PERTURBATIVE PERSPECTIVES

This starts from the linearized Hilbert-Einstein theory

$$S^{HE}[g] = \frac{1}{\kappa^2}\int d^D x \, \sqrt{-g}\, R(g) \qquad \text{(PE1)}$$

NEWTON

obtained by considering the real potentials to be the fluctuations around flat metric of $g_{\mu\nu}$, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \text{(PE2)}$$

So, the true potentials are the spin-2 field components

$$\mathbb{R}^{1,D-1} \ni x \longmapsto h_{\mu\nu} \in \mathbb{R} \qquad \text{(PE3)}$$

known as PAULI-FIERZ field.

From this perspective the valency-2 tensor $g^{\mu\nu}$ is nothing but a power series in $h_{\mu\nu}$,

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \sum_{j\geq 2} \kappa^j h^{\mu}{}_{\nu_1} h^{\nu_1}{}_{\nu_2} \cdots h^{\nu_{j-1}\nu} (-)^j \qquad \text{(PE4)}$$

Within this perspective (PE1) becomes an interacting field theory for the Pauli - Fierz field, with the full dynamics generated via VP from the local functional

$$S[h] := S^{HE}[g(h)] = S^{PF}[h] + \sum_{\ell \geq 1} \kappa^{\ell} \overset{(\ell)}{S}[h] \qquad (PE5)$$

The first term in (PE5) is the well-known Pauli - Fierz action

$$S^{PF}[h] = \int d^D x \Big(-\tfrac{1}{2}(\partial_\rho h_{\nu\varsigma})^2 + (\partial_\mu h^{\mu\nu})^2 - (\partial_\varsigma h)\partial_\mu h^{\mu\varsigma} + \tfrac{1}{2}(\partial^\mu h)^2 \Big) \qquad (PE6)$$

$h := \eta^{\mu\nu} h_{\mu\nu}$

By its very definition, it is clear that (PE6) is invariant under the linearized version of the diffeomorphism, i.e.,

$$\delta_\varepsilon h_{\mu\nu} = \partial_{(\mu} \varepsilon_{\nu)} \qquad (PE7)$$

As (PE6) is a second order, linear theory, then it is amenable from Dirac quantization prescriptions.

Indeed, the canonical analysis of (PE6) exhibits the aspects below.

a) Definitions of canonical momenta

$$\pi_{\alpha\beta} := \frac{1}{2}\left(\frac{\partial \mathcal{L}^{PF}}{\partial \dot{h}^{\alpha\beta}} + \frac{\partial \mathcal{L}^{PF}}{\partial \dot{h}^{\beta\alpha}} \right)$$

produces the relations

$$\Pi_{\alpha\beta} = -\dot{h}_{\alpha\beta} + \eta_{0(\alpha}\partial^\lambda h_{\beta)\lambda} - \eta_{\alpha\beta}\left(\partial^\rho h_{\rho 0} - \partial_0 h\right) - \tfrac{1}{2}\delta_{0(\alpha}\partial_{\beta)} h$$

which solve the general velocities

$$\dot{h}_{ij} = -\Pi_{ij} + \eta_{ij}\frac{\Pi' + \partial^k \Pi_{0k}}{D-2}, \quad \Pi' := \eta^{ij}\Pi_{ij} \tag{PE8}$$

and exhibits the primary constraints

$$\begin{cases} G^{(1)} := \Pi_{00} - \partial^j h_{j0} \approx 0 \\ G^{(1)}_j := \Pi_{0j} - \partial^k h_{jk} + \frac{1}{2}\partial_j h \approx 0 \end{cases} \tag{PE 9}$$

b) The Legendre transform of L^{PF}, which is nothing but the canonical Hamiltonian, displays the Hamiltonian density

$$\mathcal{H} = -\frac{1}{2}\Pi_{ij}\Pi^{ij} + \frac{1}{2}\frac{\Pi'\Pi'}{D-2} + \frac{\Pi'\partial^j h_{0j}}{D-2}$$

$$+ \frac{D-1}{2(D-2)}\partial^j h_{0j}\,\partial_k h^{0k} + \frac{1}{2}\partial_j h_{\mu\nu}\partial^j h^{\mu\nu}$$

$$- \partial^j h_{j\mu}\,\partial_k h^{k\mu} + \partial^j h\,\partial^l h_{jl} - \frac{1}{2}\partial^j h\,\partial_j h \tag{PE 10}$$

c) The consistency of the primary constraints exhibits the secondary constraints

$$\begin{cases} G^{(2)} := \partial^i \partial^j (h_{ij} - \eta_{ij} h^l) \approx 0 \\ G^{(2)}_j := \partial^l \Pi_{jl} + \partial_j \partial^l h_{0l} \approx 0 \end{cases} \qquad \text{(PE 11)}$$

as $G^{(1)}$ ∧ $G^{(1)}_j$ are Abelian and

$$[G^{(1)}, H] = G^{(2)} \quad , \quad [G^{(1)}_j, H] = G^{(2)}_j \qquad \text{(PE 12)}$$

d) The consistency of the secondary constraints (PE 11) closes the Dirac algorithm as

i) All the constraints are Abelian

ii) $[G^{(2)}, H] = -\partial^j G^{(2)}_j \quad , \quad [G^{(2)}_j, H] = 0$

As we are in the realm of linear theories, the quantization program can be done in two ways:

I By realizing as self-adjoint field operators the classical canonical fields $h^{\alpha\beta}$ ∧ $\Pi_{\alpha\beta}$ and identifying the physical subspace from

$$G^{(a)} |\psi\rangle = G^{(a)}_j |\psi\rangle = 0 \quad , \quad a = \overline{1,2}$$

II By implementing some canonical gauge conditions and quantizing the canonical variables of the reduced phase-space.

At this point, a good choice for the canonical gauge conditions read

$$\begin{cases} \chi^{(1)} := h^{00} \approx 0 \\ \chi^{(1)j} := h^{0j} \approx 0 \\ \chi^{(2)} := h^{1} \approx 0 \\ \chi^{(2)j} := \partial_e h^{je} - \frac{1}{2}\partial^j h \approx 0 \end{cases} \qquad h^{1} := \eta_{ij} h^{ij} \qquad \text{(PE 13)}$$

Putting together the first-class constraints and the gauge fixing conditions, the reduced phase-space can be expressed implicitly as

$$\begin{cases} \tilde{\chi}^{(1)} := h^{00} \approx 0 \\ \tilde{G}^{(1)} := \pi_{00} \approx 0 \\ \tilde{\chi}^{(1)j} := h^{0j} \approx 0 \\ \tilde{G}^{(1)}_{j} := \pi_{0j} \approx 0 \\ \tilde{G}^{(2)} := h^{1} \approx 0 \\ \tilde{\chi}^{(2)} := \pi^{1} \approx 0 \\ \tilde{\chi}^{(2)j} := \partial_e h^{je} \approx 0 \\ \tilde{G}^{(2)}_{j} := \partial^e \pi_{je} \approx 0 \end{cases} \qquad \text{(PE14)}$$

The previous representation of the reduced phase-space displays

$$\{ h^{ij}_{TT} , \pi^{TT}_{ij} \}$$

as one of its parametrization, i.e., the transversal components of h^{ij} and π_{ij}.

Of course, these parts are non-local when expressed in terms of the original fields

By using an argumentation similar to Maxwell context, one obtains that the transversal components, which are subject to

$$\partial^i \phi^{TT}_{ij} = 0 \quad , \quad \nabla^{ij} \phi^{TT}_{ij} = 0$$

are

$$\phi^{TT}_{ij} = \phi_{ij} - \frac{\nabla_{ij}}{D-1} \phi' - \partial_{(i} W_{j)} + \frac{2}{D-1} \nabla_{ij} \partial^k W_k$$

with

$$W_j = \frac{1}{\Delta} \left(\partial^l \phi_{lj} - \frac{1}{D-1} \partial_j \phi' \right) - \frac{D-3}{2(D-2)} \frac{1}{\Delta^2} \partial_j \left(\partial^{kl} \phi_{kl} - \frac{1}{D-1} \Delta \phi' \right)$$

The perturbative expansion of the Hilbert-Einstein theory leads naturally to the following question

What is the most general self-interacting theory among the gauge fields $h_{\mu\nu}$ with the free limit

$$S^{PF}[h] \quad ?$$

To answer to this question first one defines the concept of interacting theory of a given "free" limit.

Let's say we have a free theory $S_0[\phi^\alpha]$, invariant under a generic set of gauge transformations

$$\delta_\varepsilon \phi^\alpha = R^\alpha{}_A \varepsilon^A \quad , \text{i.e.} \quad \frac{\delta S_0}{\delta \phi^\alpha} R^\alpha{}_A = 0 \qquad \text{(PE15)}$$

$\bar{S}_\lambda[\phi^\alpha]$ is said to be an interacting theory associated with the generic "free" one iff

a) it possesses the same number of degrees of freedom

b) $\bar{S}_\lambda[\phi^\alpha] \xrightarrow[\lambda \to 0]{} S_0[\phi^\alpha]$

!! The previous concept is straightforwardly adopted to Hamiltonian description

The standard procedure by which one can construct interacting theories possessing gauge invariances makes use of deformation theory, i.e., one deforms the Lagrangian action

$$S_0[\phi^\alpha] \longrightarrow \bar{S}_\lambda[\phi^\alpha] = S_0[\phi^\alpha] + \lambda \overset{(1)}{S}[\phi^\alpha] + \lambda^2 \overset{(2)}{S}[\phi^\alpha] + \dots$$

and the generating set of gauge transformations

$$\delta_\varepsilon \phi^\alpha \longrightarrow \bar{\delta}_\varepsilon \phi^\alpha := \bar{R}^\alpha{}_A \varepsilon^A = R^\alpha{}_A \varepsilon^A + \lambda \overset{(1)}{R}{}^\alpha{}_A \varepsilon^A + \lambda^2 \overset{(2)}{R}{}^\alpha{}_A \varepsilon^A + \dots$$

such that

$$\frac{\delta\left(S_0 + \lambda \overset{(1)}{S} + \lambda^2 \overset{(2)}{S} + \dots\right)}{\delta \phi^\alpha}\left(R^\alpha{}_A + \lambda \overset{(1)}{R}{}^\alpha{}_A + \lambda^2 \overset{(2)}{R}{}^\alpha{}_A + \dots\right) = 0 \qquad \text{(PE16)}$$

By projecting (PE16) on various orders in the coupling constant then one obtains a tower of equations that governs the interacting theory

$$\left\{\begin{array}{l} \frac{\delta S_0}{\delta \phi^\alpha} R^\alpha{}_A = 0 \\ \frac{\delta \overset{(1)}{S}}{\delta \phi^\alpha} R^\alpha{}_A + \frac{\delta S_0}{\delta \phi^\alpha} \overset{(1)}{R}{}^\alpha{}_A = 0 \\ \vdots \\ \sum_{l=0}^{k} \frac{\delta \overset{(l)}{S}}{\delta \phi^\alpha} \overset{(k-l)}{R}{}^\alpha{}_A = 0 \\ \vdots \end{array}\right. \qquad (PE17)$$

!!! The deformation technique is very different from Utiyama's as here one deals only with gauge symmetries.

The problem of constructing consistent interactions can be elegantly reformulated and solved in the BRST context as the BRST symmetry completely captures the gauge structure of a given theory

$$S_0[\phi^\alpha],\ R^\alpha{}_A \longleftrightarrow s \longleftrightarrow S \ (\text{antifield - antibracket})$$

$$\updownarrow$$

$$\mathcal{H},\ G_A \approx 0 \longleftrightarrow s \longleftrightarrow H_B, \Omega \ (\text{Hamiltonian}$$

So, the problem of constructing consistent interacting theories reduces to a problem of deformation of the BRST objects

$$S \longrightarrow \bar{S} = S + \lambda S_1 + \lambda^2 S_2 + \dots$$

s.t.

$$(\bar{S}, \bar{S}) = 0$$

$$(S,S) = 0$$

$$(S, S_1) = 0 \iff s S_1 = 0 \qquad s \Leftarrow \delta + \gamma$$

$$2(S, S_2) + (S_1, S_1) = 0$$

$$\Updownarrow$$

$$2\, s S_2 + (S_1, S_1) = 0$$

$$\vdots$$

Or, equivalently

$$\Omega \longrightarrow \bar{\Omega} = \Omega + \lambda \Omega_1 + \lambda^2 \Omega_2 + \dots$$

$$H_B \longrightarrow \bar{H}_B = H_B + \lambda H_1 + \lambda H_2 + \dots$$

s.t.

$$[\bar{\Omega}, \bar{\Omega}] = 0$$

$$[\bar{H}_B, \bar{\Omega}] = 0$$

$$\delta + \gamma$$

$$\shortparallel$$

$$[\Omega, \Omega] = 0$$

$$[\Omega, \Omega_1] = 0 \iff s\Omega_1 = 0$$

$$2[\Omega, \Omega_2] + [\Omega_1, \Omega_1] = 0$$

$$[H_B, \Omega] = 0$$

$$[H_B, \Omega_1] + [H_1, \Omega] = 0$$

$$[H_B, \Omega_2] + [H_1, \Omega_1] + [H_2, \Omega] = 0$$

By applying the previous technique of construction consistent interactions to Pauli-Fierz model, one obtains:

i) Hilbert-Einstein theory WITH cosmological constant, if we allow only two spacetime derivatives in the interaction vertices

ii) Lovelock gravity, if we allow more spacetime derivatives in the interaction vertices

iii) No-go results concerning interacting multi-graviton theories

iv) At least standard interaction vertices, whenever one considers the inclusion of "matter" fields