Numerical Cosmology 2

Teleparallel Gravity – Foundations and Extensions to Gravity

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Aim

Goal: We study **cosmological models** using **teleparallel theories** of gravity

Proviso: An **observationally consistent cosmological model** does not imply the **underlying model is consistent**

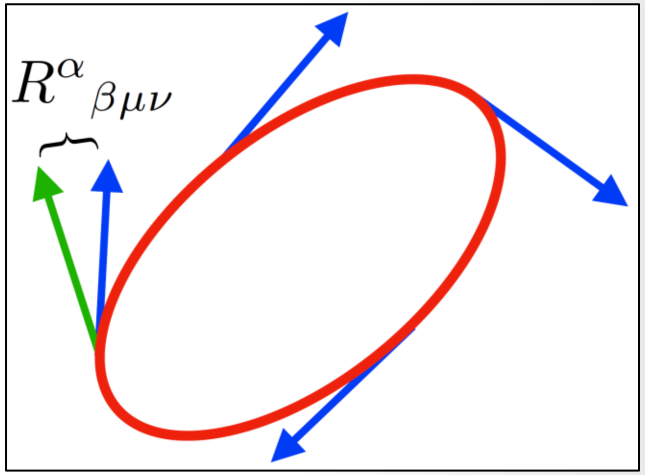
Outline

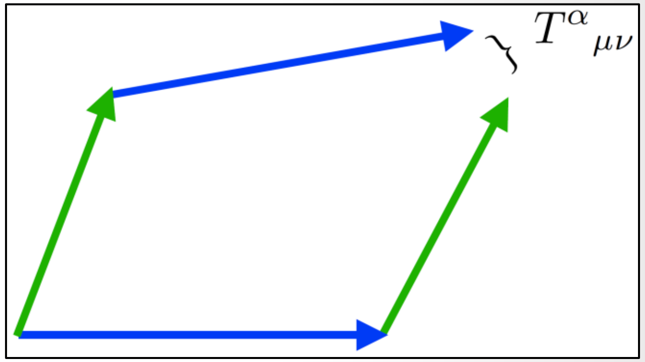
- **Standard gravity** and the motivation for modified gravity
- Introduction to **Modified Gravity**
- The **Teleparallel Gravity** (TG) formalism
- Background **cosmology** in TG
- **Perturbations** in TG and their observational consequences
- Concluding Remarks

Introduction

- Einstein 1915: **General Relativity (GR)**
  Energy-momentum source of curvature
  Levi-Civita connection: Zero Torsion, Metricity

- Einstein 1928: **Teleparallel Equivalent of GR (TEGR)**
  Teleparallel connection: Zero Curvature, Metricity

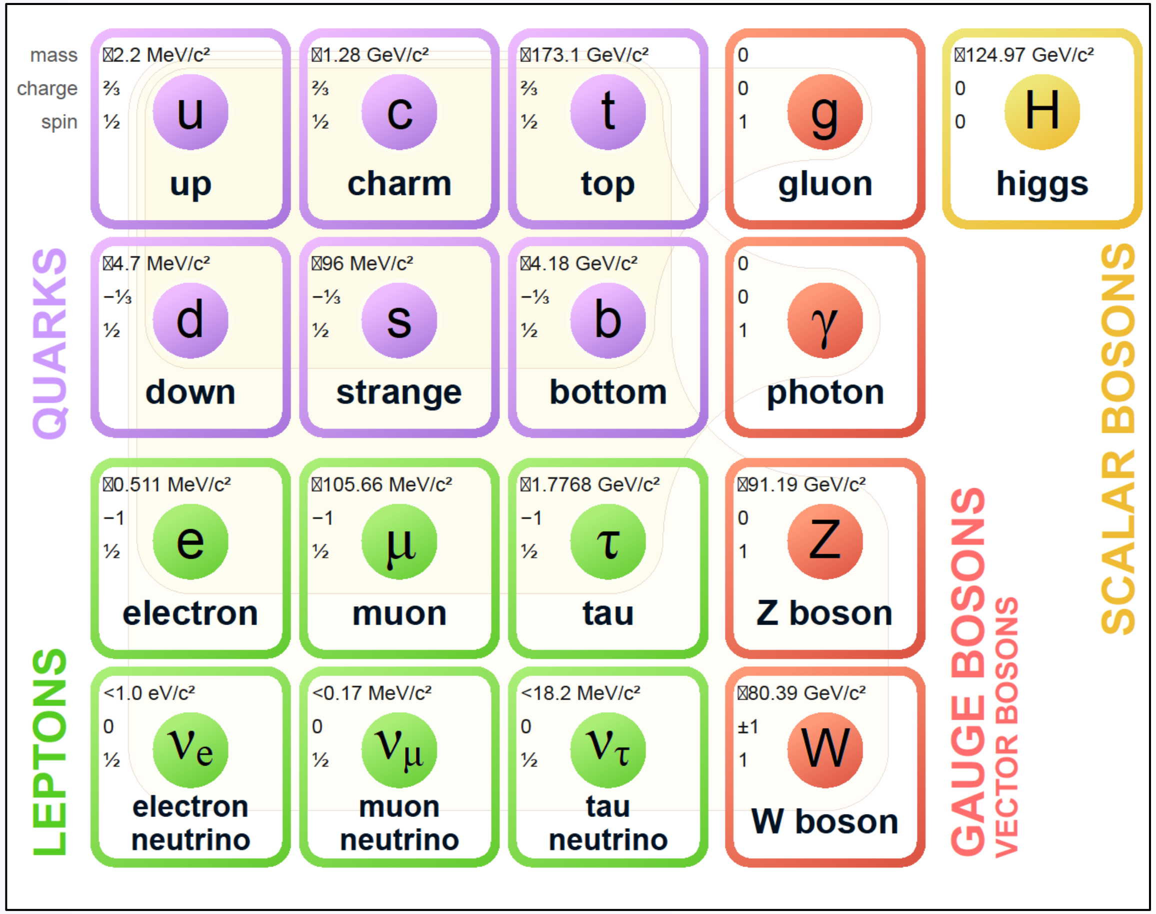




arXiv:1903.06830

Fundamental Physics

Standard Model of Particle physics + GR



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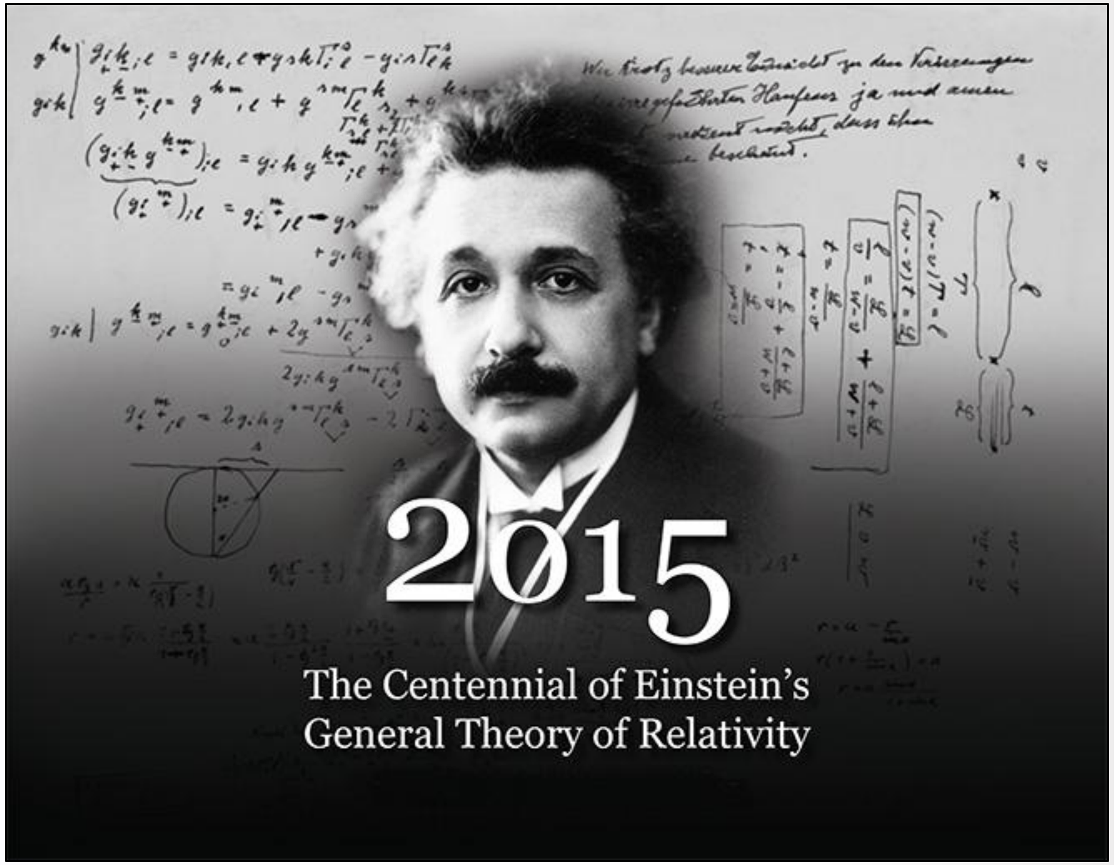
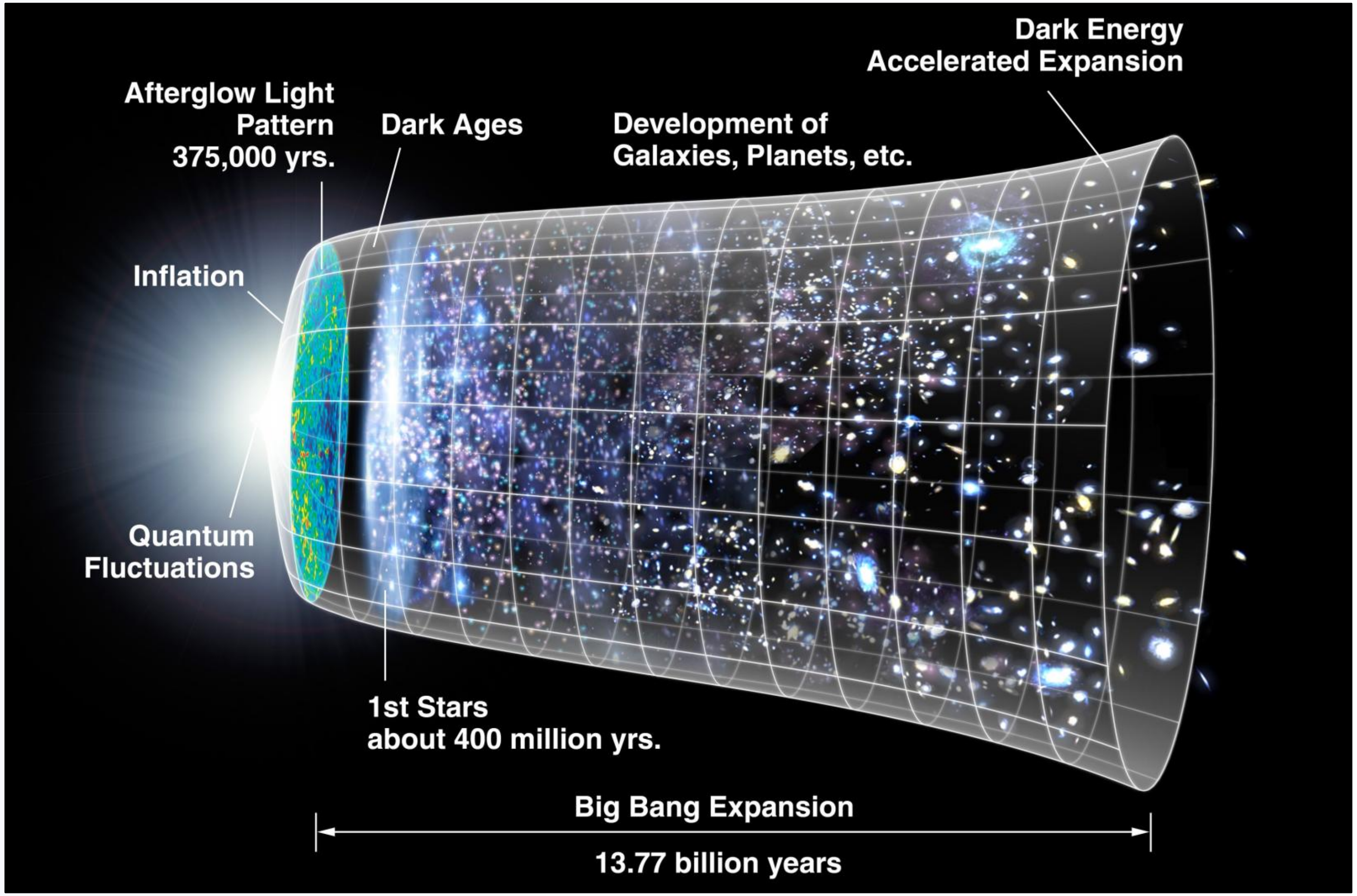
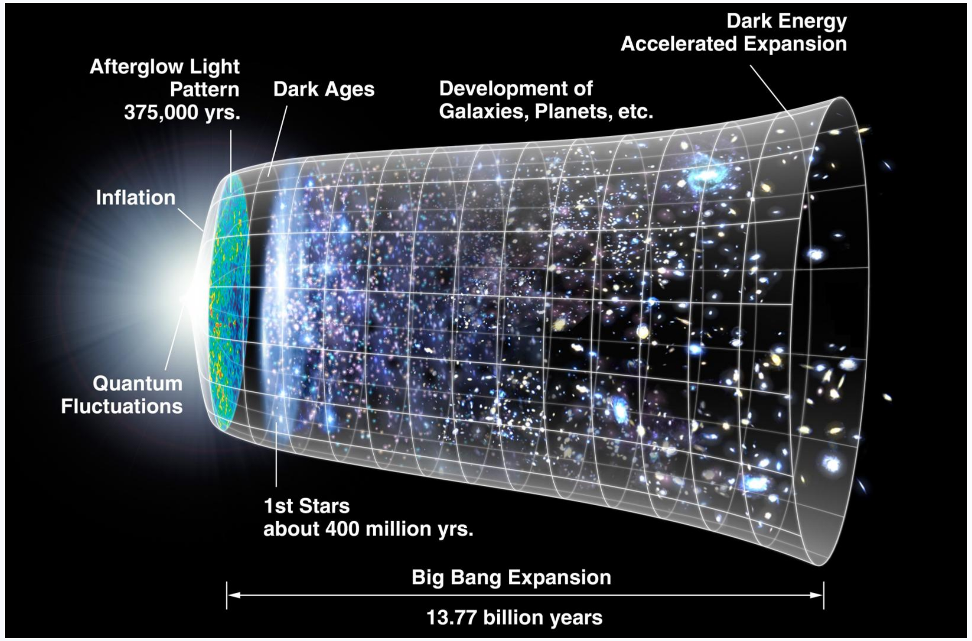
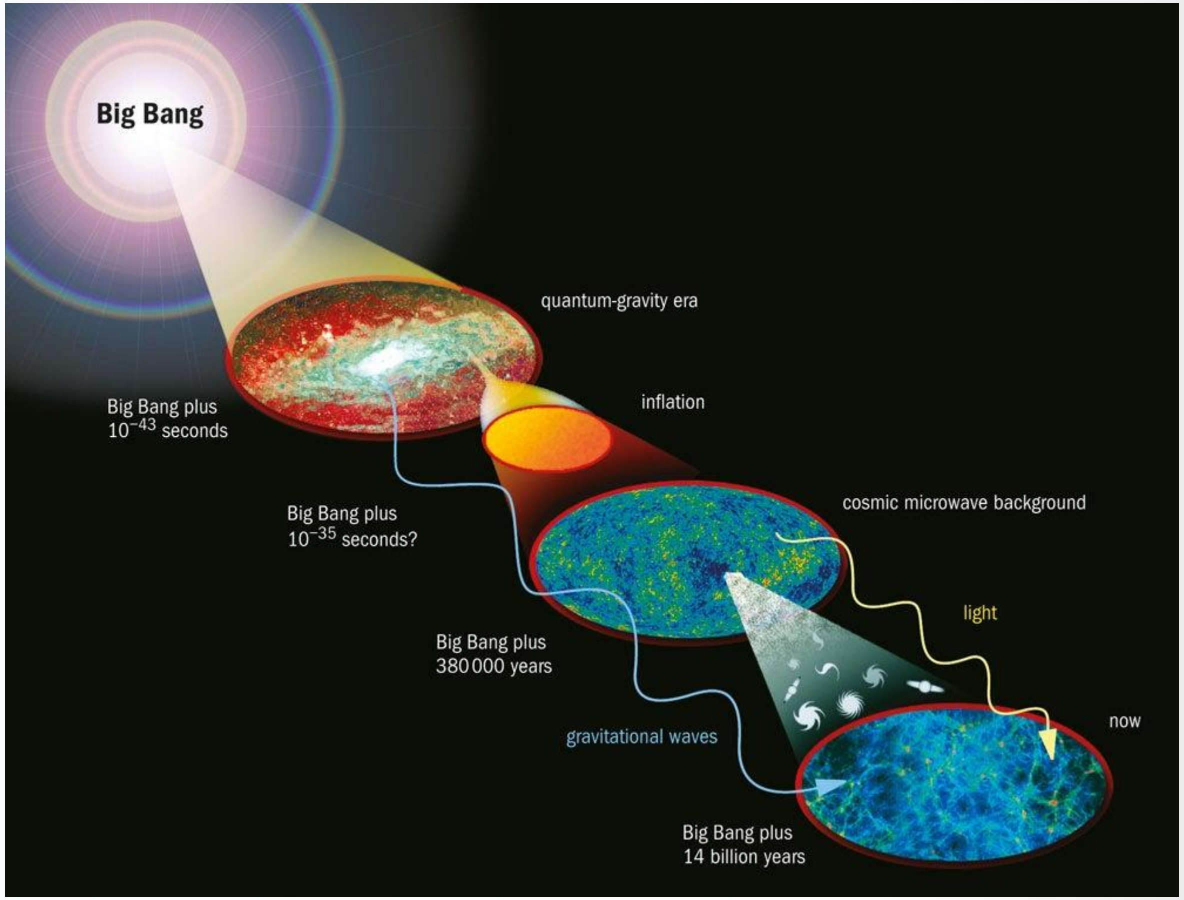


Cosmological History



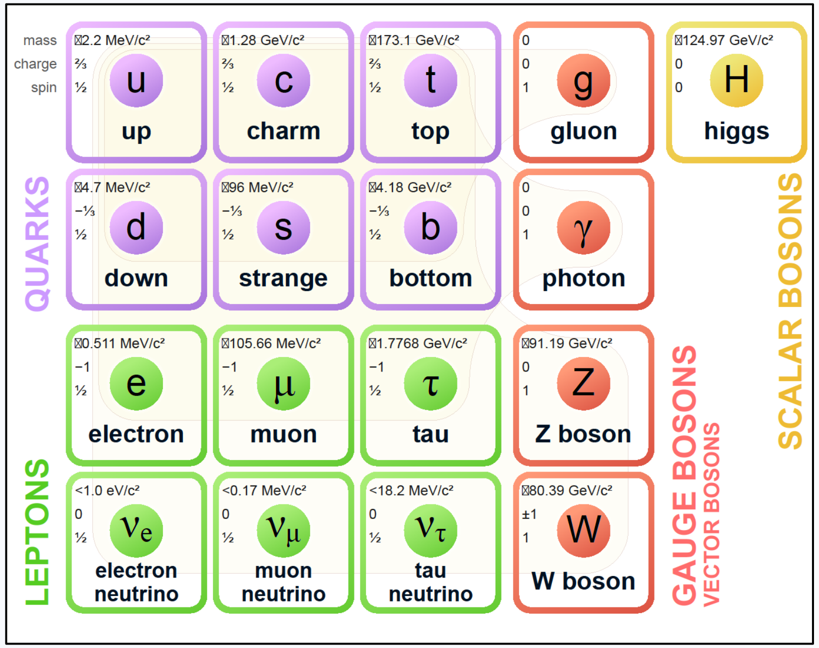
Can we describe the early universe?





No!

Adding Inflation



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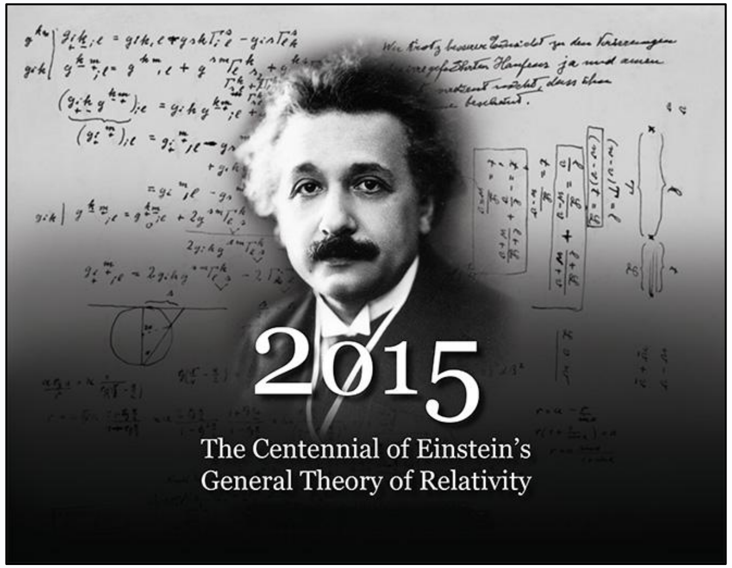

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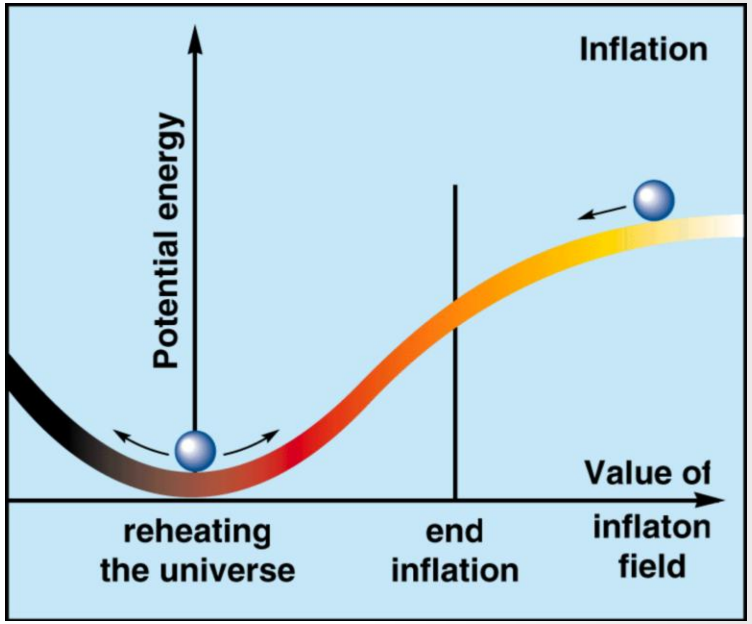


arXiv:astro-ph/9906497

General Relativity and Standard Modifications

- Einstein 1915 – **GR**:

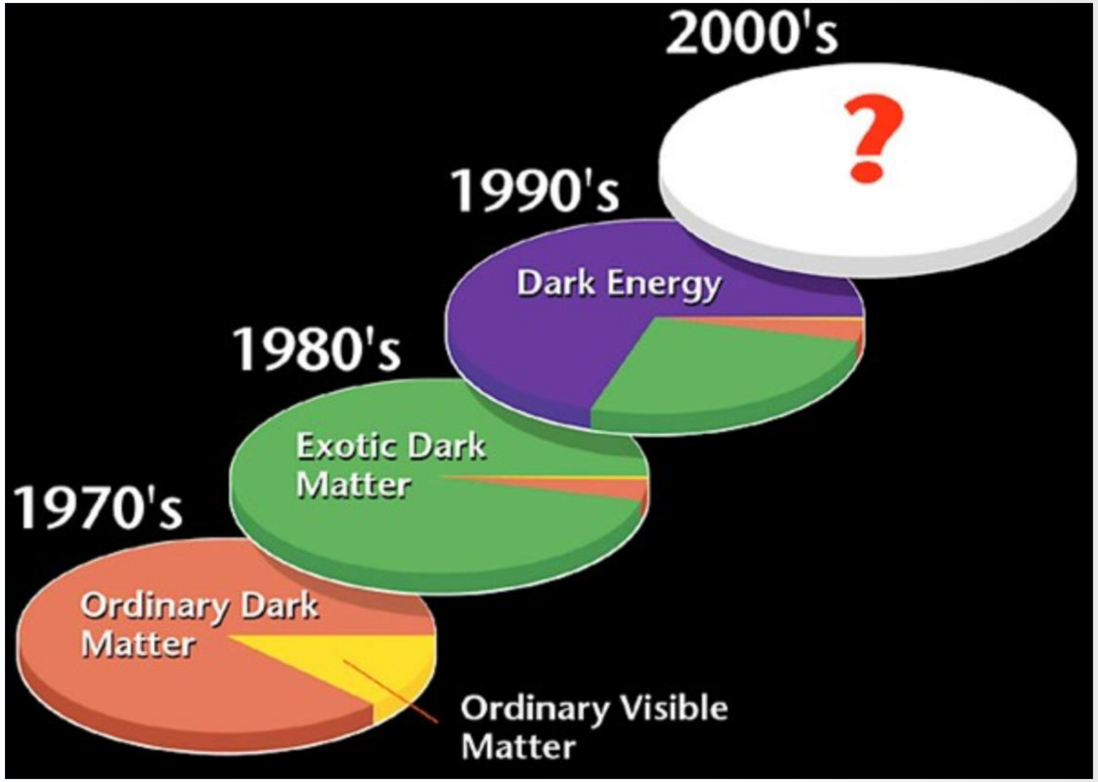
$$\boxed{S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}[\mathcal{R}] + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu},\psi)}$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}$$

with $T_{\mu\nu} := \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}}$

Late-time reasons for modified gravity

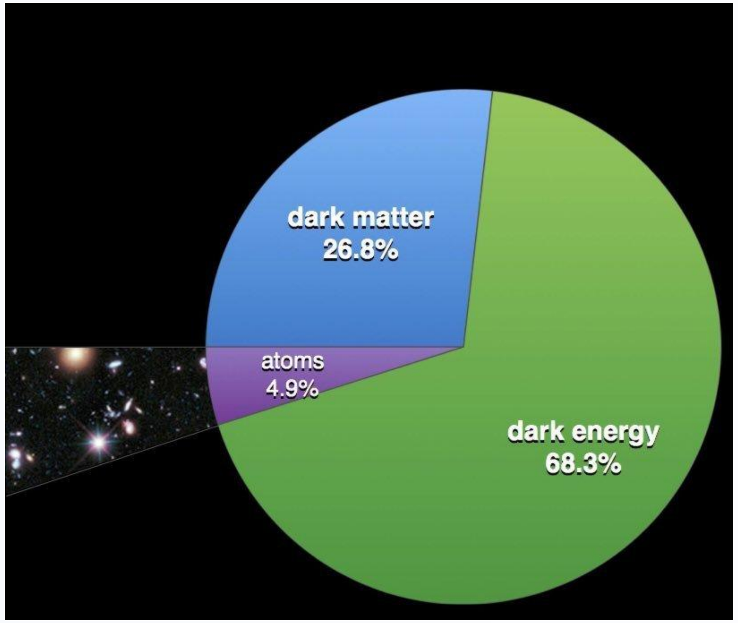
Energy budget of the universe over the decades



WMAP Science Team

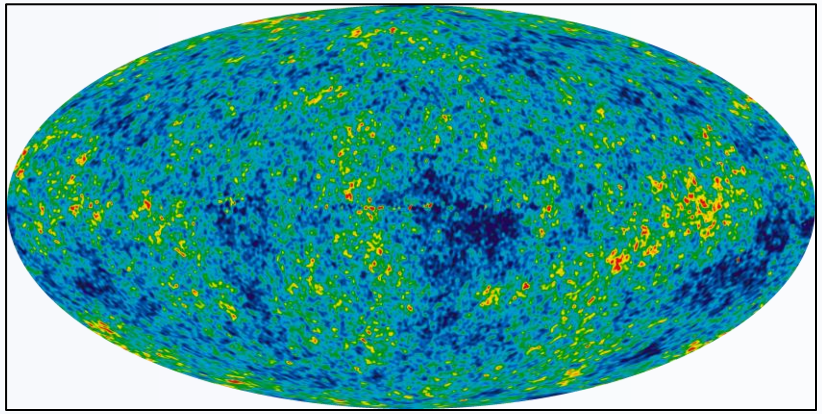
Galactic Dynamics: Flat rotation curve problem

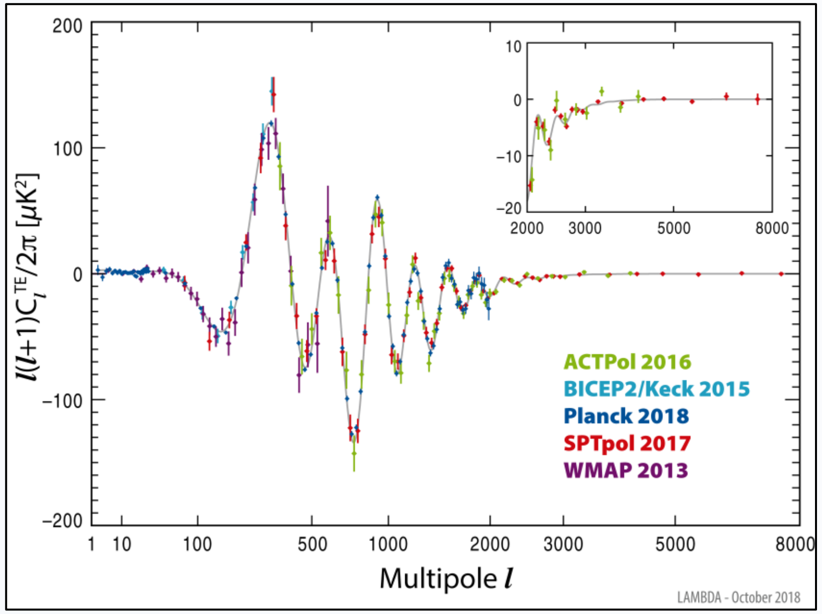
Dark Energy: Late-time acceleration



Planck Collaboration

Modified Matter I





Dark Matter Detection Attempts:

2000 – MACHO: MACHO microlensing

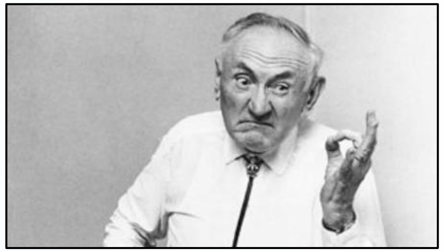
2010 – DAMA/LIBRA: WIMP particle interactions

2014-2016 – LUX: WIMP particle interactions

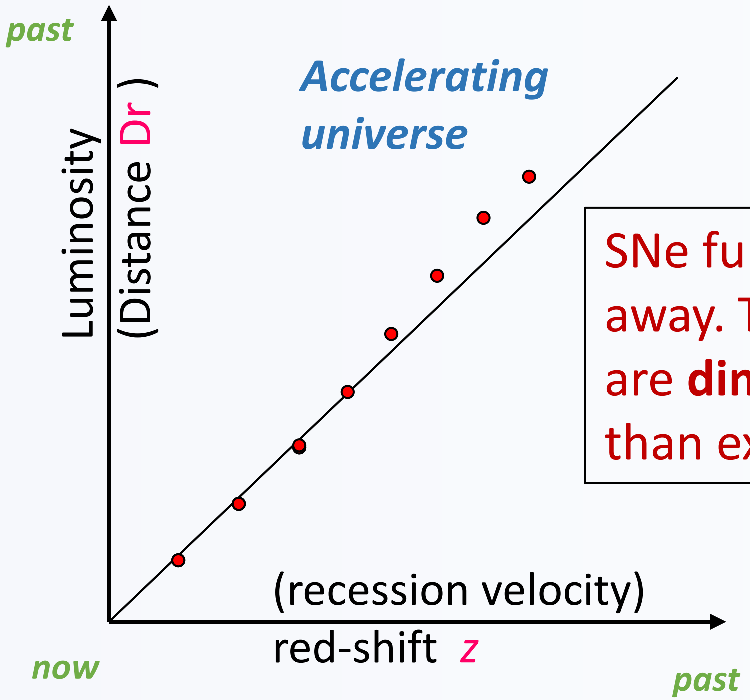
2015 – The Axion Dark Matter eXperiment (ADMX)

2016 – IceCube: Sterile neutrinos

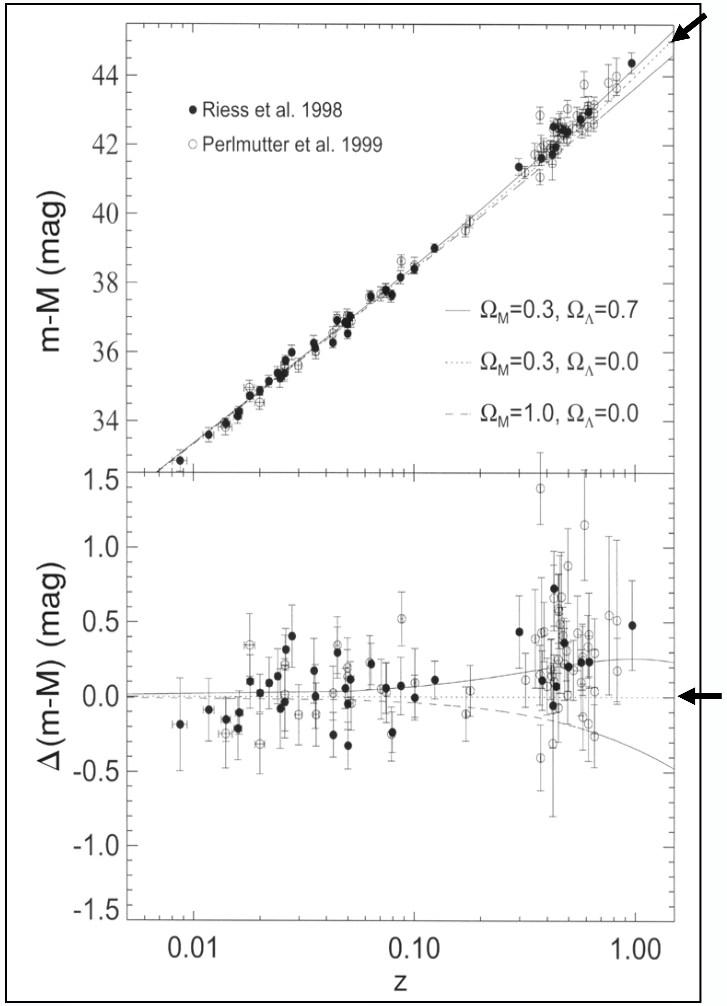
2016 – LHC: Supersymmetric particles

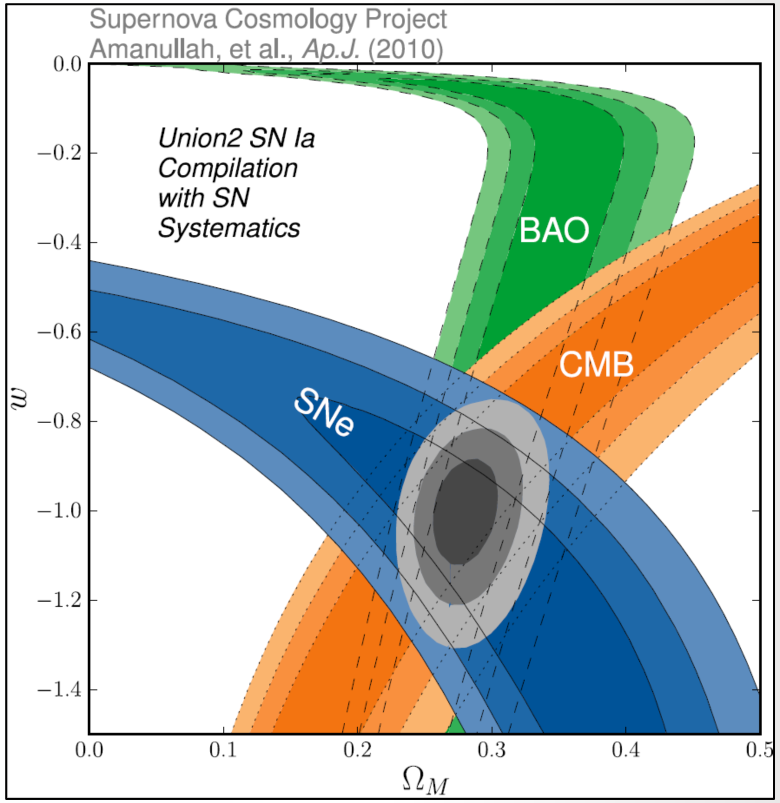


Dark Energy as the cosmological constant

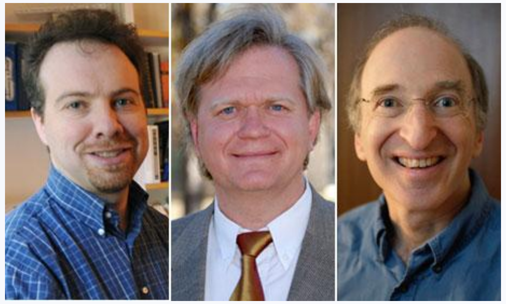


SNe further away. They are **dimmer** than expected



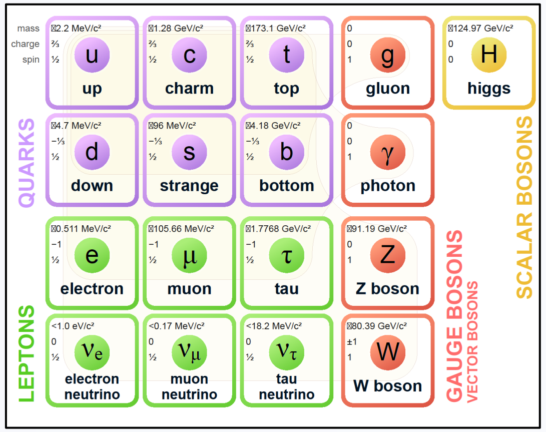


$$w_{DE} = -1.03 \pm 0.03$$



2011 Nobel Prize in Physics

Modified Matter II



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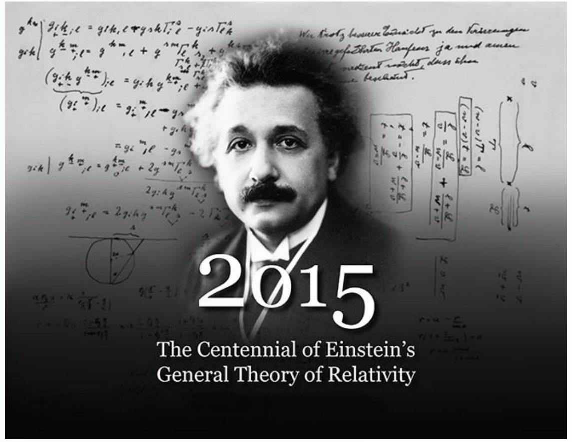

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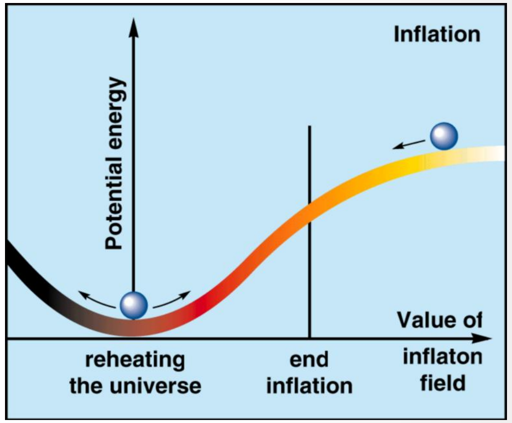

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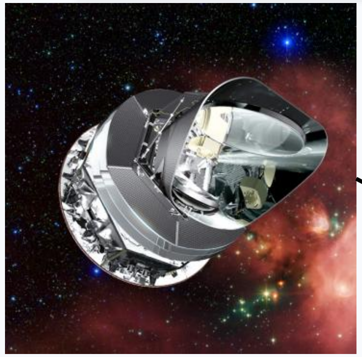
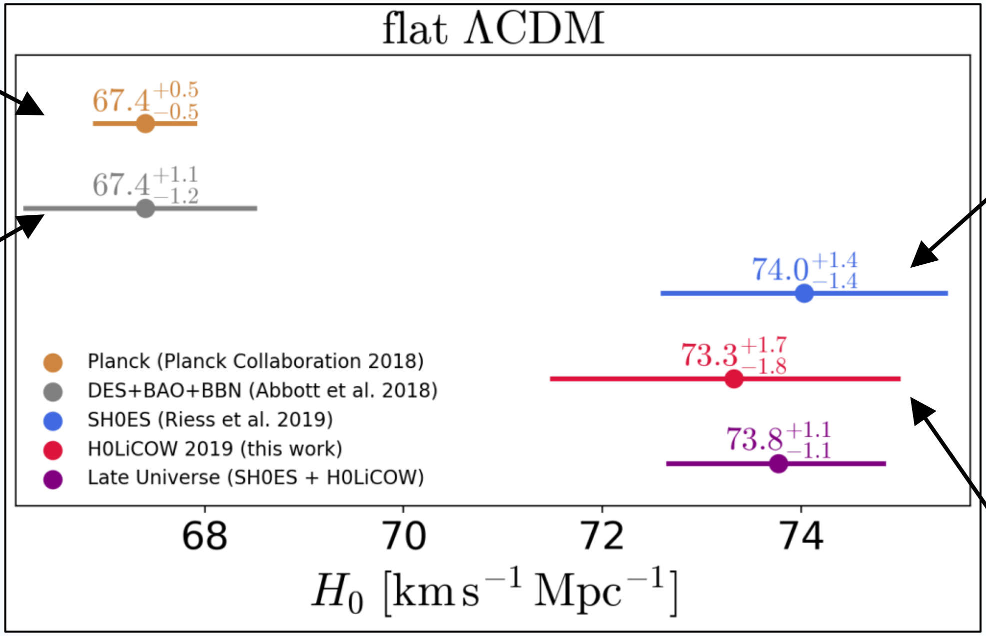
$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}[\mathcal{R} - 2\Lambda]$$

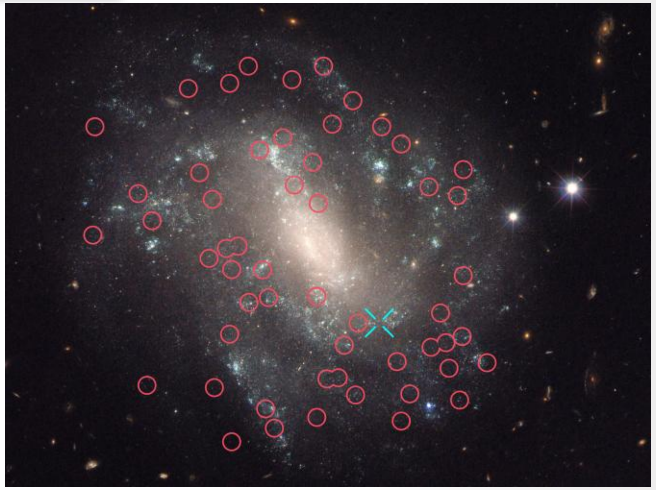
$$+\int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu},\psi)$$

The H_0 Tension



Planck Mission (predictions from the CMB)

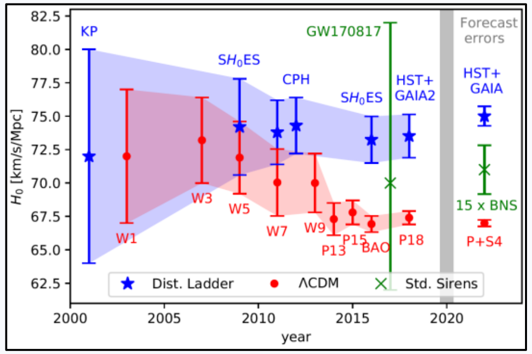




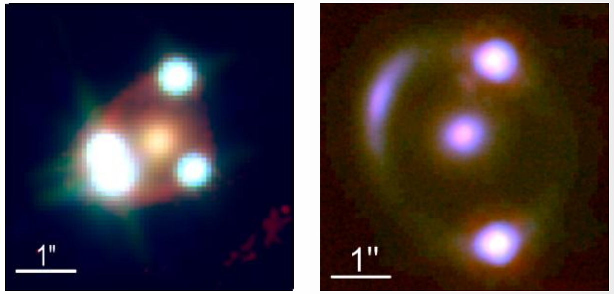
The SH0ES Project (mainly Cepheid variables)

THE DARK ENERGY SURVEY

(visible and near-infrared survey)

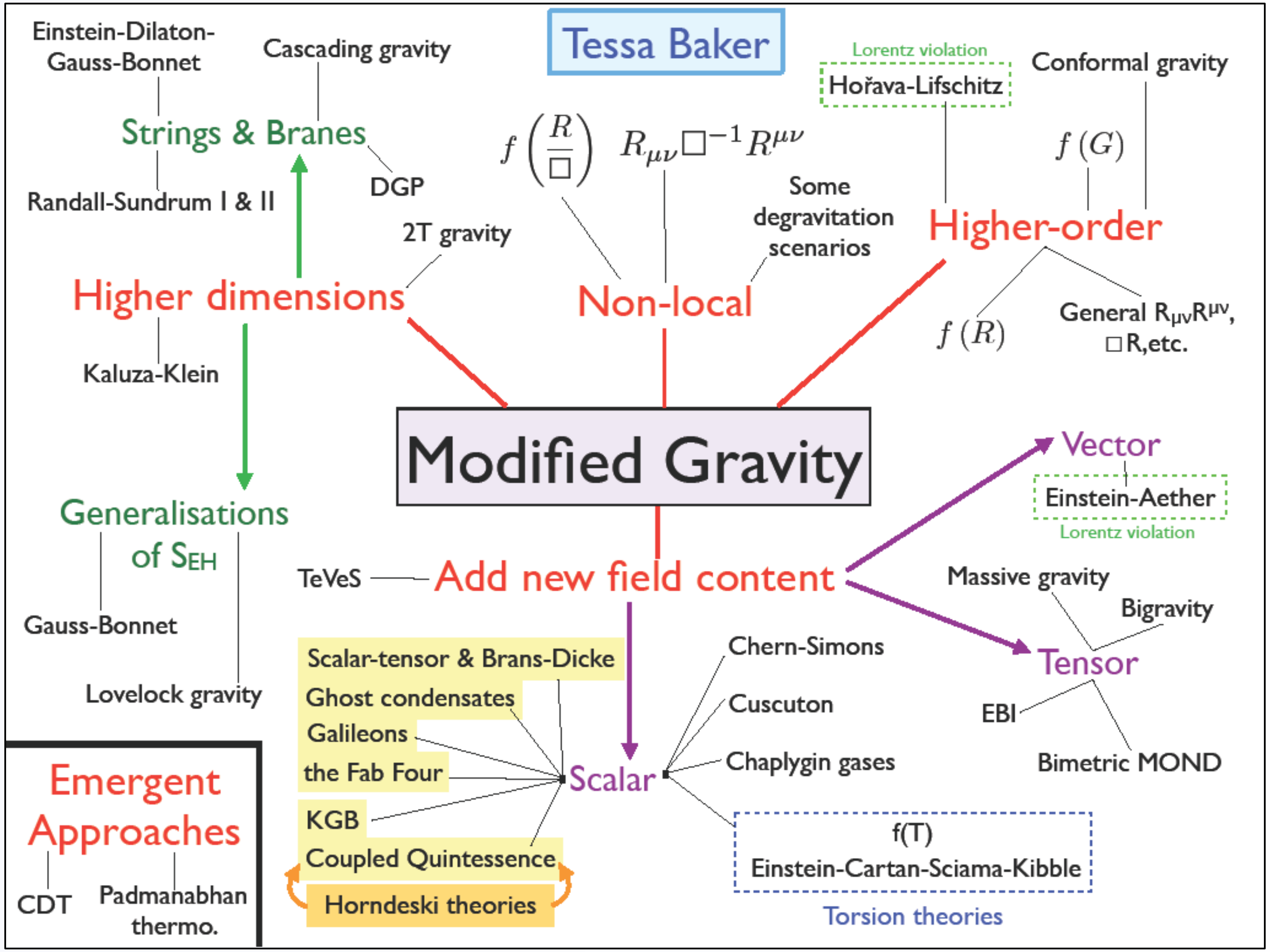


5.3σ tension



H0LICOW (strong lensing - cosmography)

The Modified Gravity Landscape



arXiv:1512.05356

Inspiration from Particle Physics

- **Gauge Principle**: Replace **global symmetries** by **local ones**
- **Group generators** produce compensating **fields**
- This results in the **standard model forces**

Can we apply this to gravity?

Gauge theory of gravity

- Formulating a **gauge theory of gravity (1956 onwards)**
- Starting from **special relativity (SR)**
  - Applying **Yang-Mills theory** to **SR**
  - Result is **Poincaré gauge theory** (**curvature** and **torsion** appear as field strengths)
- **Torsion** is the **field strength** of the **translation group**

arXiv:gr-qc/9402012

Modified Gauge Gravity

- One can always **modify gravity** (supergravity, conformal, metric affine,…)
- In all of them, **torsion** is related to the **gauge structure** of the theory
- Here, torsion opens the possibility of having a **quantum theory** of gravity

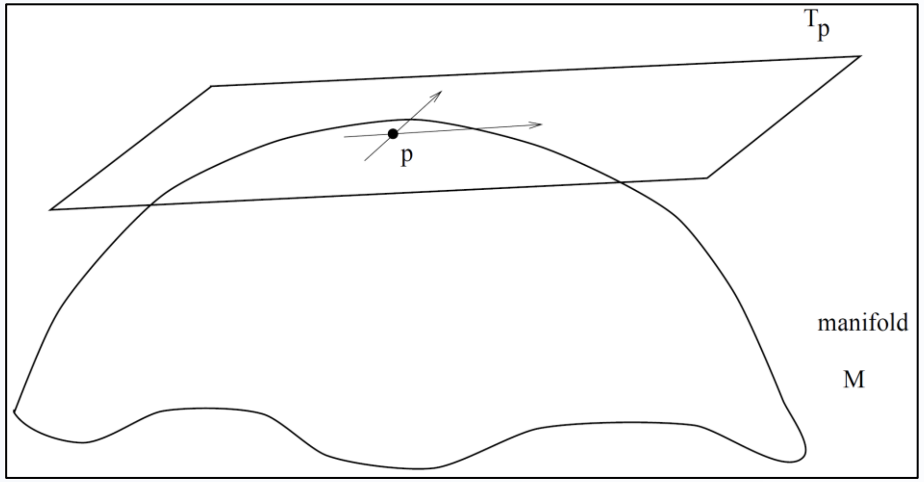
Modifying Gravity

- **Accelerating Universe (1998)**: Thousands of works in modified gravity ($f(\mathcal{R})$, Horndeski, Galileon, Lovelock, massive, Weyl,…)
- These are almost all **curvature-based**
- Can we **modify gravity** using **torsion**?

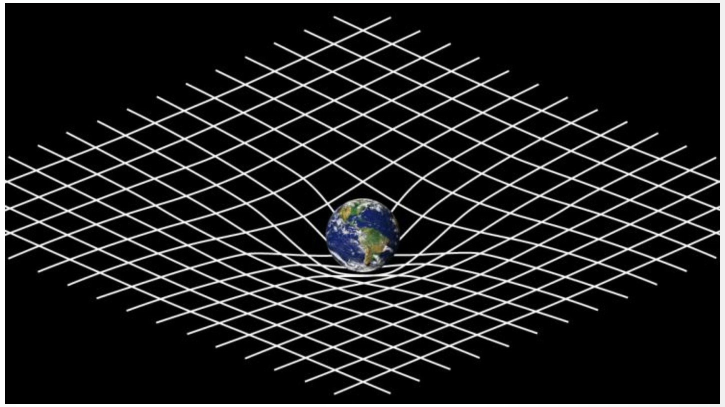
arXiv:1106.2476

Rethinking the connection

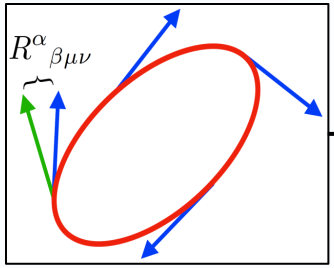
Spacetime tells matter how to move; matter tells spacetime how to curve

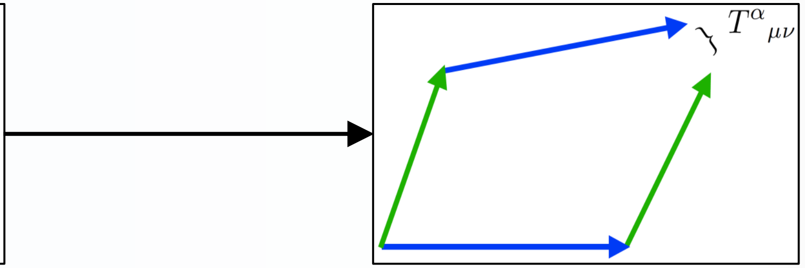


John Archibald Wheeler



<u>**Connection of gravity**</u>**:** **Curvature** is a property of the **connection**, not of the **spacetime**





The Teleparallel Equivalent of GR (TEGR)

- **TEGR**: This is the simplest torsional theory of gravity
- Tetrad ($e^a{}_\mu$): Relate the tangent space ($g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$)
- Use the **teleparallel connection** ($\Gamma^\sigma_{\mu\nu} = e_a{}^\sigma \partial_\nu e^a{}_\mu + e_a{}^\sigma \omega^a{}_{\nu\mu}$) instead of the **Levi-Civita connection** (Christoffel symbols)
- **Torsion tensor**: Measures torsion ($T^\sigma{}_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu}$)
- TEGR Action:

$$\boxed{S = -\frac{1}{16\pi G}\int d^4x\, e[T]}$$

Einstein (1928)
Phys. Rev. D 19, 3524

where $T \equiv \frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu} + \frac{1}{2}T^{\rho\mu\nu}T_{\nu\mu\rho} - T_{\rho\mu}{}^{\rho}T^{\nu\mu}{}_{\nu}$

Modified Teleparallel Gravity

- **Curvature-Torsion Relation**: $\mathcal{R} = -T + B$

$$\boxed{B \propto \nabla^{\mu} T^{\lambda}{}_{\lambda\mu}}$$

- $f(T)$ **Gravity**: Inspire by $f(\mathcal{R})$ gravity

$$S = \frac{1}{16\pi G} \int d^4x \, e[-T + f(T)] + S_{\text{mat}}$$

- Taking a flat (**FLRW**) cosmology: $g_{\mu\nu} = \text{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right)$

- **Friedmann equations**:

$$\boxed{T = 6H^2}$$

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{f(T)}{6} + \frac{T}{3} f_T$$

$$\dot{H} = -\frac{4\pi G(\rho_m + p_m)}{1 - f_T - 2T f_{TT}}$$

$f(T)$ Effective Dark Energy

- Interpreting the modification to TEGR as a **dark fluid**

$$8\pi G\, \rho_{DE} \coloneqq T f_T - \frac{f}{2}$$

$$8\pi G\, (p_{DE} + \rho_{DE}) \coloneqq -\dot{H}(f_T + 2T f_{TT})$$

- The **effective Equation-of-State** (EoS) turns out to be

$$\omega_{DE} \coloneqq \frac{p_{DE}}{\rho_{DE}} = -1 + (1+\omega)\frac{(f - T - 2T f_T)(f_T + 2T f_{TT})}{(1 - f_T - 2T f_{TT})(f - 2T f_T)}$$

- Interesting model to study **dynamical dark energy**

arXiv:1005.3039

$f(T)$ Gravity Models

Popular models of $f(T)$ gravity

1. Power-law Model: $f_1(T) = \alpha_1 (T)^b$

2. Linder Model: $f_2(T) = \alpha_2 T_0 \left(1 - e^{-p\sqrt{T/T_0}}\right)$

3. Exponential Model: $f_3(T) = \alpha_3 T_0 \left(1 - e^{-pT/T_0}\right)$

arXiv:1005.3039

Dynamical Systems Approach

- Can we extract **more information** from the Friedmann equations? These are generally **not solvable** in closed form
- Take the **ΛCDM Friedmann equations** ($\kappa^2 := 8\pi G/c^4$):

$$3H^2 = \kappa^2(\rho_m + \rho_r) + \Lambda$$
$$2\dot{H} + 3H^2 = -\left(\kappa^2/3\right)\rho_r + \Lambda$$

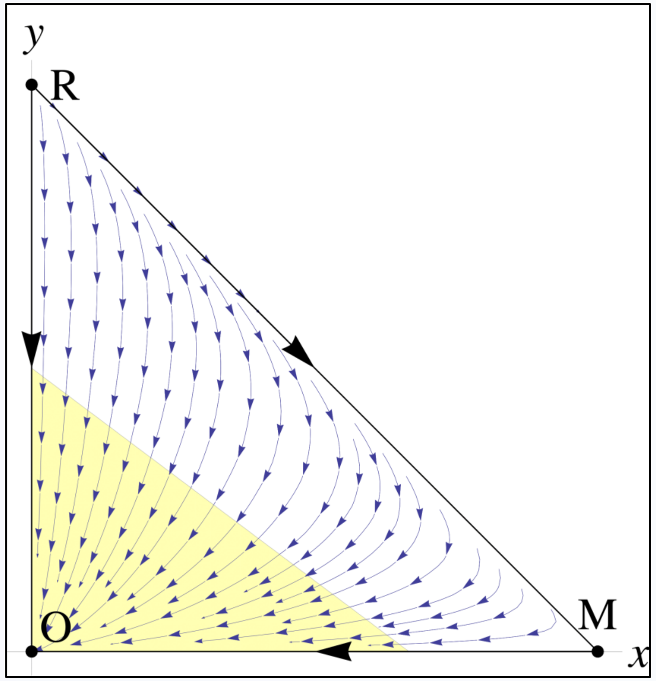
arXiv:1712.03107

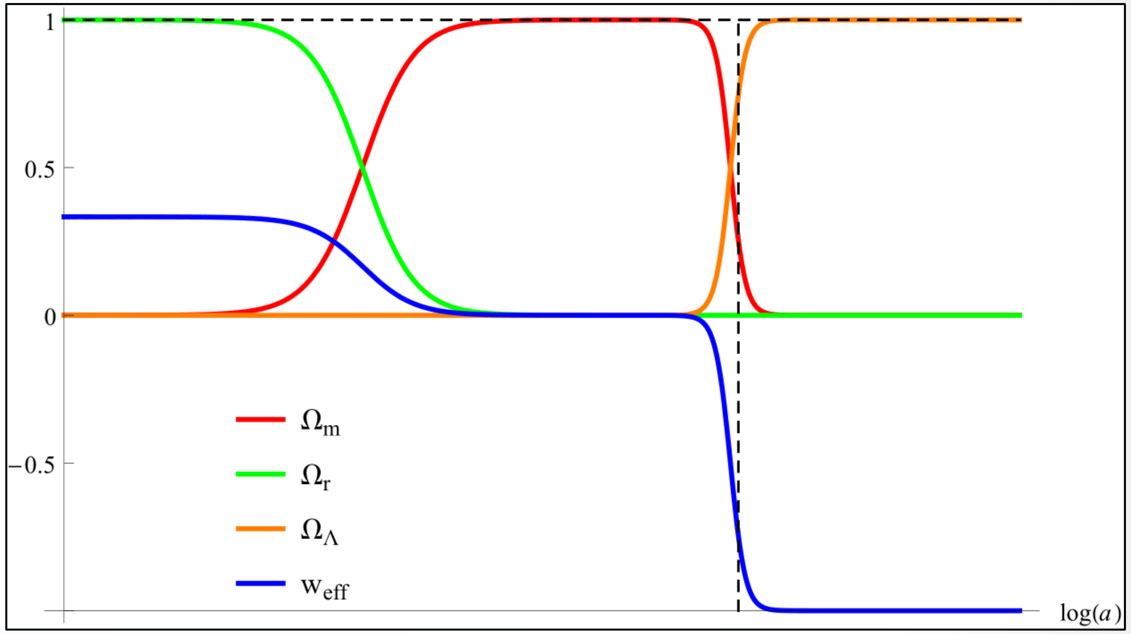
- Introducing the dimensionless variables

$$x := \Omega_m = \frac{\kappa^2 \rho_m}{3H^2} \quad y := \Omega_r = \frac{\kappa^2 \rho_m}{3H^2} \quad \Omega_\Lambda = \frac{\kappa^2 \rho_\Lambda}{3H^2}$$

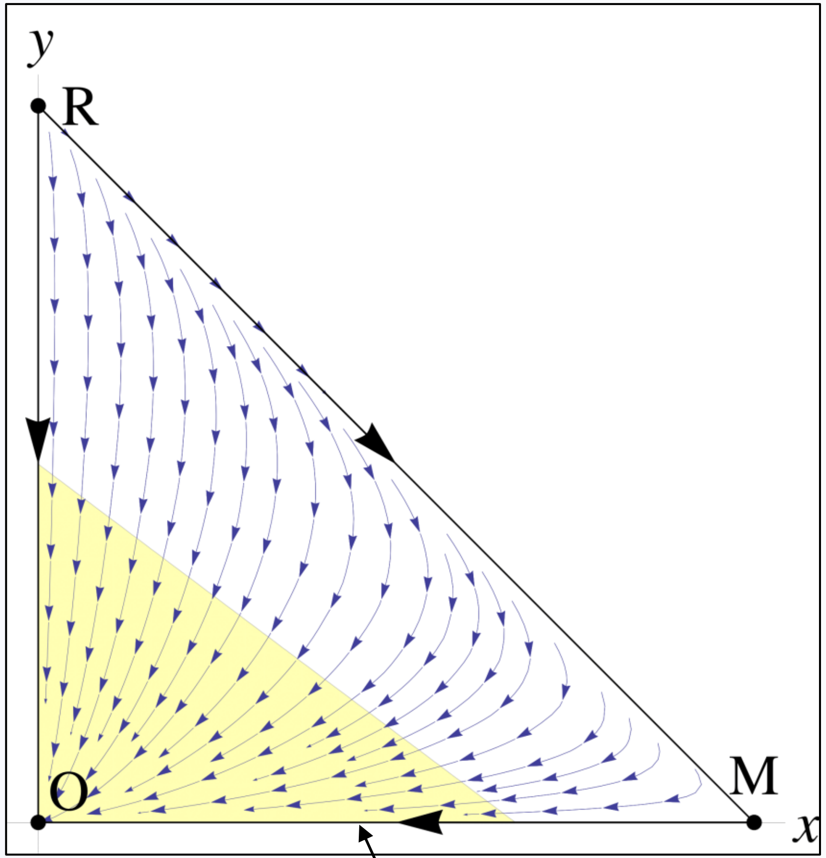
Dynamical Systems Approach in ΛCDM

- **Friedmann Equation:** $1 = x + y + \Omega_\Lambda$
- Together with $x, y \geq 0$, this gives a triangular *physical phase space*
- Dynamical system: $x' = \frac{dx}{Hdt} = x(3x + 4y - 3),\ \ y' = y(3x + 4y - 4)$





Dynamical Systems Approach



Accelerated expansion (radiation/matter → dark energy transition)

Critical point	x	y	ω_{eff}	Stability
O	0	0	-1	Saddle point
R	0	1	1/3	Unstable point
M	1	0	0	Saddle point

Dark energy dominated - Future attractor

Radiation dominated - Past attractor

Matter dominated - Past attractor

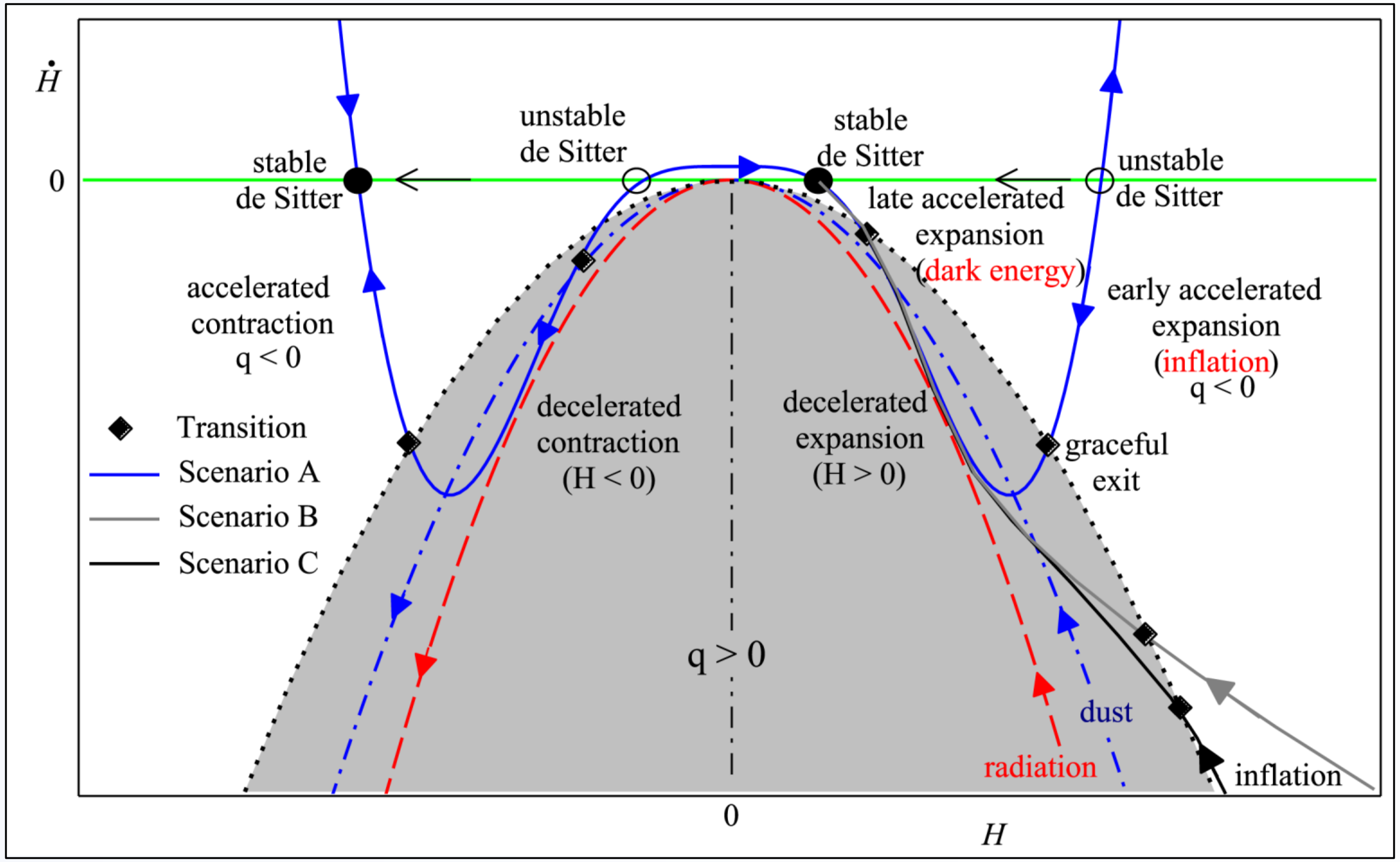
Dynamical Systems in $f(T)$ gravity

- In $f(T)$ gravity, the dynamical system is generally considered with **one additional variable** $\dot{H} = \mathcal{F}(H) = -\frac{3}{2}\left(1+\omega_{eff}\right)H^2$

arXiv:1710.10194

Model	Behaviour
$f_1(T) = \alpha_1(T)^b$	Semi-stable Minkowski transition to stable (unstable) accelerating (decelerating) de Sitter critical points For $b \ll 1$, we recover the sequence: Big Bang→Matter dominated→de Sitter acceleration
$f_2(T) = \alpha_2 T_0\left(1 - e^{-p\sqrt{T/T_0}}\right)$	Semi-stable Minkowski transition to both stable and unstable de Sitter critical points For $p > 0$, a correct transition redshift is recovered, and a consistent matter dominated era
$f_3(T) = \alpha_3 T_0\left(1 - e^{-pT/T_0}\right)$	Similar critical points as $f_2(T)$ Realistic deceleration → acceleration sequence for $p = 1/2$. Larger values of p yield better observational agreement

Dynamical Systems in $f(T)$ gravity



H	Behaviour
Large	Big bang or bouncing cosmology transitions to a decelerating cosmology
Intermediate	Correct radiation era
Small	Consistent transition redshift to late-time accelerated phase
Fate	General all cosmologies evolve to a de Sitter fixed point

arXiv:1710.10194

$f(T)$ Gravity Field Equations

- **Extended Action**:

$$S = \frac{1}{16\pi G}\int d^4x\, e[-T + f(T)] + S_{\rm mat}$$

- ***f*(*T*) gravity** field equations

$$W_C{}^{\alpha}:\quad \frac{1}{4}E_C{}^{\alpha}(T + f(T)) - (\partial_\rho T) f_{TT} E_C{}^{\lambda} S_\lambda{}^{\rho\alpha} - (1 + f_T)\left(E_C{}^{\lambda} T^{\lambda}{}_{\mu\nu} S_\lambda{}^{\alpha\nu} + e^{-1}\partial_\rho\left(e E_C{}^{\lambda} S_\lambda{}^{\rho\alpha}\right)\right) = 4\pi G \theta_C{}^{\alpha}$$

- The **contortion** and **superpotential tensors** are defined by

$$K^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \mathring{\Gamma}^{\lambda}_{\mu\nu} = -\frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}\right)$$

$$S_\lambda{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_{\lambda} + \delta^{\mu}_{\lambda} T^{\alpha\mu}{}_{\alpha} - \delta^{\mu}_{\lambda} T^{\alpha\mu}{}_{\alpha}\right)$$

arXiv:2106.13793

The Weitzenbock Gauge

- The spin connection, $\omega^{a}{}_{\nu\mu}$, represents the **six degrees** of freedom associated with the **Lorentz transformations**

- The **tetrad** and **spin connection** field equations:

$$W_{(\mu\nu)} = 4\pi G\,\theta_{(\mu\nu)} \qquad \text{and} \qquad W_{[\mu\nu]} = 4\pi G\,\theta_{[\mu\nu]} = 0$$

- The **Weitzenbock guage** is defined by spin connection components being zero ($\omega^{a}{}_{\nu\mu} = 0$)

$$\omega^{a}{}_{\nu\mu} = \Lambda^{A}{}_{C}\partial_{\mu}\Lambda_{B}{}^{C}$$

arXiv:2106.13793

Scalar Perturbations

- $\boldsymbol{f(T)}$ **gravity** leaves imprints at the **perturbative level**

$$e^0{}_\mu = \delta^0_\mu(1+\psi)\,, e^i{}_\mu = \delta^i_\mu a(1-\phi) \;\Rightarrow\; ds^2 = (1+2\psi)dt^2 - a^2(1-2\phi)\delta_{ij}dx^i dx^j$$

- **Matter perturbation** components

$$\theta_C{}^\gamma = -\begin{bmatrix} \rho_m + \delta\rho_m & (\rho_m + p_m)\delta^i v \\ -a^2\delta^i_I(\rho_m + p_m)\delta v & -\delta^I{}_i(p_m + \delta p_m) \end{bmatrix}$$

- **Matter over-density perturbations** also contribute through

$$\delta_m = \frac{\delta\rho_m}{\rho_m} - 3a^2Hv$$

where $V = a^2 v$ is the scalar mode velocity

arXiv:2106.13793

- For **dark matter**: $p_m = \delta p_m = 0$

Scalar Perturbations

- $f(T)$ **gravity** leaves imprints at the **perturbative level**

$$e^{0}{}_{\mu} = \delta^{0}_{\mu}(1+\psi)\,, e^{i}{}_{\mu} = \delta^{i}_{\mu}a(1-\phi) \;\Rightarrow\; ds^2 = (1+2\psi)dt^2 - a^2(1-2\phi)\delta_{ij}dx^i dx^j$$

- **Mészáros equation perturbation evolution** equation

$$\ddot{\delta}_m + 2H\mathcal{H}\dot{\delta}_m + 4\pi G_{\text{eff}}\rho_m\delta_m = 0$$

- For the **sub-horizon limit** ($k \gg aH$) :

$$\mathcal{H} \to 1$$

$$G_{\text{eff}} \to \frac{G}{1+f_T}$$

- This represents a future de Sitter point where $a(t) \to e^{H_{dS}t}$

arXiv:1010.3512
arXiv:2308.15995

Growth Factors

- **Growth Factor** $(k \gg aH)$:

$$D(z) = \frac{\delta_m(a)}{\delta_m(a_i)}$$

where a_i is a reference scale factor value (normally $a_i = 0.1$ for matter era initial conditions)

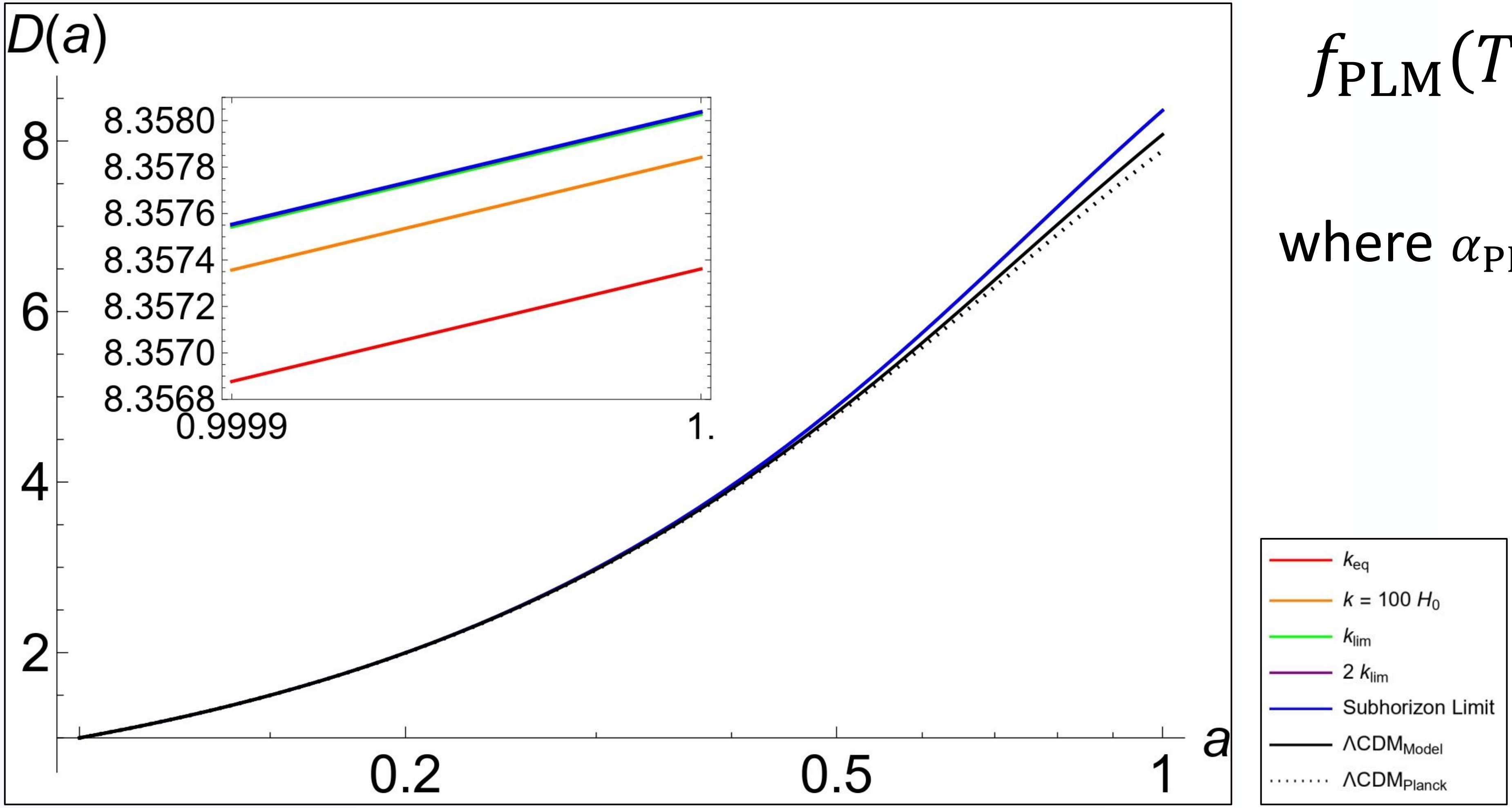
- **Clustering growth rate** can be assumed to be linear:

$$g(a) = \frac{d\ln\delta_m}{d\ln a} \simeq \Omega_m(a)^{\gamma(a)}$$

- Resulting in a **growth factor**

$$D(a) = \exp\left(\int_1^a \frac{\Omega_m(\tilde{a})^{\gamma(\tilde{a})}}{\tilde{a}} d\tilde{a}\right)$$

Growth Factor



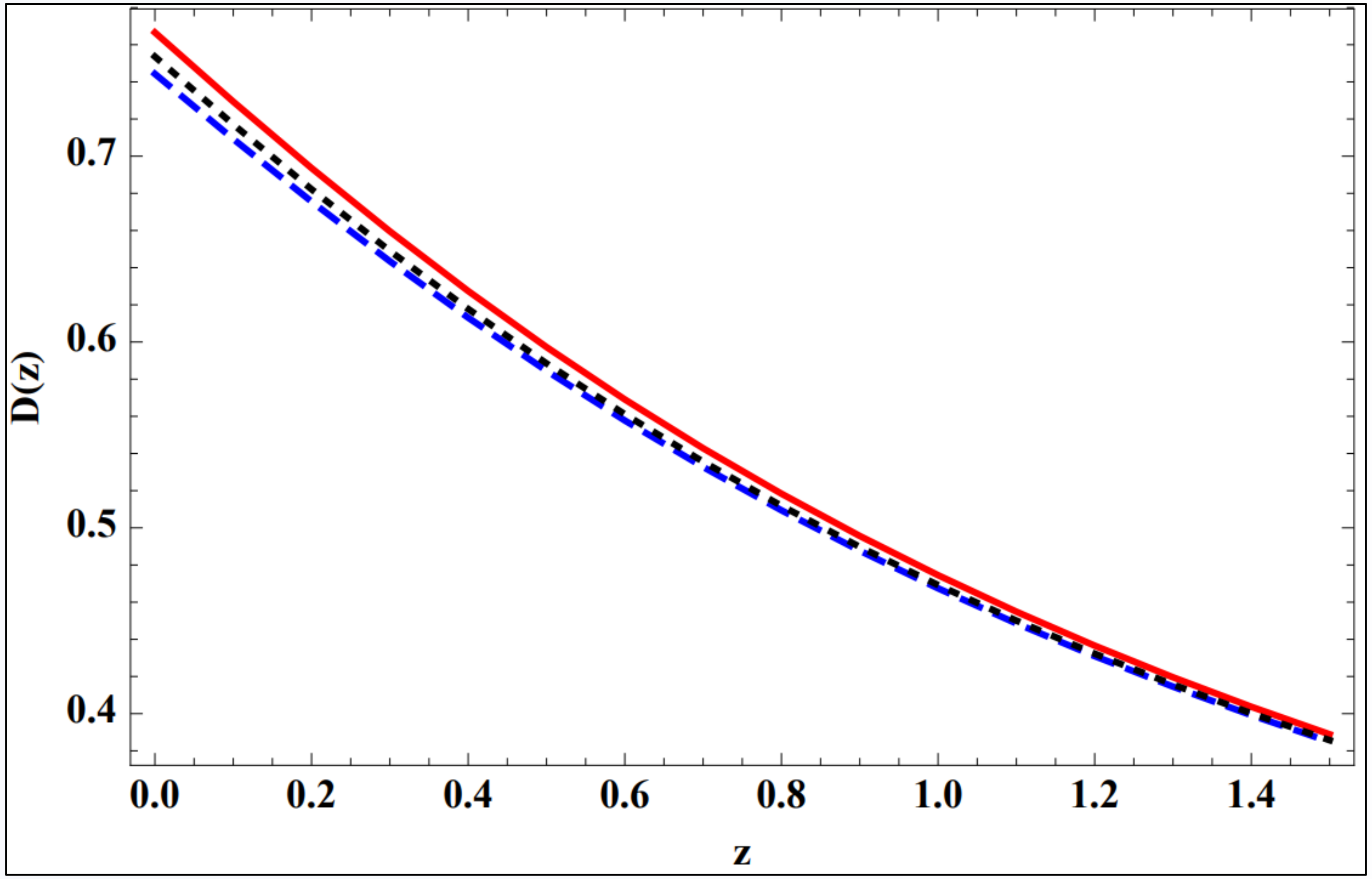
$$f_{\mathrm{PLM}}(T) = \alpha_{\mathrm{PLM}}(T)^{\beta_{\mathrm{PLM}}}$$

where $\alpha_{\mathrm{PLM}} = \dfrac{(6H_0^2)^{1-\beta_{\mathrm{PLM}}}(1-\Omega_{m,0})}{2\beta_{\mathrm{PLM}}-1}$

$H_0 = 68.5$
$\beta_{\mathrm{PLM}} = -0.22$
$\Omega_{m,0} = 0.35$

arXiv:2308.15995

Growth Factor



$$f_{\mathrm{PLM}}(T) = \alpha_{\mathrm{PLM}}(T)^{\beta_{\mathrm{PLM}}}$$

where $\alpha_{\mathrm{PLM}} = \frac{(6H_0^2)^{1-\beta_{\mathrm{PLM}}}(1-\Omega_{m,0})}{2\beta_{\mathrm{PLM}}-1}$

Numerical solution of the growth factor evolution equation for ΛCDM (**red**), **power-law** f_1 $(n = 0.1)$ gravity (**blue**), using the effective $f(T)$ EoS in GR (**black**)

$f(T,B)$ Gravity

$B \propto \nabla^{\mu} T^{\lambda}{}_{\lambda\mu}$

- **Curvature-Torsion Relation**: $\mathcal{R} = -T + B$
- Naturally, $f(T) \neq f(-T+B) = f(\mathcal{R})$
- $f(T,B)$ organically decouples **second-order** (T) and **fourth-order** (B) contributions
- Only **linear** B is a boundary term
- $\boldsymbol{f(T,B)}$ **Gravity**: Generalizing $f(\mathcal{R})$ gravity

$$S = \frac{1}{16\pi G}\int d^4x\, e\left[-T + \boldsymbol{f(T,B)}\right] + S_{\mathrm{mat}}$$

$f(T,B)$ Background Cosmology

- As before, the **Friedmann equations** are

$$3H^2 = 8\pi G(\rho_m + \rho_{\text{eff}})$$
$$3H^2 + \dot{H} = -8\pi G(p_m + p_{\text{eff}})$$

- Therefore, the **effective fluid** will have

$$8\pi G\rho_{\text{eff}} = 3H^2(3f_B + 2f_T) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f$$
$$8\pi G p_{\text{eff}} = \frac{1}{2}f - \left(3H^2 + \dot{H}\right)(3f_B + 2f_T) - 2H\dot{f}_T + \ddot{f}_B$$

- Giving an **effective EoS**

$$\omega_{\text{eff}} = -1 + \frac{\ddot{f}_B - 3H\dot{f}_B - 2\dot{H}f_T - 2H\dot{f}_T}{3H^2(3f_B + 2f_T) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f}$$

arXiv:1508.05120
arXiv:1609.08373

$$T = 6H^2$$
$$B = 6\left(3H^2 + \dot{H}\right)$$
$$\mathcal{R} \equiv -T + B = 6\left(2H^2 + \dot{H}\right)$$

$f(T,B)$ Gravity Models

Popular models of $f(T,B)$ gravity:

1. General Taylor Model: $f_1(T,B) = f(T_0,B_0) + f_T(T_0,B_0)(T-T_0) + f_B(T_0,B_0)(B-B_0) + \frac{1}{2}f_{TT}(T_0,B_0)(T-T_0)^2 + \frac{1}{2}f_{BB}(T_0,B_0)(B-B_0)^2 + O\left(T^3,B^3\right)$

$$\simeq A_0 + A_1 T + A_2 T^2 + A_3 B^2 + A_4 TB$$

2. Power-law Model: $f_2(T,B) = b_0 B^k + t_0 T^m$
3. Mixed power-law Model: $f_3(T,B) = f_0 B^k T^m$

arXiv:1909.10328

Dynamical Systems in $f(T,B)$

- For $f(T,B)$ gravity, the **time-dependence** can be encapsulated in

$$\lambda = \frac{\ddot{H}}{H^3} = \frac{H'^2}{H^2} + \frac{H''}{H} \qquad \left(H' \equiv \frac{dH}{d\ln a} = \frac{dH}{Hda}\right)$$

- Choosing **dynamical variables**:

$$x := f_B, \qquad y := f_B', \qquad z := \frac{\dot{H}}{H^2} = \frac{H'}{H}, \qquad w := -\frac{f}{6H^2}$$

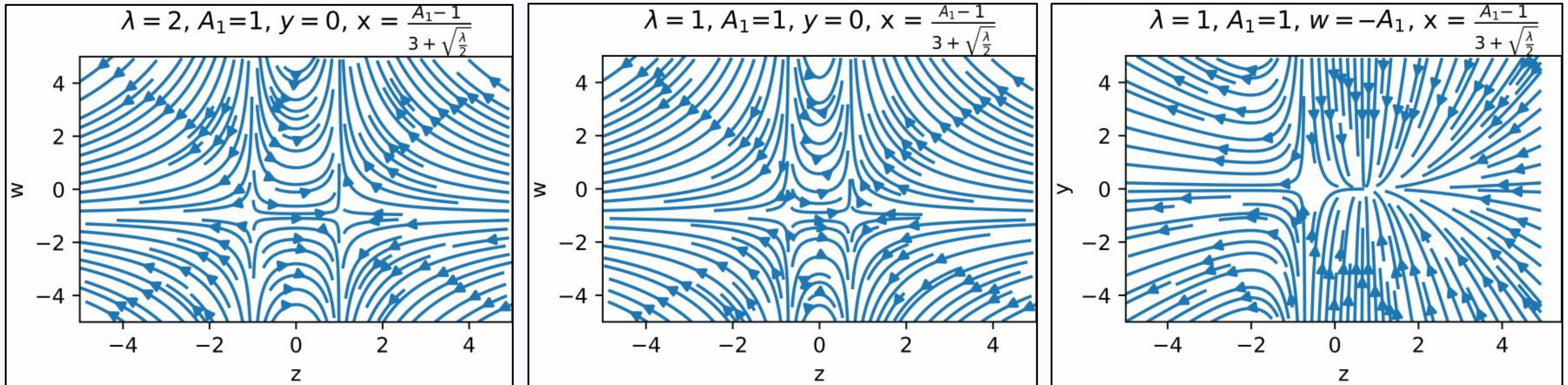
- Gives the **autonomous system** for $f(T,B)$ gravity as

arXiv:2005.14191

$$z' = \lambda - 2z^2$$
$$x' = y$$
$$w' = -6zx - 2zf_T - \lambda x - 2zw$$
$$y' = 3w + (9+3z)x + f_T(6+2z) + 2f_T' - zy - 3 - 2z$$

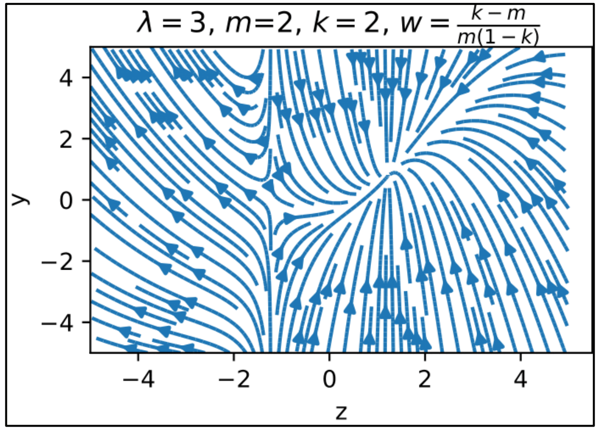
Dynamical Systems in $f(T,B)$

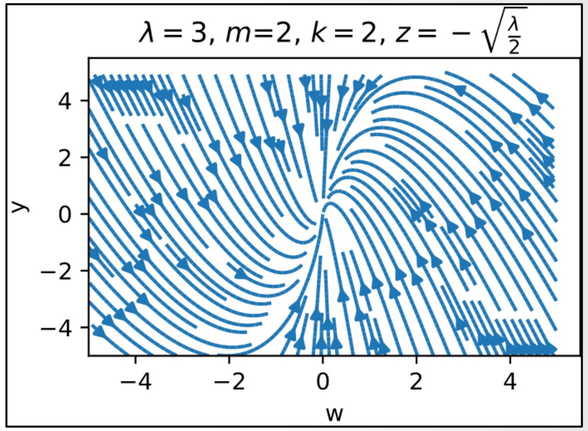
- For $f_1(T,B) \simeq A_0 + A_1 T + A_2 T^2 + A_3 B^2 + A_4 TB$, we define $$f_T \coloneqq -(3+z)x - 2w - A_1$$
- Dynamical analysis results in that for $\lambda \neq 0$, **all critical points are saddle-like**
- This model produces the **correct epochs** and **agrees with observational constraints**

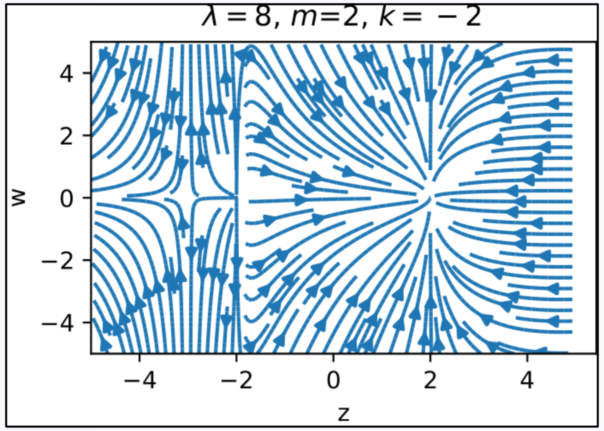


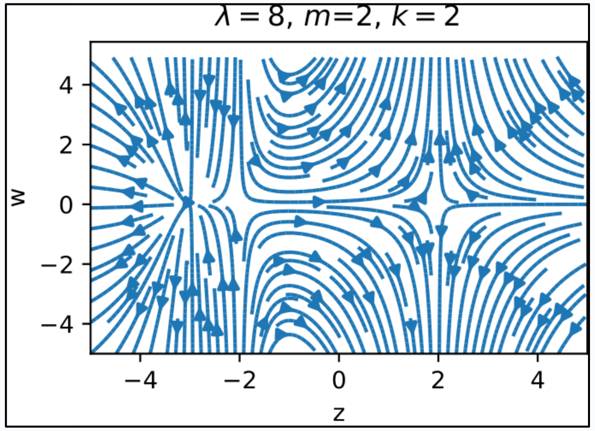
Dynamical Systems in $f(T,B)$

- For $f_2(T,B) = b_0 B^k + t_0 T^m$:
- For $f_3(T,B) = f_0 B^k T^m$:









Both models can **produce ΛCDM** but for some cases crosses the **phantom line**

Scalar Perturbations

- $f(T)$ **gravity** leaves imprints at the **perturbative level**

$$e^0{}_\mu = \delta^0_\mu(1+\psi)\,, e^i{}_\mu = \delta^i_\mu a(1-\phi) \;\Rightarrow\; ds^2 = (1+2\psi)dt^2 - a^2(1-2\phi)\delta_{ij}dx^i dx^j$$

- **Matter over-density perturbations** also contribute through

$$\delta_m = \frac{\delta\rho_m}{\rho_m}$$

- **Four Branching** cases occurs that depend on

$$\Pi := f_B + f_T$$
$$\Upsilon := f_{BB} + 2f_{TB} + f_{TT}$$
$$\Xi := f_{TB}^2 - f_{TT}f_{BB}$$

arXiv:2009.02168

Scalar Perturbations

- **Branch A $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi \neq 0\}$:**
  - $G_{eff} = -G\,\dfrac{4\Upsilon}{36H^2(f_{BB}f_{TT}+2\Xi)+3\Upsilon f_T}$
  - **Example:** $f(T,B) = f_1(T) + f_2(T)f_3(B) + f_4(B)$

- **Branch B $\{\Pi \neq \text{const}, \Upsilon \neq 0, \Xi = 0\}$:**
  - **Examples:** $f_a(T,B) = f(T)$, $f_b(T,B) = -T + f(B)$
  - $(G_b)_{eff} = -\dfrac{G}{f_T}$; $(G_b)_{eff} = G\,\dfrac{4H\left(2\dot{f}_B+H\right)}{\left(\dot{f}_B+2H\right)^2}$

arXiv:2009.02168

Scalar Perturbations

- **Branch C $\{\Pi \neq \text{const}, \Upsilon = 0, \Xi \neq 0\}$:**
  - $G_{eff} = -G\,\frac{4}{3(f_T + 12H^2 f_{TB})}$

- **Branch D $\{\Pi = \text{const}, \Upsilon = 0, \Xi = 0\}$:**
  - **Unique case where** $f(T,B) = f(\mathcal{R})$
  - $G_{eff} = \frac{4G}{3f_{\mathcal{R}}}$

arXiv:2009.02168

Tensor Perturbations

- Taking **tensor perturbations** for **tetrads fields**

$$e^0{}_\mu = \delta^0_\mu \,, e^i{}_\mu = \delta^i_\mu + \frac{1}{2}\delta^j_\mu \delta^{ki} h_{jk} \Rightarrow \quad ds^2 = dt^2 - a^2\left(\delta_{ij} + h_{ij}\right) dx^i dx^j$$

- Produces a **gravitational wave propagation equation** (GWPE)

$$\ddot{h}_{ij} + (3 + \alpha_M) H \dot{h}_{ij} - (1 + \alpha_T)\frac{k^2}{a^2} h_{ij} = 0$$

in the Fourier domain

- $\alpha_T = c_T^2 - 1$ is the **tensor excess speed** and $\alpha_M = \frac{1}{HM_*^2}\frac{dM_*^2}{dt}$ is the **Planck mass run rate** (M_*^2 is the effective **Planck mass**)

Gravitational Wave Propagation Equation

- **Conformal time ($dt = ad\eta$):** $ds^2 = dt^2 - a^2\delta_{ij}dx^i dx^j \rightarrow ds^2 = a^2(dt^2 - \delta_{ij}dx^i dx^j)$
- The **GWPE** becomes $h''_{ij} + (2+\alpha_M)\mathcal{H}h'_{ij} + \left(c_T^2 k^2 + a^2\mu^2\right)h_{ij} = 0$

where $\mathcal{H} = a'/a$ is **conformal Hubble parameter** and μ is the **graviton mass**

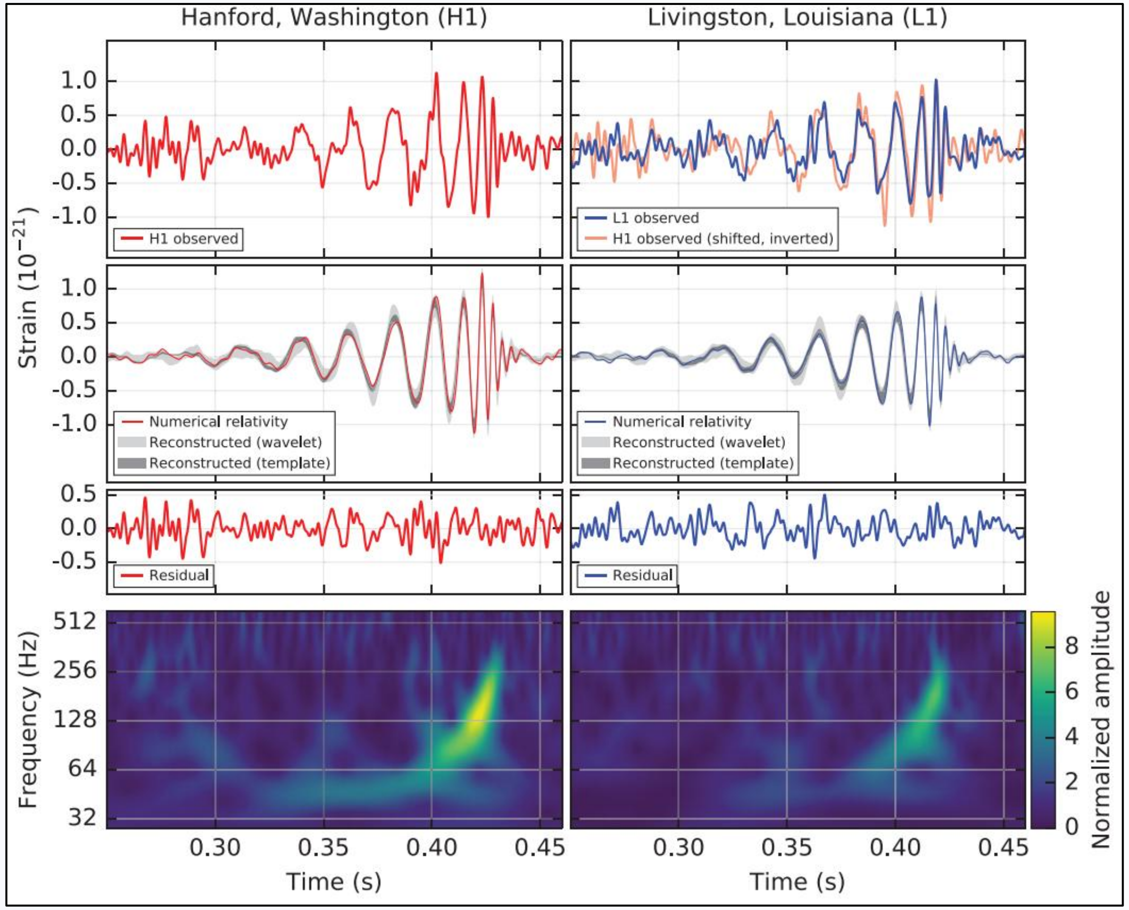
- This gives a solution

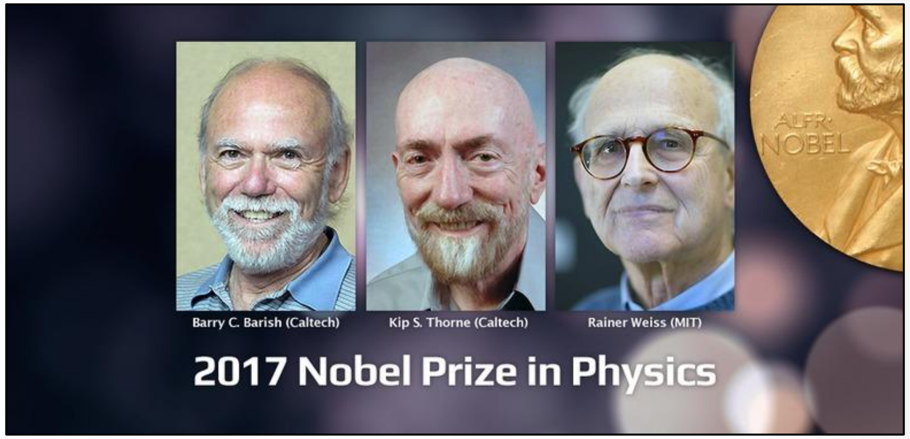
$$\boxed{h_{GW} \sim h_{GR}\underbrace{e^{-\frac{1}{2}\int \alpha_M \mathcal{H} d\eta}}_{\text{Amplitude modulation}}\underbrace{e^{ik\int\left(\alpha_T + \frac{a^2\mu^2}{k^2}\right)^{1/2} d\eta}}_{\text{Phase change}}}$$

arXiv:1807.09241
arXiv:1710.04825

GW Observations

Can we use **GW observations** to detect **modified gravity**?



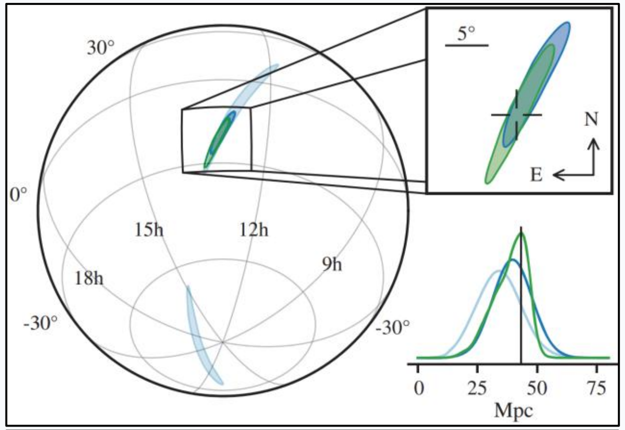


The Era of Multi-messenger Astronomy

GW170817

GRB170817A

LIGO-Virgo localization



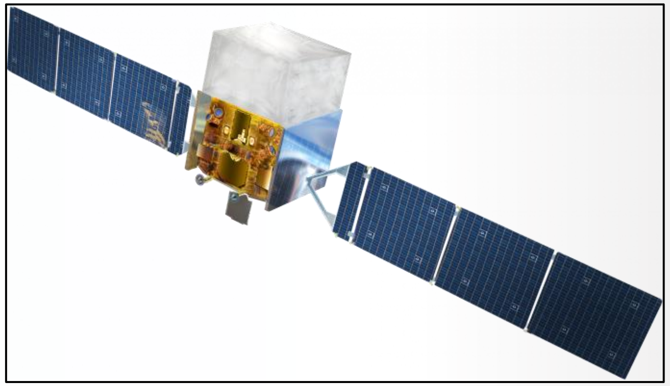
Virgo observatory

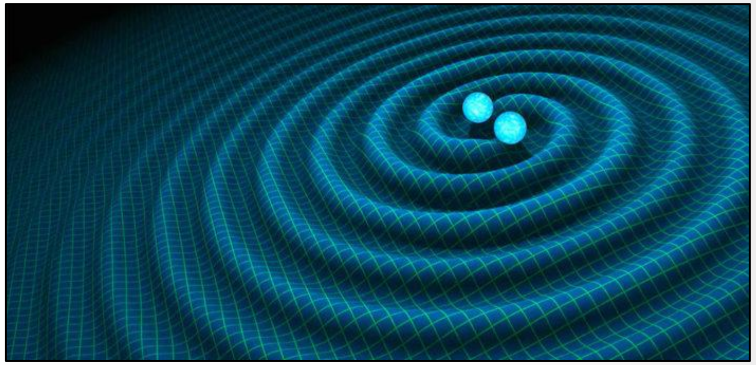
$$M_{\mathrm{Tot}} = 2.74^{+0.04}_{-0.01} M_{\odot}$$

$$\Delta T = 1.7s$$

$$c_T = c^{+7\times 10^{-16}}_{-3\times 10^{-15}}$$

Fermi Telescope

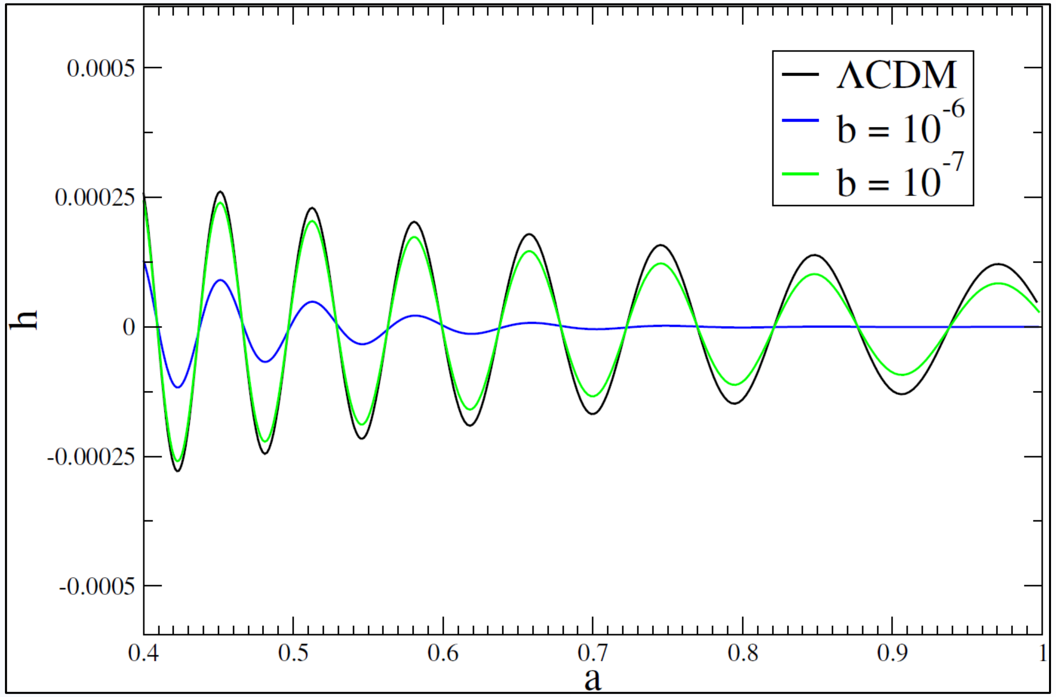




Modified GWPE

- **$f(T)$ Gravity:** $\ddot{h}_{ij} + \left(3H - \frac{\dot{f}_T}{f_T}\right)\dot{h}_{ij} - \frac{k^2}{a^2}h_{ij} = 0$
- **$f(T,B)$ Gravity:** $\ddot{h}_{ij} + \left(3H - \frac{1}{f_T}\left[\dot{f}_T - f_{TB}\left(6\dddot{H} + 6\ddot{H} + 39H\dot{H}\right)\right]\right)\dot{h}_{ij} - \frac{k^2}{a^2}h_{ij} = 0$

$f_1(T) = \alpha_1(T)^b$



$c_T = c$
in these case

arXiv:1008.1250
arXiv:1810.03942

Matter coupling

- Do **non-minimal matter coupling** terms give interesting cosmologies
- This produces an action

$$\boxed{\Theta = g_{\mu\nu}\Theta^{\mu\nu}}$$

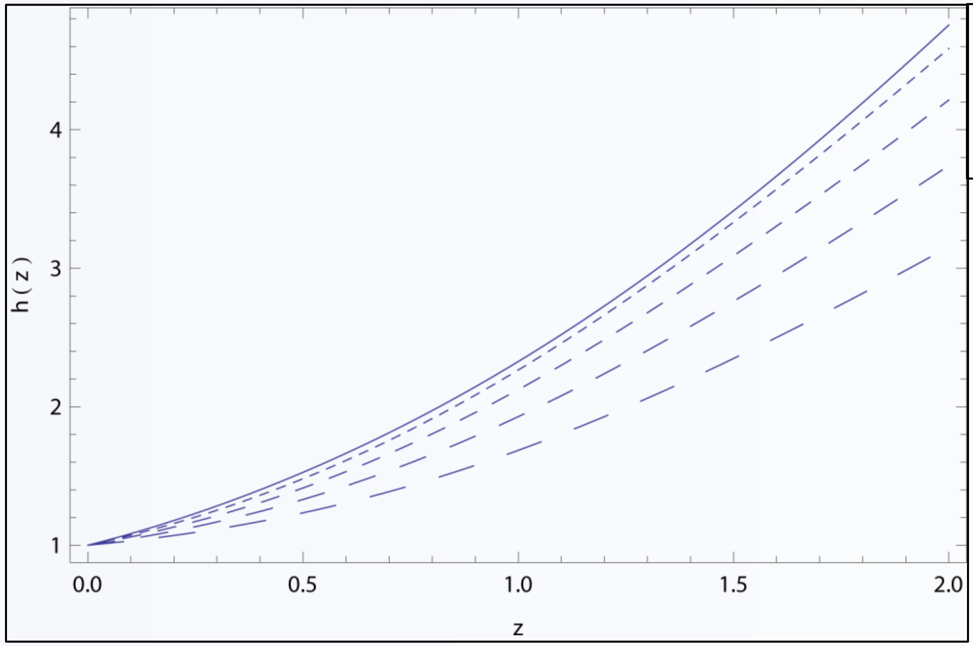
$$S = \frac{1}{16\pi G}\int d^4x\, e\left[-T + f(T,\Theta)\right] + S_{\text{mat}}$$

- Hence resulting in Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\rho_m - \frac{f(T) + 12H^2 f_T}{6} + f_\Theta\left(\frac{\rho_m + p_m}{3}\right)$$

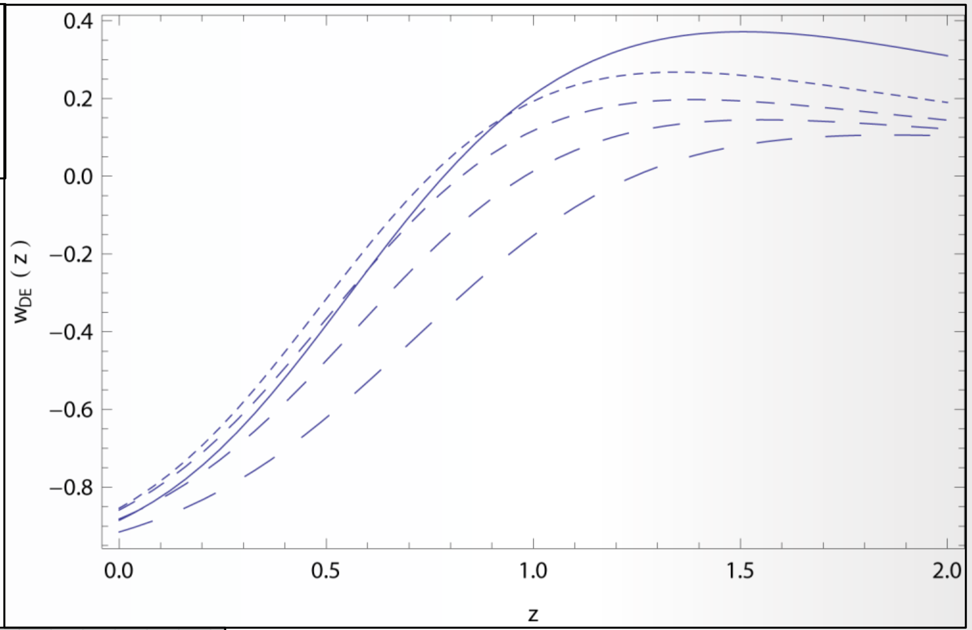
$$\dot{H} = -4\pi G(\rho_m + p_m) - \dot{H}\left(f_T - 12H^2 f_{TT}\right) - H(\dot{\rho}_m - 3\dot{p}_m)f_{T\Theta} - f_\Theta\left(\frac{\rho_m + p_m}{2}\right)$$

Matter coupling

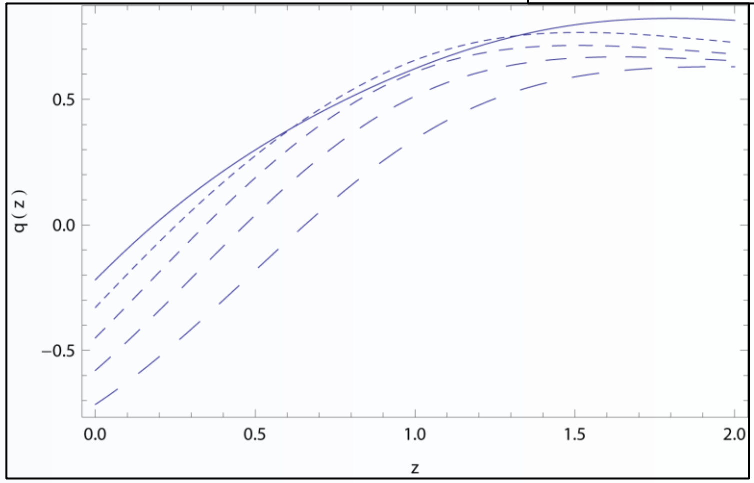


$$f(T,\Theta) = \alpha T\Theta = \alpha T\rho_m + \Lambda$$

$$\alpha = \frac{\alpha_0}{H^2}; \Lambda = \lambda H_0^2$$



Solid curves: $\alpha = -0.01; \Lambda = -3$
Dotted curves: $\alpha = -0.02; \Lambda = -3.5$
Short-dashed curves: $\alpha = -0.03; \Lambda = -4$
Dashed curves: $\alpha = -0.04; \Lambda = -4.5$
Long-dashed curves: $\alpha = -0.05; \Lambda = -5$



arXiv:1405.0519

Horndeski Gravity

Horndeski Gravity: Produces the most **general second-order theory** that contains only **one scalar field** (in **standard gravity**)

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left[\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right]$$

where

$$\mathcal{L}_2 = \mathrm{G}_2(\phi, X)$$
$$\mathcal{L}_3 = \mathrm{G}_3(\phi, X)\Box\phi$$
$$\mathcal{L}_4 = \mathrm{G}_4(\phi, X)\mathcal{R} + \mathrm{G}_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]$$
$$\mathcal{L}_5 = \mathrm{G}_5(\phi, X)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}\mathrm{G}_{5,X}(\phi, X)\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi\right]$$

Teleparallel Horndeski Gravity (TeleDeski)

- **TeleDeski Goal**: What is the **TG analog** of **Horndeski theory**?
- **Conditions**: (i) Field equations must be second-order; (ii) terms cannot be parity-violating; (iii) contributions can be at most quadratic in torsion
- **Extra contribution**: $\mathcal{L}_{\text{Tele}} = G_{\text{Tele}}(\phi, X, T, T_{Ax}, T_{vec}, I_2, J_i)$ [I_2 - linear coupling with matter, J_i - quadratic coupling with matter]

arXiv:1904.10791

GWs in TeleDeski

$$\ddot{h}_{ij} + (3+\alpha_M)H\dot{h}_{ij} - (1+\alpha_T)\frac{k^2}{a^2}h_{ij} = 0$$

- **TeleDeski GWPE:**

$$\alpha_T = \frac{2X}{M_*^2}\left(2G_{4,X} - 2G_{5,\phi} - G_{5,X}(\ddot{\phi} - \dot{\phi}H) - 2G_{\text{Tele},J_8} - \frac{1}{2}G_{\text{Tele},J_5}\right) = 0$$

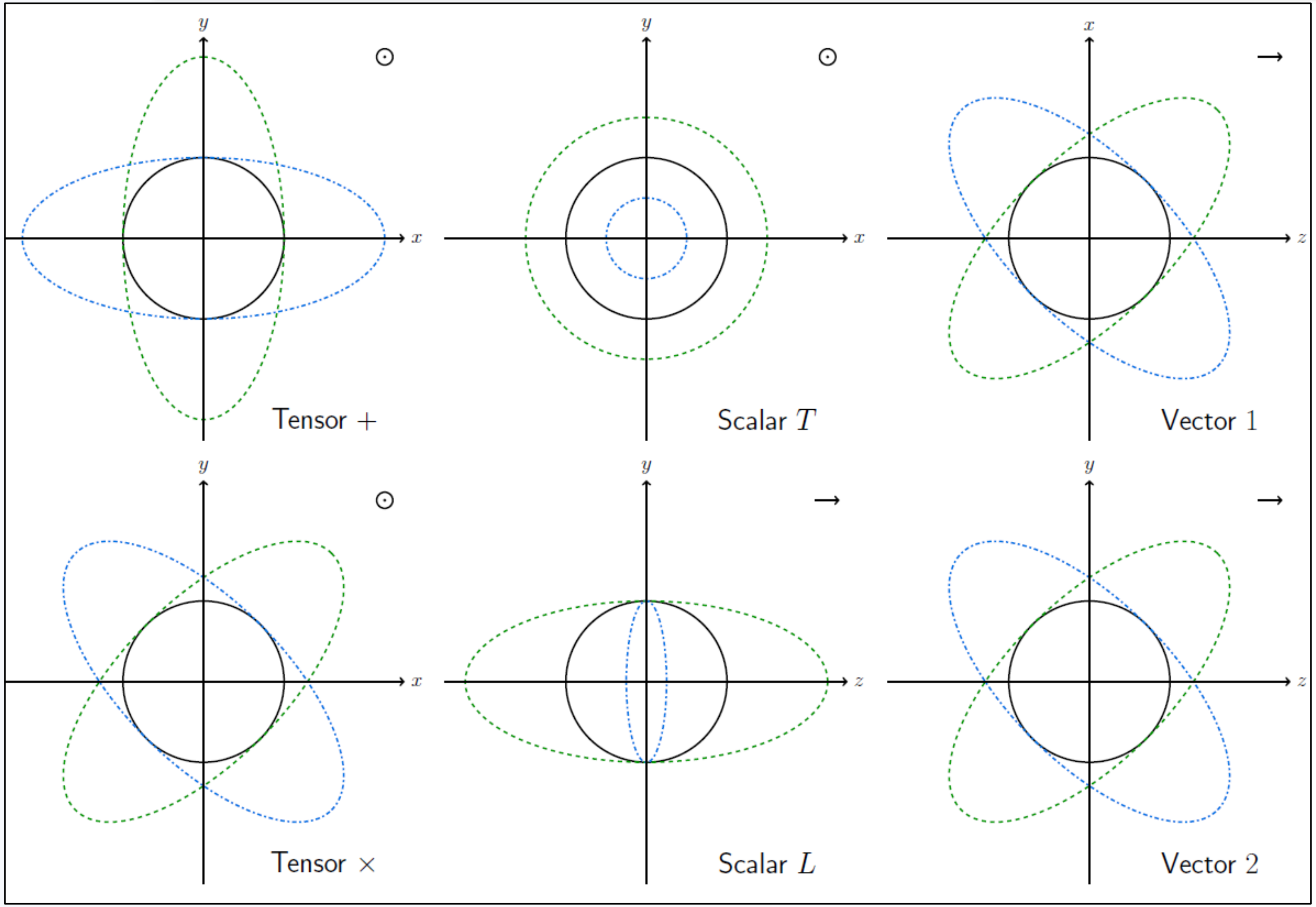
where $M_*^2 = 2\left(G_4 - 2XG_{4,X} + XG_{5,\phi} - \dot{\phi}XHG_{5,X} + 2XG_{\text{Tele},J_8} + \frac{1}{2}XG_{\text{Tele},J_5} - G_{\text{Tele},T}\right)$

- **Running Planck mass**: Continues to observe $\alpha_M = \frac{1}{HM_*^2}\frac{dM_*^2}{dt}$

- **New possibilities**: Opens new possibilities for reviving Horndeski gravity

arXiv:1907.10057

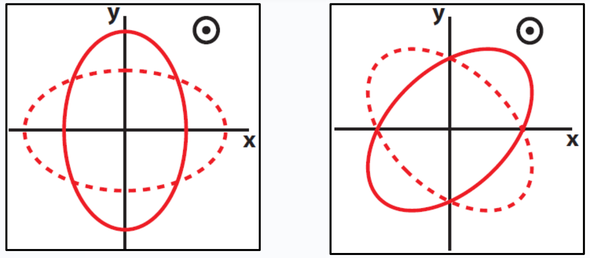
GW Polarization Modes



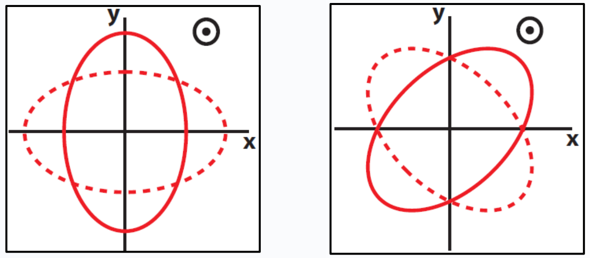
Metric tensor theories have at most **6 polarizations**

TG Polarizations

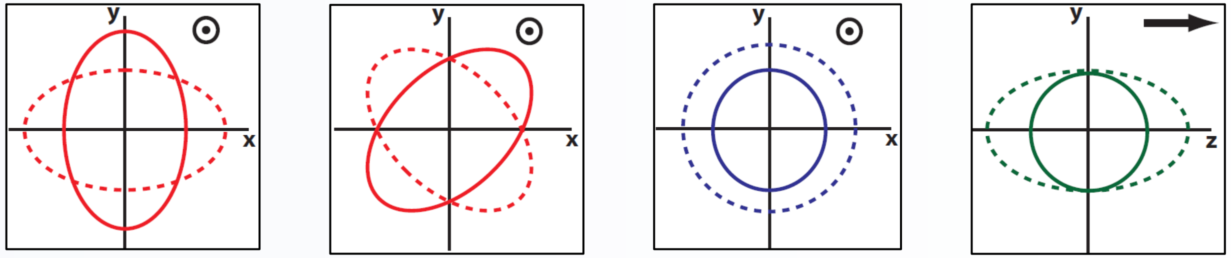
- **GR:**



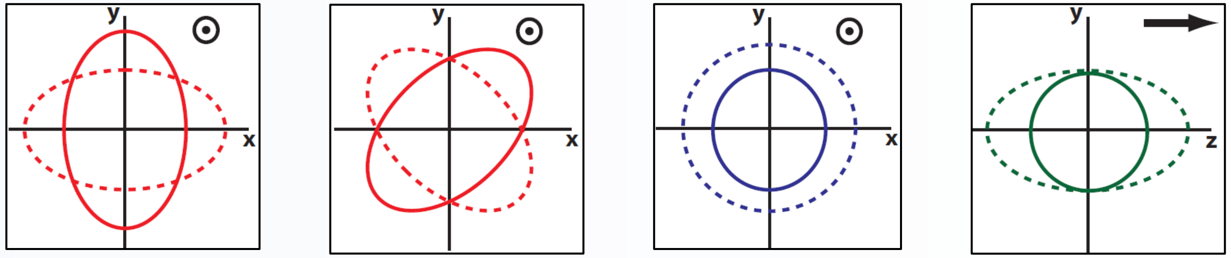
- **$f(T)$ Gravity:**



- **$f(T,B)$ Gravity:**



- **$f(\mathcal{R})$ Gravity:**



Scalar Perturbations in TeleDeski

- **Perturbation:** $e^A{}_\mu = \begin{bmatrix} 1+\psi & 0 \\ 0 & a(1-\phi) \end{bmatrix}$
- **Second order action:**

$$S^{(2)}_{\text{Scalar}} = \int dt \frac{d^3k}{(2\pi)^3}\left[\mathcal{A}\phi^2 + 6\mathcal{D}\psi^2 - 6\mathcal{F}\dot{\psi}^2 - 6\mathcal{G}\phi\dot{\psi} - k^2\left(-\mathcal{B}\phi^2 - 2\mathcal{E}\psi^2 + 4\mathcal{H}\phi\psi\right)\right]$$

where

$$\mathcal{A} = XG_{2,\mathrm{X}} + 2X^2G_{2,\mathrm{XX}} - 2XG_{3,\phi} - 2X^2 + G_{3,\phi\mathrm{X}}XG_{\mathrm{Tele,X}} + 2X^2G_{\mathrm{Tele,XX}}$$

$$\mathcal{B} = G_{\mathrm{Tele,T_{vec}}} + \frac{2}{9}X\left(2G_{\mathrm{Tele,J_8}} + 2XG_{\mathrm{Tele,J_6}} - 5G_{\mathrm{Tele,J_5}} + 3G_{\mathrm{Tele,J_3}}\right)$$

$$\mathcal{C} = -G_{\mathrm{Tele,T_{vec}}} + XG_{\mathrm{Tele,I_2I_2}} + \frac{1}{3}X\left(4G_{\mathrm{Tele,J_8}} + G_{\mathrm{Tele,J_5}}\right)$$

$$\mathcal{D} = \frac{\partial}{\partial t}\left(\dot{\phi}G_{\mathrm{Tele,I_2}}\right)$$

$$\mathcal{E} = G_4 - X\left(G_{5,\phi} - \ddot{\phi}G_{5,X}\right) - G_{\mathrm{Tele,T}} + 2G_{\mathrm{Tele,T_{vec}}} + \frac{1}{9}X\left(-2G_{\mathrm{Tele,J_8}} + 2XG_{\mathrm{Tele,J_6}} - 5G_{\mathrm{Tele,J_5}} - 6G_{\mathrm{Tele,J_3}}\right)$$

$$\mathcal{F} = G_4 - 2XG_{4,X} + XG_{5,\phi} - G_{\mathrm{Tele,T}} + \frac{3}{2}\left(G_{\mathrm{Tele,T_{vec}}} - X\,G_{\mathrm{Tele,I_2I_2}}\right)$$

$$\mathcal{G} = -\dot{\phi}XG_{3,X} + \dot{\phi}G_{4,\phi} + 2\dot{\phi}XG_{4,\phi X} + \frac{1}{2}\dot{\phi}G_{\mathrm{Tele,I_2}} + \dot{\phi}XG_{\mathrm{Tele,XI_2}}$$

$$\mathcal{H} = G_4 - 2XG_{4,X} + XG_{5,\phi} - G_{\mathrm{Tele,T}} + G_{\mathrm{Tele,T_{vec}}} + \frac{1}{9}X\left(2G_{\mathrm{Tele,J_8}} - 2XG_{\mathrm{Tele,J_6}} + 5G_{\mathrm{Tele,J_5}} + \frac{3}{2}G_{\mathrm{Tele,J_3}}\right)$$

arXiv:2301.04457

Stability conditions

- **Ghost Stability condition**: Wrong kinetic sign

$$-\frac{2\mathcal{F}(3\mathcal{C}+2\mathcal{F})}{\mathcal{C}} > 0$$

- **Laplacian Stability Condition**: Positive scalar mode propagation

$$c_s^2 = \frac{-2\mathcal{C}+\mathcal{B}\mathcal{C}\mathcal{E}}{\mathcal{B}\mathcal{F}(3\mathcal{C}+2\mathcal{F})} > 0$$

arXiv:2301.04457

Conclusion

- **TG offers an interesting alternative** to traditional ways to modify gravity
- TG satisfies a number of desired qualities, and offers a more **consistent picture of modified gravity**
- TG can fix problems in **traditional modified gravity**

Thank You