Canonical quantum gravity and cosmological applications

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Outline

1. The Hartle-Hawking "no boundary proposal" for the wave function of the universe

2. Loop quantum gravity

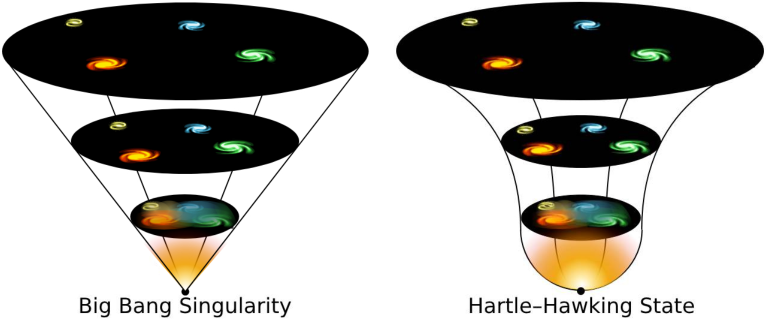
The Hartle-Hawking "no boundary proposal"

The Hartle-Hawking proposal was intended to explain the quantum origin of the universe. It was intended to describe cosmological solution of the (properly rigorized) Wheeler-DeWitt equation before the Planck epoch ($t < t_P \simeq 10^{-43}$ s)

The Hartle-Hawking proposal

- Originally formulated by Hartle and Hawking in their “Euclidean approach to quantum gravity” based on naive “Wick rotation” to imaginary time.
- As observed by Penrose and others, such naive Wick rotations are not possible in general and proper connection between the Euclidean and Lorentzian versions of General Relativity requires much more subtle mathematics (see Witten, Kontsevich, Maldacena et al.). This is closely related to ideas of the Schwinger-Keldish “in-in” approach to building a “folded” version of the S-matrix and its generalizations which connect Euclidean and Lorentzian formulations of Quantum Physics (i.e. Quantum Statistical Mechanics and ordinary Quantum Mechanics). See work by Maldacena et al.
- The proposal can be shown to satisfy the Wheeler-de Witt equation with appropriate assumptions of the proper definition of the Wheeler-DeWitt D’Alembertian (or its Euclidean version, which is known as the Wheeler-DeWitt Laplacian). Much more subtle analysis is needed to fully settle the issue. such as pre-quantization through the BV-BRST approach.
- The Hartle-Hawking proposal relies on the assumption that the quantum universe has no boundary in the past – in contradistinction with classical solutions of the Einstein equations, which were shown by Hawking and Ellis to develop an initial (cosmological) singularity (“the Big Bang singularity”) under mild causality and positivity assumptions.

- The proposal involves summation and path integration over all Euclidean or Lorentzian metrics which are nonsingular and boundary-free in the past, but (if necessary) with appropriate boundary conditions in the spatial directions.
- It is best understood for universes with closed spatial section (issues with IR regularization).
- In practice, it is understood explicitly only in very simple models with high degree of symmetry and almost exclusively in the semiclassical approximation extracted through saddle point/oscillatory integral methods and their "complex contour" generalizations which connect to the Stokes phenomenon and Picard-Lefschetz theory.
- Higher orders in the saddle point expansions connect to "resurgence" and results related to Borel resummation of semiclassical expansions
- Numerous unsolved problems exist in making sense of this proposal in general. They relate to areas of functional analysis such as semiclassical expansions, WKB theory and the theory of resonances in mathematical scattering theory.



Remark

The Hartle-Hawking state is supposed to satisfy the (properly-formulated) Wheeler-DeWitt equation. As such, it can be used as a guideline to make mathematical sense of that equation. A fundamental idea in the subject is that a proper understanding of the WdW equation and of the Hartle-Hawking state requires a good theory of complexified Riemannian manifolds and a complexified version of General Relativity. This involves developing a proper theory of involutions and of "real slices", among other issues.

Loop quantum gravity

Loop quantum gravity is an attempt to carry out the canonical quantization of gravity by re-casting the latter in terms of variables reminiscent of non-Abelian gauge theory. It is affected by many technical issues which have never been properly resolved.

Ashtekhar-Barbero variables (gauge group $\mathrm{SU}(2)$):
A. Ashtekar, "New Variables for Classical and Quantum Gravity," Physical Review Letters 57 (1986) 2244–224
J. Barbero, "Real Ashtekar variables for Lorentzian signature space-times," Physical Review D 51 (1995) 5507–5510, arXiv:gr-qc/9410014.
Loop quantum cosmology:
A. Ashtekar and P. Singh, "Loop quantum cosmology: a status report," Classical and Quantum Gravity 28 (2011) 213001, arXiv:1108.0893 [gr-qc].
B. Bolliet and A. Barrau, "Conceptual issues in loop quantum cosmology," arXiv:1602.04452 [gr-qc].

Remark

Part of the physics literature but especially social media, youtube etc. abound in pseudo-scientific claims made about LQG, including intentionally misleading statements whose main purpose is to serve as propaganda attacks against String Theory.

Fundamental criticisms of LQG

- It is unclear how to recover General Relativity in the classical limit. The current "solution" proposed to deal with this issue is by coarse-graining the "quantum geometry" predicted by LQG while attempting to treat the matter fields using QFT on curved spacetimes (in the sense of Hawking and Ellis). There are many proposals of such attempts, but none of them have addressed the underlying issues to the satisfaction of most mathematical physicists.

Remark

By contrast, this is trivial in String Theory, where it follows from the computation of the one-loop beta function of a nonlinear sigma model.

- The formulation in terms of loop quantum variables involves a nonlinear Hamiltonian constraint, which must be regularized appropriately when quantizing the theory. It is unclear how to do this mathematically. Some proposals towards dealing with this problems include "spin foams" (Perez) and "group field theory" (Baratin & Oritin), but they are far from having addressed the issue to the satisfaction of mathematical physicists.
- There are very serious issues with understanding the continuum limit of LQG in all approaches: Lorentz violations, fine tuning, hidden parameters etc. etc. etc. These are particularly severe in applications to "loop quantum cosmology".

The Dirac algebra of the ADM approach and the Ashtekhar-Barberi variables

The constraint functions $\mathcal{H}$ and $\mathcal{H}_i$ of the ADM approach form the Dirac algebra of hypersurface deformations:

$$\begin{aligned}
\{H[N], H[N']\} &= H^i[h^{ij}(N\partial_j N' - N'\partial_j N)] \\
\{H[N], H_i[\vec{N}']\} &= -H[L_{\vec{N}'} N] \\
\{H_i[\vec{N}], H_i[\vec{N}']\} &= -H_i[(L_{\vec{N}'} N_i)_{i=1,2,3}] \ .
\end{aligned}$$

Since $\mathcal{H}$ and $\mathcal{H}_i$ generate gauge symmetries, only those classical observables $\mathcal{O}$ which Poisson commutes with $\mathcal{H}$ and $\mathcal{H}_i$ are gauge-invariant. These are called (classical) Dirac observables.

Introduce the Ashtekhar-Barberi variables E^i_a and K^a_i via:

$$hh^{ij} = E^{ia} E^j_a \quad , \quad \sqrt{h} K^i_j = K_{ia} E^{ja} \ ,$$

where internal $\mathrm{SU}(2)$ indices $a, b = 1, 2, 3$ are raised and lowered trivially using the Kronecker symbol δ^a_b. The new non-vanishing Poisson brackets are:

$$\{K^a_i(x), E^j_b(y)\} = \delta^{(3)}(x, y)\delta^a_b \delta^j_i \ .$$

Since we added degrees of freedom, we introduce an additional constraint, called the Gauss law:

$$G_{ab}[\Lambda] \overset{\mathrm{def.}}{=} \frac{1}{2} \int_\Sigma \mathrm{d}^3 x \Lambda^{ab} (K_{ia} E^i_b - K_{ib} E^i_a) \approx 0 \ .$$

The Gauss function $G_{ab}[\Lambda]$ generates internal $\mathrm{SU}(2)$ gauge transformations with parameters Λ^{ab} via Poisson brackets with K_i^a and E_a^i, under which classical observables should be invariant.

It can be shown that this reformulation reproduces the ADM Poisson brackets up to terms proportional to the Gauss constraint, which vanish when the Gauss law is imposed. Hence this reformulation describes the same dynamics up to $\mathrm{SU}(2)$ gauge transformations and hence describes the same physics if one considers only classical observables which are invariant under such transformations.

Introduce the new canonical variables:

$$A_i^a = \Gamma_i^a + \beta K_i^a \ , \ \tilde{E}_a^i = \frac{1}{\beta} E_a^i \ ,$$

where Γ_i^a is the spin connection defined by the vielbein e_a^i of h:

$$h^{ij} = e_a^i e^{ja} \ , \ \nabla_i^h e_j^a = \partial_i e_j^a - \Gamma_{ij}^k e_k^a + \epsilon^{abc} \Gamma_i^b e_j^c \ ,$$

where $\beta \in \mathbb{R}^\times$ is the Barbero-Imirizi parameter. Then $A^a = A_i^a \mathrm{d}x^i$ is an $\mathrm{SU}(2)$ connection defined on Σ, usually called the Ashtekhar connection.

The theory is now reformulated in terms of the parallel transport of this connection, which is completely encoded by its holonomy and fluxes.

One can quantize the Ashtekhar-Barberi formulation using standard gauge theory techniques, which in the holonomy-flux formulation leads to a spin network description. However, the Gauss constraint leads to problems in the quantum interpretation, as already mentioned above. There are other problems as well with the spin network interpretation.