Canonical quantum gravity and cosmological applications

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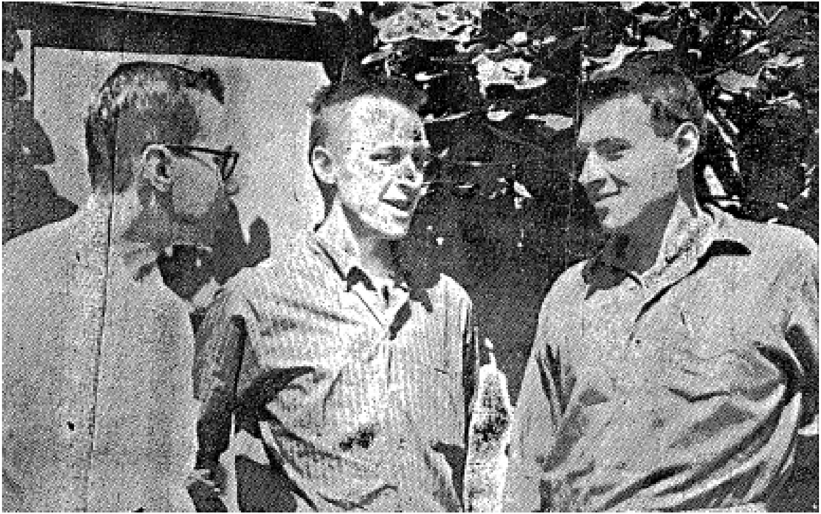
Outline



The Arnowitt–Deser–Misner (ADM) formalism was one of the first attempts to a canonical formulation of general relativity (first published in 1959).

The ADM formalism

To build a canonical formulation of gravity, one assumes that the Lorentzian metrics g of interest are globally hyperbolic and hence that spacetime is foliated by spacelike hypersurfaces Σ_t transverse to a timelike curve which we parameterize by a variable $t \in I$ with I a non-degenerate interval. We denote by $h(t)$ the positive-definite metric induced by g on Σ_t. Choosing local coordinates $\vec{x} = (x^1, x^2, x^3)$ on Σ_t, such metrics can be written in the ADM form:

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + h_{ij}(t,x)(\mathrm{d}x^i + N^i(t,x))(\mathrm{d}x^j + N^j(t,x))$$

where N is called the lapse function and $\vec{N} = (N^1, N^2, N^3)$ is called the shift vector. Geometrically, the leaves Σ_t are diffeomorphic to a model leaf Σ and the spacetime manifold $\mathcal{M}$ is diffeomorphic with $I \times \Sigma$. We can take N to be a real-valued function defined on $I \times \Sigma$) and $\vec{N}$ to be a section of the bundle $p_2^*(T\Sigma)$), where $p_2 : I \times \Sigma \to \Sigma$ is the canonical projection.

Remark

These assumptions are more restrictive than they might appear. For example, they preclude a wide class of four-manifold topologies which would otherwise be allowed for spacetimes in general relativity. Moreover, it is unclear how to justify the global hyperbolicity requirement, since the spacetime metrics considered are supposed to be used in path integrals etc. and hence are not assumed to satisfy the Einstein equations at the outset. A proper path integral formulation would allow at least nowhere-differentiable objects N,$\vec{N}$ and h.

For simplicity, set $8\pi G = 1$. We consider the action of gravity coupled to unspecified matter and with cosological constant Λ:

$$S = S_{matter} + \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x \left[\sqrt{g}(R - 2\Lambda) + S_\partial\right] \ ,$$

where S_∂ is a boundary term which determines the boundary conditions to be imposed on the model leaf Σ. Plugging in the ADM form of the metric and using the Gauss-Codazzi relation between extrinsic and total curvature, one can write the action as:

$$S = S_{matter} + \frac{1}{2}\int_I \mathrm{d}t \int_\Sigma \mathrm{d}^3x N\sqrt{h}\left[K_{ij}K^{ij} - (K_i^i)^2 + R(h) - 2\Lambda\right]$$

where $R(h)$ is the scalar curvature of h, $K_i^i \stackrel{\text{def.}}{=} h^{ij}K_{ji}$ and K is the extrinsic curvature of Σ_t in $(\mathcal{M}, g)$:

$$K_{ij} = \frac{1}{2N}\left[-\dot{h}_{ij} + 2\nabla^h_{(i}N_{j)}\right] \ ,$$

where $\dot{} \stackrel{\text{def.}}{=} \partial_t$. The system has gauge symmetries coming from diffeomorphisms of Σ and of the interval I. Hence the Hamiltonian formulation is constrained.

The parameterized Hamiltonian form of the action obatined by ADM is:

$$S = \int_I \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left[\dot{h}_{ij}\pi^{ij} - N\mathcal{H} - N^i\mathcal{H}_i \right] ,$$

where:

$$\pi^{ij} \stackrel{\mathrm{def.}}{=} \frac{\delta\mathcal{L}}{\dot{h}_{ij}} = -\frac{\sqrt{h}}{2}(K^{ij} - h^{ij}K^l_l)$$

are the momenta conjugate to h_{ij} and:

$$\begin{aligned} \mathcal{H} &\stackrel{\mathrm{def.}}{=} 2G_{ijkl}\pi^{ij}\pi^{kl} - \frac{1}{2}\sqrt{h}(R(h) - 2\Lambda) + \mathcal{H}_{matter} \\ \mathcal{H}_i &\stackrel{\mathrm{def.}}{=} -2\nabla^h_j\pi^{ij} + \mathcal{H}^i_{matter} . \end{aligned}$$

Here:

$$G_{ijkl} \stackrel{\mathrm{def.}}{=} \frac{1}{2\sqrt{h}}\left[h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right]$$

is the de Witt metric (a.k.a. "supermetric"), which can be viewed as an indefinite metric on the bundle $\mathrm{Sym}^2(T\Sigma)$. It induces a metric on $\mathrm{Sym}^2(T^*\Sigma)$ via the musical isomorphism. In turn, the latter induces a metric defined on the space of all Riemannian metrics defined on Σ (which was historically called "superspace" – no connection to the notion of superspace used in supersymmetry !).

In the parameterized Hamiltonian form of the action, the shift and lapse functions appear as Lagrange multipliers which impose the momentum constraints $\mathcal{H}_i \approx 0$ and the Hamiltonian constraint $\mathcal{H} \approx 0$ through their equations of motion. These constraints are equivalent with the $0i$ and 00 components of the Einstein equations. The parameterized Hamiltonian is:

$$H = H_{matter} + \int_\Sigma \mathrm{d}^3x(\pi^{ij}\dot{h}_{ij} - \mathcal{L}_{grav}) = H_{matter} + \int_\Sigma \mathrm{d}^3x(N\mathcal{H} + N^i\mathcal{H}_i) \ .$$

Remark

To pass from the parameterized Hamiltonian to the canonical one, we have to solve the momentum and Hamiltonian constraints and impose coordinate conditions which eliminate the reparameterization freedom along Σ and I. This has been done explicitly only in very particular cases — though it can always be achieved formally.

The formal canonical quantization procedure replaces h_{ij} and π^{ij} by the operators:

$$h_{ij} \longrightarrow h_{ij} \cdot$$
$$\pi^{ij} \longrightarrow -\mathbf{i}\frac{\delta}{\delta h_{ij}}$$

and similarly for the matter fields and their conjugate momenta. This produces the *quantum momentum constraints*:

$$\hat{\mathcal{H}}^i\Psi = \left[2\mathbf{i}\nabla_j^h\frac{\delta}{\delta h_{ij}} + \hat{H}^i_{matter}\right]\Psi = 0$$

and the *Wheeler-DeWitt equation*:

$$\hat{\mathcal{H}}\Psi = \left[-G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} - \sqrt{h}(R(h) - 2\Lambda) + \hat{H}_{matter}\right]\Psi = 0 \ ,$$

where $\hat{H}_{matter}$ is the quantum Hamiltonian of matter fields. Here Ψ is the "wavefunction of the universe", which is a functional $\Psi(h_{ij}, \Phi)$ of the Riemannian metric $h \in \mathrm{Met}(\Sigma)$ and the matter fields Φ defined on Σ. The momentum constraints implement diffeomorphism invariance along Σ while the Wheeler-DeWitt equation implements diffeomorphism invariance along I.

To make mathematical sense of the Wheeler-De Witt equation is a highly nontrivial task, since Ψ is a functional of field configurations:

- One has an ordering ambiguity when interpreting the Wheeler=DeWitt D'Alembertian:
$$\nabla_{WDW} \stackrel{\text{def.}}{=} G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} ,$$
which is compounded by the fact that this is a *functional* version of the D'Alembert operator
- One has the usual problems of constructing an appropriate Hilbert space for field-theoretical wave-functionals Ψ
- One has the added complication of having to deal with gauge invariance under reparameterization, which can be eased by using the BV-BRST approach
- The fact that the de Witt metric G is not positive-definite produces difficulties with the probabilistic interpretation of the theory, some of which are already familiar from the Klein-Gordon equation.

These problems remain unsolved in general. The physics literature is concerned almost exclusively with very special examples, usually related to highly symmetric models (e.g. the "minisuperspace approximation" used in quantum cosmology). Experience with functional analysis suggests that the correct formulation is through scattering theory on manifolds (Melrose). In general, this has to be done in an infinite-dimensional setting, which requires new ideas and techniques.

What is “quantum cosmology” ?

“Quantum cosmology” is an attempt to build a Hilbert space interpretation of canonical quantization of gravity following the old ideas of ADM. At present, it suffers from many problems due to poor understanding of its mathematical foundations, and it is mosty limited to semiclassical considerations and the study of simple examples. Some ideas and directions:

- The Hartle-Hawking “no boundary proposal” for the “wave function of the universe”
- Complex solutions of the Einstein equations and various notions of the space of “complexified metrics” and “choices of integration contours” in a putative “path integral over metrics” (which is understood only in simple cases and though leading orders of the saddle point approximation).
- Applications of Picard-Lefschetz theory to saddle point expansions; the Stokes phenomenon.
- Connections to the Schwinger-Keldish (“in-in”) formalism.
- Connections to cosmological perturbation theory.
- Connections to scattering theory on manifolds (poorly understood).
- The BV-BRST approach to pre-quantisation of canonical gravity.
- Attempts to deal with various issues though loop quantum gravity.
- Connections to secondary calculus ?

- The proper formulation of initial value problems in GR in various spacetime dimensions, with various kinds of fields and in various supergravity theories (starting with the work of Y. Choquet-Bruhat on the gravity-Yang-Mills system and latter). Quite a few prpblems are still open in the presense of spinor fields and for various extended supergravity in higher dimensions.
- Well-posedness of initial value problems in GR (Sergiu Klainerman et al)
- Infra-red regularization for non-compact spatial section and topological aspects
- Summing over three-manifold topolgies in the "no-boundary proposal of Hartle and Hawking".
- Open problems in cosmological perturbation theory: the proper meaning of the "adiabatic approximation" and the "cosmological effective actions"; a systematic and more geometric treatment that would allow going beyong second order in the cosmological effective action; dealing correctly with spinor matter; the proper treatment of graviton loops and loop corrections in general; the role of the "Bunch-Davis" conditions and how they could be relaxed; alpha-vacua and related topics in algebraic QFT on curved spacetimes.
- Dynamical systems aspects; the role played by the WKB approximation and decoherence in quantum cosmology; inertial manifolds.