rho(\xi_i)} \Big(\lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p) \Big) \\ & + \sum_{i<j} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p) \,, \end{aligned}$$

Example - Koszul bracket

- Anchor: $\theta(\lambda_1)\lambda_2 = \theta(\lambda_1, \lambda_2)\,, \quad \Big(\theta(\lambda_1)\Big)^\mu = \lambda_{1\nu}\theta^{\nu\mu}$
- Koszul bracket
$$[\lambda_1, \lambda_2]_\theta = \mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2))$$
- θ has to be Poisson bi-vector, in order for the structure $\{T^\star\mathcal{M},\ [,]_\theta,\ \theta\}$ to be Lie algebroid
$$\theta([\lambda_1, \lambda_2]_\theta) = [\theta(\lambda_1), \theta(\lambda_2)]_L\,, \quad [\theta, \theta]_S = 0$$
$$[\theta, \theta]_S|^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$$
- Lie derivative:
$$\hat{\mathcal{L}}_\lambda f = \mathcal{L}_{\theta(\lambda)} f\,, \quad \hat{\mathcal{L}}_{\lambda_1}\lambda_2 = [\lambda_1, \lambda_2]_\theta$$
- exterior derivative:
$$(d_\theta f)^\mu = \theta^{\mu\nu}\partial_\nu f\,, \quad (d_\theta\xi)^{\mu\nu} = \theta^{\mu\rho}\partial_\rho\xi^\nu - \theta^{\nu\rho}\partial_\rho\xi^\mu - \xi^\rho\partial_\rho\theta^{\mu\nu}$$

Generalized geometry

- Generalized tangent bundle: $T\mathcal{M} \oplus T^\star\mathcal{M}$
- $O(D, D)$ invariant inner product

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle \xi_1 \oplus \lambda_1, \xi_2 \oplus \lambda_2 \rangle = i_{\xi_1}\lambda_2 + i_{\xi_2}\lambda_1 = \xi_1^\mu \lambda_{2\mu} + \xi_2^\mu \lambda_{1\mu}$$

- Let $T\mathcal{M}$ and $T^\star\mathcal{M}$ be vector bundles of two Lie algebroids with brackets $[,]_L$ and $[,]_{L^\star}$. We can define the antisymmetric bracket:

$$\begin{aligned}[\Lambda_1, \Lambda_2] = \quad & \Big([\xi_1, \xi_2]_L + \mathcal{L}^\star{}_{\lambda_1}\xi_2 - \mathcal{L}^\star{}_{\lambda_2}\xi_1 - \frac{1}{2} d^\star(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\ & \oplus \Big([\lambda_1, \lambda_2]_{L^\star} + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)\end{aligned}$$

- Courant bracket

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}} = [\xi_1, \xi_2]_L \oplus \Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)$$

Courant algebroid

- Courant algebroid $\Big(E, \langle,\rangle, [,], \rho\Big)$. We can define a derivative of a function by $\langle \mathcal{D}f, \Lambda\rangle = \mathcal{L}_{\rho(\Lambda)} f$. The compatibility conditions:

1. $\rho[\Lambda_1, \Lambda_2] = [\rho(\Lambda_1), \rho(\Lambda_2)]_L$
2. $[\Lambda_1, f\Lambda_2] = f[\Lambda_1, \Lambda_2] + (\mathcal{L}_{\rho(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2\rangle \mathcal{D}f$
3. $\mathcal{L}_{\rho(\Lambda_1)}\langle \Lambda_2, \Lambda_3\rangle = \langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle + \langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_3\rangle\rangle$
4. $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$
5. $\mathrm{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{D}\mathrm{Nij}(\Lambda_1, \Lambda_2, \Lambda_3)$

- Jacobiator

$$\mathrm{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = [[\Lambda_1, \Lambda_2], \Lambda_3] + [[\Lambda_2, \Lambda_3], \Lambda_1] + [[\Lambda_3, \Lambda_1], \Lambda_2]$$

- Nijenhuis operator

$$\mathrm{Nij}(\Lambda_1, \Lambda_2, \Lambda_3) = \frac{1}{6}\Big(\langle [\Lambda_1, \Lambda_2], \Lambda_3\rangle + \langle [\Lambda_2, \Lambda_3], \Lambda_1\rangle + \langle [\Lambda_3, \Lambda_1], \Lambda_2\rangle\Big)$$

Dirac structures

- Dirac structures are subbundles of maximal dimension that are **isotropic** with respect to the inner product and **involutive** with respect to the bracket of the Courant algebroid. A Courant algebroid becomes a Lie algebroid on Dirac structures.
- Isotropy: $\langle \Lambda_1, \Lambda_2 \rangle = 0$
- Dirac structures have a following form:

$$\mathcal{V}_B(\Lambda) = \xi^\mu \oplus 2B_{\mu\nu}\xi^\nu \,, \quad \mathcal{V}_\theta(\Lambda) = \kappa\theta^{\mu\nu}\lambda_\nu \oplus \lambda_\mu \,.$$

- **Example** - Standard Courant algebroid: $\Big(T\mathcal{M} \oplus T^\star\mathcal{M}, \langle , \rangle, [,]_C, \rho = \mathrm{Id}\Big)$
- Pre-symplectic manifolds

$$[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)]_\mathcal{C} = \mathcal{V}_B([\Lambda_1, \Lambda_2]_\mathcal{C}) \,, \quad dB = 0$$

- Poisson manifolds

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_\mathcal{C} = \mathcal{V}_\theta([\Lambda_1, \Lambda_2]_\mathcal{C}) \,, \quad [\theta, \theta]_S = 0$$

Bosonic string σ-model

- Action

$$\mathcal{S} = \kappa \int_\Sigma d\sigma d\tau \Big[B_{\mu\nu}(x) + \frac{1}{2} G_{\mu\nu}(x) \Big] \partial_+ x^\mu \partial_- x^\nu \,, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma$$

- Canonical momentum

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu} \dot{x}^\nu - 2\kappa B_{\mu\nu} x'^\nu \,.$$

- Hamiltonian

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu\nu} \pi_\nu + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu - 2x'^\mu B_{\mu\rho} (G^{-1})^{\rho\nu} \pi_\nu$$
$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$$

- Hamiltonian (matrix form):

$$\mathcal{H}_C = \frac{1}{2\kappa} (X^T)^M H_{MN} X^N$$
$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 2(BG^{-1})_\mu{}^\nu \\ -2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} \,, \quad X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,.$$

Symmetries of closed bosonic string

- Symmetry generator $\mathcal{H}_{(G,B)} + \{\mathcal{G}, \mathcal{H}_{(G,B)}\} = \mathcal{H}_{(G+\delta G, B+\delta B)}$
- Diffeomorphisms:

$$\mathcal{G}_\xi = \int_0^{2\pi} d\sigma \xi^\mu \pi_\mu$$

$$\begin{aligned} \delta_\xi G_{\mu\nu} &= \mathcal{L}_\xi G_{\mu\nu} = \xi^\rho \partial_\rho G_{\mu\nu} + \partial_\mu \xi^\rho G_{\rho\nu} + \partial_\nu \xi^\rho G_{\rho\mu} \,, \\ \delta_\xi B_{\mu\nu} &= \mathcal{L}_\xi B_{\mu\nu} = \xi^\rho \partial_\rho B_{\mu\nu} - \partial_\mu \xi^\rho B_{\rho\nu} + B_{\mu\rho} \partial_\nu \xi^\rho \,. \end{aligned}$$

- Algebra of diffeomorphisms

$$\left\{ \mathcal{G}_{\xi_1}, \mathcal{G}_{\xi_2} \right\} = -\mathcal{G}_{[\xi_1, \xi_2]_L} \,.$$

- Local gauge transformations:

$$\mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \lambda_\mu \kappa x'^\mu$$

$$\begin{aligned} &\delta_\lambda G_{\mu\nu} = 0 \\ &\delta_\lambda B_{\mu\nu} = (d\lambda)_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu \end{aligned}$$

Symmetries of closed bosonic string

- T-duality relates momenta and winding numbers.

$$P^\mu = \int d\sigma \pi_\mu \,, \quad W^\mu = \int d\sigma \kappa x'^\mu$$

- Two symmetry generators are related by T-duality

$$\kappa x'^\mu \cong \pi_\mu \,.$$

- Motivation to consider a following generator:

$$\mathcal{G}_\Lambda = \mathcal{G}_\xi + \mathcal{G}_\lambda = \int d\sigma \langle \Lambda, X \rangle$$

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix} \,, \quad X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix}$$

- Generator algebra closes on Courant bracket:

$$\Big\{ \mathcal{G}_{\Lambda_1}, \mathcal{G}_{\Lambda_2} \Big\} = -\mathcal{G}_{[\Lambda_1, \Lambda_2]_\mathcal{C}}$$

- Courant bracket is an extension of a Lie bracket which is invariant under the T-duality.

Twisted Courant bracket

- The twisted Courant brackets can describe different string fluxes.
- Transformations e^T that keep the inner-product invariant

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle e^T \Lambda_1, e^T \Lambda_2 \rangle \,.$$

- Method for obtaining the twisted Courant bracket from generator algebra:

$$\hat{X}^M = (e^T)^M_{\ N}\, X^N \,, \quad \hat{\Lambda}^M = (e^T)^M_{\ N}\, \Lambda^N$$

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X \rangle = \int d\sigma \langle e^T \Lambda, e^T X \rangle = \int d\sigma \langle \hat{\Lambda}, \hat{X} \rangle = \mathcal{G}^{(T)}_{\hat{\Lambda}}$$

- Poisson bracket algebra:

$$\left\{ \mathcal{G}^{(T)}_{\hat{\Lambda}_1}, \mathcal{G}^{(T)}_{\hat{\Lambda}_2} \right\} = -\mathcal{G}^{(T)}_{[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}} \,, \quad [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = e^T [e^{-T} \hat{\Lambda}_1, e^{-T} \hat{\Lambda}_2]_{\mathcal{C}}$$

- Anchor and derivative transformation laws:

$$\rho = \pi \circ e^{-T} \,, \quad \mathcal{D} f = e^T df$$

- All five compatiblity conditions are apriori satisfied.

B-twisted Courant bracket

- B-shifts $e^{\hat{B}}$

$$\hat{B} = \begin{pmatrix} 0 & 0 \\ 2B & 0 \end{pmatrix}, \quad e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix}$$

- We can diagonalize Hamiltonian:

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N = \frac{1}{2\kappa}\hat{X}^M G_{MN} \hat{X}^N$$

$$\hat{X}^M = (e^{\hat{B}})^M{}_N X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix}, \quad G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix}$$

- Non-canonical currents i_μ

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar{\sigma}), \quad B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$

- B-twisted Courant bracket

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = [\xi_1, \xi_2]_L \oplus \left(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + dB \right)$$

θ-twisted Courant bracket

- T-dual background fields

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}\,, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}$$

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}\,, \quad \theta^{\mu\nu} = \frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$$

- θ-transformations $e^{\hat\theta}$

$$\hat\theta = \begin{pmatrix} 0 & \kappa\theta \\ 0 & 0 \end{pmatrix}\,, \quad e^{\hat\theta} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix}$$

- Hamiltonian in the diagonal form

$$^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}\hat X^M \, ^\star G_{MN}\hat X^N\,, \quad \hat X^M = \begin{pmatrix} \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} k^\mu \\ \pi_\mu \end{pmatrix}$$

$$^\star G_{MN} = \begin{pmatrix} ^\star(G^{-1})_{\mu\nu} & 0 \\ 0 & ^\star G^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix}$$

- Non-canonical currents k^μ

$$\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma)$$

$$Q_\rho{}^{\mu\nu} = \partial_\rho\theta^{\mu\nu}\,, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$$

θ-twisted Courant bracket

- θ-twisted Courant bracket

$$\begin{aligned} \xi &= [\xi_1,\xi_2]_L - \kappa[\xi_2,\lambda_1\theta]_L + \kappa[\xi_1,\lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1,\lambda_2,.) \\ &\quad -\kappa\theta\Big(\mathcal{L}_{\xi_2}\lambda_1 - \mathcal{L}_{\xi_1}\lambda_2 + \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\ \lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + \kappa[\lambda_1,\lambda_2]_\theta \end{aligned}$$

- Dirac structures

$$[\mathcal{V}_B(\Lambda_1),\mathcal{V}_B(\Lambda_2)]_{\mathcal{C}_\theta} = \mathcal{V}_B[\Lambda_1,\Lambda_2]_{\mathcal{C}_\theta}\,, \quad d(BG^{-1}G_E)=0$$

$$[\mathcal{V}_\theta(\Lambda_1),\mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_\theta} = 0\,, \quad \forall\theta$$

θ-twisted Courant bracket

- Exchange of the canonical variables and background fields with their respective T-duals:

$$\pi_\mu \leftrightarrow \kappa x'^\mu , \quad 2B_{\mu\nu} \leftrightarrow \kappa\theta^{\mu\nu}$$
$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu$$

- Isomorphism between Courant algebroids that correspond to the T-dual transformations

$$\varphi = (e^{\hat{\theta}} e^{-\hat{B}})^M{}_N = \begin{pmatrix} \delta^\mu_\nu - 2\kappa(\theta B)^\mu{}_\nu & \kappa\theta^{\mu\nu} \\ -2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} .$$

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = \Big([\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta}\Big) .$$

Conclusions

1. The Courant bracket is a T-dual invariant extension of the Lie bracket. It can be obtained from the Poisson bracket algebra of the generator governing diffeomorphisms and local gauge transformations.
2. The Courant bracket can be twisted with an arbitrary $O(D, D)$ transformation. In particular, twisting it with the Kalb-Ramond field B gives rise to the H-flux, while twisting with the non-commutativity parameter θ yields Q and R fluxes.
3. The Courant bracket twisted by B is T-dual to the Courant bracket twisted by θ.

Thank you for your attention