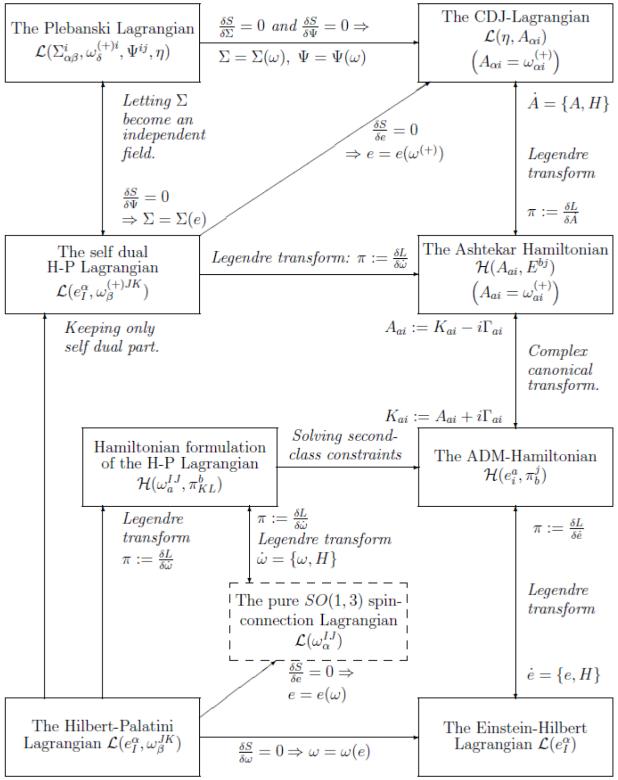
Various Formulations of Gravitational Interaction

E. M. Cioroianu,

Bucharest, 2024

OUTLINE

1. STATE OF THE ART
2. SETTING THE ARENA: LOCAL FIELD THEORIES, GAUGE TRANSFORMATIONS, AUXILIARY VARIABLES
3. SECOND-ORDER FORMULATIONS
   a) HILBERT - EINSTEIN
   b) LOVELOCK
4. FIRST-ORDER FORMULATIONS
   a) PALATINI
   b) HILBERT - PALATINI
   c) ADM
5. HIGHER-ORDER FORMULATIONS
6. PERTURBATIVE PERSPECTIVE
7. REFERENCES



→ EINSTEIN'S TUTOR : The story of Emmy Noether and the invention of modern physics

(L. Phillips, Public Affairs, N.Y., 2024)

2. Setting the arena: Local field theories, Gauge transformations, Auxiliary Variables

Bundle $E \xrightarrow{\pi} M$ usually trivial, $E = A_M := A \times M$

with A – v.s | associative, commutative | graded commutative algebra

field = (local) section in E | ϕ^α

dynamics : generated via variational from a local functional

$$\delta S = 0$$

$$S[\phi^\alpha] = \int_M d^D x \; \mathcal{L}([\phi]_x)$$

$[\phi] = (\phi^\alpha, \phi^\alpha_\mu, \phi^\alpha_{\mu\nu}, \ldots)$

$[\phi]_x = (\phi^\alpha(x), \partial^x_\mu \phi^\alpha, \partial^x_{\mu\nu}\phi^\alpha, \ldots)$

$$\mathcal{L} \in \mathcal{F}(J^\infty E)$$

$\mathcal{F} \equiv \mathcal{C}^\infty$

* If a is a local function, $a \in \mathcal{F}(J^\infty E)$, then

$$\frac{\delta a}{\delta \phi^\alpha} = 0 \iff a = \partial_\mu j^\mu, \quad j^\mu \in \mathcal{F}(J^\infty E)$$

* (Algebraic Poincaré lemma)

If ω is a local horizontal p-form, $0 < p < D$

$$a = \frac{1}{p!} a_{\mu_1 \ldots \mu_p}(x, [\phi]) \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$$

$$d_H a = 0 \iff a = d_H b$$

$$d_H := dx^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} + \phi^\alpha_\mu \frac{\partial}{\partial \phi^\alpha} + \ldots$$

Among the fields, one distinguishes

- dynamical fields $\iff$ EOM $\subset J^k E$, $k \neq 0$
- auxiliary fields $\iff$ EOM $\subset E$

Let $S[z^A, y^\alpha]$ be a field theory with the fields z^A, y^α.

The fields y^α are said to be auxiliary iff can be expressed from their E.O.M. only by algebraic manipulations

$$\frac{\delta S}{\delta y^\alpha} = 0 \iff y^\alpha = y^\alpha([z^A]) \qquad (1)$$

<u>Lemma</u> Auxiliary fields can be removed from theory without affecting the dynamics, i.e.

$$\begin{cases} \dfrac{\delta S}{\delta z^A} = 0 \\ \dfrac{\delta S}{\delta y^\alpha} = 0 \end{cases} \iff \begin{cases} \dfrac{\delta \bar{S}}{\delta z^A} = 0 \\ y^\alpha = y^\alpha([z^A]) \end{cases} \qquad (2)$$

where

$$\bar{S}[z^A] := S[z^A, y^\alpha([z^A])] \qquad (3)$$

A generic field theory exhibits rigid, and could exhibit gauge symmetries.
If one considers the local theory

$$S[\phi^\alpha] = \int d^D x \, \mathcal{L}([\phi]_x) \qquad (4)$$

the infinitesimal transformation on E,

$$\begin{cases} x^\mu \to x'^\mu = x^\mu + \varepsilon X^\mu(x,[\phi]) \\ \phi^\alpha \to \phi'^\alpha = \phi^\alpha + \varepsilon \Phi^\alpha(x,[\phi]) \end{cases} \qquad (5)$$

is said to be an infinitesimal rigid symmetry of (4) iff it leaves (4) invariant, i.e.

$$\mathcal{L}_{\hat{X}}(d^D x \, \mathcal{L}) = d \, j^{[D-1]} \qquad (6)$$

where

$$\hat{X}$$

is the prolongation of

$$X^\mu \frac{\partial}{\partial x^\mu} + \Phi^\alpha \frac{\partial}{\partial \phi^\alpha} \in TE$$

to

$$T(J^\infty E)$$

Another kind of symmetry for a local field theory is represented by gauge symmetry. This can be prescribed independently for each point of the base manifold and leave invariant (4), i.e.

$$\phi^\alpha \to \phi'^\alpha = \phi^\alpha + R^\alpha{}_A \varepsilon^A \qquad (7)$$

local

In (7) one used de Witt condensed notations

$$\begin{cases} \alpha \Leftrightarrow (\alpha, x) \\ f_\Delta \, k^\Delta \Longleftrightarrow \int d^Dx \, f_\Delta(x) \, k^\Delta(x) \end{cases} \qquad (8)$$

Remarks:

a) The existence of a rigid symmetry is equivalent to a conservative current

$$\partial_\mu j^\mu \approx 0$$

$\approx$ means modulo EOM $\Sigma : \frac{\delta S}{\delta \phi^\alpha} = 0$

Σ - THE COVARIANT PHASE-SPACE

b) The existence of a gauge symmetry is equivalent to a redundant description of the covariant phase-space

$$\frac{\delta S}{\delta \phi^\alpha} R^\alpha = 0$$

Auxilliary variables and symmetries

It can be shown that if $\delta_\varepsilon Z^A = Z^A \varepsilon$, $\delta_\varepsilon y^\alpha = Y^\alpha \varepsilon$ are some infinitesimal symmetries (rigid | gauge), i.e.

$$\frac{\delta S}{\delta Z^A} \delta_\varepsilon Z^A + \frac{\delta S}{\delta y^\alpha} \delta_\varepsilon y^\alpha = 0 \qquad (9)$$

then

$$\frac{\delta \bar{S}}{\delta Z^A} \bar{\delta}_\varepsilon Z^A = 0 \quad , \quad \bar{\delta}_\varepsilon Z^A := Z^A\left([Z], y^\alpha = y^\alpha([Z])\right) \qquad (10)$$

Conversely, if (10) holds then they can extended to a rigid | gauge symmetry of the original theory, only by adding trivial (≈ 0) rigid | gauge transformations.

Conclusions

a) Auxilliary variables do not affect the number of independent symmetries) degrees of freedom

b) Do modify the dynamics

e.g.

$$S^L[q] = \int dt \; L(q, \dot{q}) \longleftrightarrow S^H[q,p] = \int dt\left(p\dot{q} - H(q,p)\right)$$

SECOND-ORDER DYNAMICS   FIRST-ORDER DYNAMICS

In the general context considered, a field theory $S[\phi]$ is said to be of k-th order iff

$$\Sigma : \quad \frac{\delta S}{\delta \phi^\alpha} = 0 \qquad , \qquad \begin{array}{l} \Sigma \subset J^k E \\ \Sigma \not\subset J^{k-1} E \end{array}$$

2. SECOND-ORDER FORMULATIONS

The most famous second-order formulation is due to Hilbert and describe gravitational field via the potentials

$$x \longmapsto g_{\mu\nu}(x)$$

with $g \in \Gamma(\vee^2 T^* M)$ a non-degenerate, symmetric, rank-2 tensor

The dynamics comes from variational principle based on

$$S^{HE}[g] = \int d^D x \, \sqrt{-g}\, (R - 2\Lambda) \qquad (11)$$

Action (11) is invariant under reparametrization

$$x^\mu \to x^\mu + \varepsilon X^\mu(x) \qquad , \qquad X = X^\mu(x) \frac{\partial}{\partial x^\mu}$$

as

$$\mathcal{L}_X \left(d^D x \, \sqrt{-g}\, (R - 2\Lambda) \right) = \varepsilon \, d^D x \, \partial_\mu \left(\sqrt{-g}\, (R - 2\Lambda) X^\mu \right) \quad (12)$$

It has been shown

Variational problems involving combined tensor fields

By H. Rund $\boxed{D=4}$

that the most general theory invariant under reparametrizations that depends on metric and its derivatives up to order two is that written in (11)

a) Let $S_{(1)}[q] = \int dt \, L_{(1)}(t, q, \dot{q})$, $q = (q^a)$, be a local functional associated with trivial bundle $M_{\mathbb{R}} := M \times \mathbb{R} \xrightarrow{pr_2} \mathbb{R}$, $L_{(1)} \in \mathcal{F}(J^1 M_{\mathbb{R}})$

Find the necessary and sufficient conditions for time-reparametrization invariance of $S_{(1)}$.

b) Do the same thing for $S_{(k)}[q] = \int dt \, L_{(k)}$, $L_k \in \mathcal{F}(J^k M_{\mathbb{R}})$

By its very definition, (11) is a gauge theory, being invariant under

$$\delta_\varepsilon g_{\mu\nu} := \mathcal{L}_\varepsilon g_{\mu\nu} \quad , \quad \varepsilon := \varepsilon^\mu \partial_\mu$$

$$= g_{\mu\nu,\rho}\, \varepsilon^\rho + g_{\rho(\mu} \partial_{\nu)} \varepsilon^\rho \qquad (13)$$

By direct computation (13) can be expressed as

$$\delta_\varepsilon g_{\mu\nu} = \varepsilon_{(\mu;\nu)} \tag{14}$$

with the covariant derivatives associated with Levi-Civita connection

Show that (14) closes off-shell.

It has been shown that for $\boxed{\underline{\text{any}}}$ density

$$\mathcal{L} \in \mathcal{F}(J^2 V^2 \mathcal{T}^* M) \tag{15}$$

$$(E(\mathcal{L}))^{\mu\nu} := \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \in \mathcal{F}(J^4 V^2 \mathcal{T}^* M) \tag{16}$$

verifies

$$(E(\mathcal{L}))^{\mu\nu}{}_{;\nu} = 0 \tag{17}$$

In view of this one and the transformation law of $\mathcal{L}$ under general coordinate transformations

$$x^\mu \to \bar{x}^\mu = \bar{x}^\mu(x) \tag{18}$$

that imposes, via ordinary derivations, the purely algebraic relations

$$\begin{cases} \Pi^{\mu(\nu|\lambda\rho)} = 0 & (\Rightarrow \Pi^{\mu\nu|\lambda\rho} = \Pi^{\lambda\rho|\mu\nu}) \\ \Pi^{\mu(\nu|\lambda)} = 0 & (\Leftrightarrow \Pi^{\mu\nu|\rho} = 0) \\ \Pi^{\mu\nu} - \frac{1}{3} R^{(\mu}{}_{\sigma\alpha\beta} \Pi^{\nu)\alpha|\beta\sigma} = \frac{1}{2} g^{\mu\nu} \mathcal{L} \end{cases} \qquad (19)$$

where $\Pi^{\mu\nu|\lambda\rho}$, $\Pi^{\mu\nu|\lambda}$, and $\Pi^{\mu\nu}$ are the tensor densities constructed out of the derivatives

$$\Lambda^{\mu\nu|\lambda\rho} := \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\lambda\rho}}, \quad \Lambda^{\mu\nu|\lambda} := \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\lambda}}, \quad \Lambda^{\mu\nu} := \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \qquad (20)$$

Precisely, the following relations take place

$$\begin{cases} \Pi^{\mu\nu|\lambda\rho} = \Lambda^{\mu\nu|\lambda\rho} \qquad (21) \\ \Pi^{\mu\nu|\lambda} = \Lambda^{\mu\nu|\lambda} + \Gamma^{\lambda}_{\rho\sigma} \Lambda^{\mu\nu|\rho\sigma} + 2\, \Gamma^{\mu}_{\rho\sigma} \Lambda^{\nu(\rho|\sigma)\lambda} \\ \Pi^{\mu\nu} = \Lambda^{\mu\nu} + \Gamma^{(\mu}_{\lambda\rho,\sigma} \Lambda^{\nu)\lambda|\rho\sigma} + \Gamma^{(\mu}_{\lambda\rho} \Pi^{\nu)\lambda|\rho} - \Lambda^{\sigma\theta|\rho(\mu} \Gamma^{\nu)}_{\lambda\sigma} \Gamma^{\lambda}_{\rho\theta} \\ \qquad - \Lambda^{\sigma\theta|\rho(\mu} \Gamma^{\nu)}_{\rho\lambda} \Gamma^{\lambda}_{\sigma\theta} - \Gamma^{\mu}_{\rho\lambda} \Gamma^{\nu}_{\sigma\theta} \Lambda^{\rho\sigma|\lambda\theta} \end{cases}$$

In terms of previous tensor densities, the equations of motion become

$$(E(L))^{\mu\nu} \equiv \Pi^{\mu\nu|\lambda\rho}{}_{;\lambda;\rho} + \Pi^{\mu\nu} = 0 \qquad (22)$$

As we mentioned, in $D=4$

$$(E(L))^{\mu\nu} \in \mathcal{F}(J^2 V^2 \tau^* M) \text{ iff } \mathcal{L} \propto \mathcal{L}^{HE} \qquad (23)$$

The last part of this section envisages the generalisation of (23) to an arbitrary space-time dimension, $D \geq 4$.

The answer to this inquiry is given in two steps.

I) Find the general tensor density

$$E^{\mu\nu} \in \mathcal{F}(J^2 V^2 \tau^* M) \qquad (24)$$

that enjoys

$$E^{\mu\nu} = E^{\nu\mu} \quad \wedge \quad E^{\mu\nu}{}_{;\nu} = 0 \qquad (25)$$

II) In what circumstances there exists a scalar density $\mathcal{L} \in \mathcal{F}(J^2 V^2 \tau^* M)$ such that

$$E^{\mu\nu} = (E(\mathcal{L}))^{\mu\nu} \qquad (26)$$

Divergence-Free Tensorial Concomitants

DAVID LOVELOCK[1]) (Bristol, England)

The Einstein Tensor and Its Generalizations*

DAVID LOVELOCK

The first task lifts the result of Rund to tensor densities of valency 2. Precisely, if one uses the notations

$$\left\{\begin{aligned} (\Lambda^{\mu\nu})^{\alpha\beta|\gamma\delta} &:= \frac{\partial E^{\mu\nu}}{\partial g_{\alpha\beta|\gamma\delta}} \\ (\Lambda^{\mu\nu})^{\alpha\beta|\gamma} &:= \frac{\partial E^{\mu\nu}}{\partial g_{\alpha\beta|\gamma}} \\ (\Lambda^{\mu\nu})^{\alpha\beta} &:= \frac{\partial E^{\mu\nu}}{\partial g_{\alpha\beta}} \end{aligned}\right. \qquad (27)$$

then, via definitions similar to (21), one constructs tensor densities

$$\Pi^{\mu\nu\|\alpha\beta|\gamma\delta}, \quad \Pi^{\mu\nu\|\alpha\beta|\gamma}, \quad \Pi^{\mu\nu\|\alpha\beta} \qquad (28)$$

$$\Pi^{\mu\nu\|\alpha\beta|\gamma\delta} = (\Lambda^{\mu\nu})^{\alpha\beta|\gamma\delta} + \ldots, \quad \Pi^{\mu\nu\|\alpha\beta|\gamma} = (\Lambda^{\mu\nu})^{\alpha\beta|\gamma} + \ldots$$

$$\Pi^{\mu\nu\|\alpha\beta} = (\Lambda^{\mu\nu})^{\alpha\beta} + \ldots \qquad (\text{see } (21))$$

Using again the transformation law for the tensor density $E^{\mu\nu}$ under local chart changes

$$x^\mu \to \bar{x}^\mu = \bar{x}^\mu(x)$$

only by using ordinary calculus, one obtains purely algebraic relations similar to (19), i.e.,

$$\left\{\begin{array}{l} \Pi^{\mu\nu\|\alpha(\beta|\gamma\delta)} = 0 \quad \Rightarrow \quad \Pi^{\mu\nu\|\alpha\beta|\gamma\delta} = \Pi^{\mu\nu\|\gamma\delta|\alpha\beta} \\ \Pi^{\mu\nu\|\alpha(\beta|\gamma)} = 0 \quad \Leftrightarrow \quad \Pi^{\mu\nu\|\alpha\beta|\gamma} = 0 \\ \Pi^{\mu\nu\|\alpha\beta} - \frac{1}{3}\left(R^\alpha{}_{\rho\lambda\sigma}\,\Pi^{\mu\nu\|\beta\lambda|\sigma\rho} + R^\beta{}_{\rho\lambda\sigma}\,\Pi^{\mu\nu\|\alpha\lambda|\sigma\rho}\right) \\ \qquad = \frac{1}{2} g^{\alpha\beta} E^{\mu\nu} - \frac{1}{2} g^{\mu(\alpha} E^{\beta)\nu} - \frac{1}{2} g^{\nu(\alpha} E^{\beta)\mu} \end{array}\right. \qquad (29)$$

Using definition of covariant derivatives out of metric connection, definitions (27) and results (29) it has been proved the equivalence

$$E^{\mu\nu}{}_{;\nu} = 0 \quad \Longleftrightarrow \quad \Pi^{\mu\nu\|\alpha\beta|\lambda\rho} + \Pi^{\mu\lambda\|\alpha\beta|\rho\nu} + \Pi^{\mu\rho\|\alpha\beta|\nu\lambda} = 0 \quad (30)$$

Relations (30) play the role of integrability conditions for $E^{\mu\nu}$. These are used to integrate (24), as follow

First, from (29) and (30), it results that tensor densities

$$\Pi^{\mu\nu\|\alpha\beta|\lambda\rho}$$

is completely symmetric in the pairs

$$(\mu\nu),\ (\alpha\beta)\ \wedge\ (\lambda\rho)$$

One defines the family of tensor densities

$$\left\{ \Pi^{\mu\nu\|\mu_1\mu_2|\mu_3\mu_4;\mu_5\mu_6|\mu_7\mu_8\cdots\mu_{4k-3}\mu_{4k-2}|\mu_{4k-1}\mu_{4k}} \ :\ k \geq 0 \right\} \quad (31)$$

$$\Pi^{\mu\nu\|\mu_1\mu_2|\mu_3\mu_4\cdots\mu_{4k+1}\mu_{4k+2}|\mu_{4k+3}\mu_{4k+4}} :=$$

$$:= \frac{\partial \Pi^{\mu\nu\|\mu_1\mu_2|\mu_3\mu_4\cdots\mu_{4k-3}\mu_{4k-2}|\mu_{4k-1}\mu_{4k}}}{\partial g_{\mu_{4k+1}\mu_{4k+2}|\mu_{4k+3}\mu_{4k+4}}} \quad (32)$$

The previous definitions show that tensor densities (31) enjoy

a) are symmetric in any pair of indices

$$(\mu_{2l+1}\mu_{2l})$$

b) are symmetric at the permutation

$$(\mu\nu) \leftrightarrow (\mu_{2l+1}\mu_{2l})$$

c) verify a cyclic identity in

$$(\nu, \mu_{2l+1}, \mu_{2l})$$

These properties show that the components of the tensor densities (31) vanish whenever at least three of their indices coincide. This further implies that (31) contains at most

$$m_D := \left[\frac{D+1}{2} \right]$$

elements

By integrating successively (32) one obtains

$\boxed{!!\, D=4}$

$$E^{\mu\nu} = \sum_{k=0}^{m_D - 1} c_k \, \Pi^{\mu\nu \| \mu_1 \mu_2 | \mu_3 \mu_4 ; \mu_5 \mu_6 | \mu_7 \mu_8 \cdots \mu_{4k-3} \mu_{4k-2} | \mu_{4k-1} \mu_{4k}}$$

$$\times \prod_{l=1}^{k} R_{\mu_{4l-3} \mu_{4l-2} \mu_{4l-1} \mu_{4l}} \tag{33}$$

with

$$c_k \in \mathbb{R}$$

and

$$\Pi^{\mu\nu \| \mu_1 \mu_2 | \mu_3 \mu_4 ; \mu_5 \mu_6 | \mu_7 \mu_8 \cdots \mu_{4k-3} \mu_{4k-2} | \mu_{4k-1} \mu_{4k}} \in \mathcal{F}\left(\vee^2 T^* M \right)$$

tensor densities of valence $4k+2$

Using an iterative formula, it has been shown that the general solution to (25) reads

$$E^{\mu\nu} = \sqrt{-g}\sum_{k=0}^{m_D-1} a_k\, g^{\tau(\mu}\delta^{\nu)}_{[\tau}\,\delta^{\lambda_1}_{\rho_1}\cdots\delta^{\lambda_{2k}}_{\rho_{2k}]}\prod_{l=1}^{k} R_{\lambda_{2l-1}\lambda_{2l}}{}^{\rho_{2l-1}\rho_{2l}} \qquad (34)$$

$$a_k \in \mathbb{R}$$

Returning to the second question, i.e., the existence of a scalar density $\mathcal{L}$ such that (34) are the variational derivatives of $\mathcal{L}$

$$E^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} \qquad (35)$$

from (34) one results that

$$\mathcal{L} = 2\sqrt{-g}\sum_{k=0}^{m_D-1} a_k\,\delta^{\lambda_1}_{[\rho_1}\cdots\delta^{\lambda_{2k}}_{\rho_{2k}]}\prod_{l=1}^{k} R_{\lambda_{2l-1}\lambda_{2l}}{}^{\rho_{2l-1}\rho_{2l}} \qquad (36)$$

Of course (36) is not unique.

FIRST - ORDER FORMULATIONS

According to general definition, these take into account evolution generated from VPs based functionals involving first-order jet bundles

a) PALATINI

The source of this formulation comes from Hilbert - Einstein

$$\mathcal{L}^{HE} = \sqrt{-g}\,(R-2\Lambda) = \sqrt{-g}\, g^{\mu\nu}\left(R_{\mu\nu} - \frac{2}{D}\Lambda g_{\mu\nu}\right)$$

$$= \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\lambda}{}_{\mu[\nu,\lambda]} + \Gamma^{\rho}{}_{\mu[\nu}\Gamma^{\lambda}{}_{\lambda]\rho} - \frac{2\Lambda}{D} g_{\mu\nu}\right) \quad (P1)$$

which is invariant under the infinitesimal gauge transformations

$$\delta_\varepsilon g_{\mu\nu} = \varepsilon_{(\mu;\nu)} \quad (P2)$$

In the light of the concrete expression of metric connection coefficients,

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left(g_{\mu(\nu,\lambda)} - g_{\nu\lambda,\mu}\right), \quad \Gamma^{\mu}{}_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda} \quad (P3)$$

it results that

$$\begin{cases} \delta_\varepsilon \Gamma_{\mu\nu\lambda} = \varepsilon_{\mu,\nu\lambda} - \partial_{(\nu}\left(\Gamma^{\rho}{}_{\lambda)\mu}\varepsilon_\rho\right) + \left(\Gamma^{\rho}{}_{\nu\lambda}\varepsilon_\rho\right)_{,\mu} \\ \delta_\varepsilon \Gamma^{\mu}{}_{\nu\lambda} = \varepsilon^{\mu}{}_{,\nu\lambda} - \Gamma^{\rho}{}_{\nu\lambda}\varepsilon^{\mu}{}_{,\rho} + \Gamma^{\mu}{}_{\rho(\nu}\varepsilon^{\rho}{}_{,\lambda)} + \Gamma^{\mu}{}_{\nu\lambda,\rho}\varepsilon^{\rho} \end{cases} \quad (P4)$$

In Palatini perspective, the field spectrum is

$$\hat{g}^{\mu\nu} := F_g\, g^{\mu\nu} \,, \quad \Gamma^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{\lambda\nu}$$

$$S^P[\hat{g}, \Gamma] := \int d^Dx \quad \hat{g}^{\mu\nu}\left(R_{\mu\nu}(\Gamma) - \frac{2\Lambda}{D} g_{\mu\nu}(\hat{g})\right) \qquad (P5)$$

where $g_{\mu\nu}(\hat{g})$ are then rational functions in $\hat{g}$ that verify

$$g_{\mu\nu}(\hat{g})\, \hat{g}^{\nu\rho} = \delta^{\rho}_{\nu} \left(-\det \hat{g}\right)^{1/D-2} \qquad (P6)$$

By construction, (P5) is invariant under generating set of gauge transf.

$$\begin{cases} \delta_\varepsilon \hat{g}^{\mu\nu} = \varepsilon^{\rho}{}_{;\rho}\, \hat{g}^{\mu\nu} - \hat{g}^{\rho(\mu}\, \varepsilon^{\nu)}{}_{;\rho} \\ \delta_\varepsilon \Gamma^{\mu}{}_{\nu\lambda} = \varepsilon^{\mu}{}_{,\nu\lambda} - \Gamma^{\rho}{}_{\nu\lambda}\, \varepsilon^{\mu}{}_{,\rho} + \Gamma^{\mu}{}_{\rho(\nu}\, \varepsilon^{\rho}{}_{,\lambda)} + \Gamma^{\mu}{}_{\nu\lambda,\rho}\, \varepsilon^{\rho} \end{cases} \qquad (P7)$$

!! The previous covariant derivatives involve only the independent fields $\Gamma^{\mu}{}_{\nu\rho}$.

<u>Remark</u>

In (P5) $\hat{g}^{\mu\nu}$ or $\Gamma^{\mu}{}_{\nu\lambda}$ can be conceived as auxiliary variables, as their EOM can be solved purely algebraic

Indeed, by direct computation one obtains

$$\frac{\delta S^P}{\delta \Gamma^\mu{}_{\alpha\beta}} = \hat{g}^{\alpha\beta}\, \Gamma^\rho{}_{\rho\mu} + \frac{1}{2}\hat{g}^{\rho\lambda}\, \Gamma^{(\alpha}{}_{\rho\lambda}\, \delta^{\beta)}_\mu$$

$$- \hat{g}^{\rho(\alpha}\, \Gamma^{\beta)}{}_{\rho\mu} - \partial_\mu \hat{g}^{\alpha\beta} + \frac{1}{2}\partial_\rho\left(\hat{g}^{\rho(\alpha}\, \delta^{\beta)}_\mu\right) = 0 \qquad (D8)$$

By contracting (D8) with $\delta^\mu{}_\beta$, it produces

$$\hat{g}^{\rho\lambda}\, \Gamma^\alpha{}_{\rho\lambda} = -\partial_\mu \hat{g}^{\mu\alpha}$$

and further

$$\hat{g}^{\rho(\alpha}\, \Gamma^{\beta)}{}_{\rho\mu} = -\partial_\mu \hat{g}^{\alpha\beta} + \frac{1}{D-2}\, \hat{g}_{\alpha\beta}\, \partial_\mu \hat{g}^{\alpha\beta} \qquad (D9)$$

To connect it with Hilbert-Einstein formulation, one returns to metric field as independent variable in terms of which (D9) becomes

$$\Gamma_{(\rho\lambda)\mu} = g_{\rho\lambda,\mu} \qquad (D10)$$

This means that

$$\bar{S}^P[g] := S^P[g, \Gamma(g)]$$

$$\equiv S^{HE}[g]$$

From (P5) one can eliminate $g^{\mu\nu}$ on theirs EOM

$$\frac{\delta S^P}{\delta g^{\mu\nu}} \equiv \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}\right) = 0 \qquad (P11)$$

with $g_{\mu\nu} = g_{\mu\nu}(g^{\alpha\beta})$ rational functions that enjoy

$$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$$

Solving (P11) w.r.t $g^{\mu\nu}$, one obtains

$$S^{S}[\Gamma] \equiv \bar{S}^P[g(\Gamma),\Gamma]$$

$$= \int d^D x \, (-) \frac{4\Lambda(D-1)}{D-2} \left((-)^{D-1} \frac{D-2}{\Lambda} \det\left(R_{\mu\nu}(\Gamma)\right)\right)^{1/2} \qquad (P12)$$

which is due to Schrödinger

b) HILBERT - PALATINI FORMULATION

Here, the bundle is the frame bundle associated with a manifold M, FM

$$\phi^\alpha \to e^a{}_\mu \,,\quad \omega_{\mu ab} = -\omega_{\mu ba}$$

$$e^a \,,\ \omega_{ab} \in \Lambda^1 M$$

$$e^a{}_\mu \eta_{ab} e^b{}_\nu = g_{\mu\nu} \,,\qquad \eta_{ab} = \mathrm{diag}(-,+,\dots,+) \qquad (HP1)$$

With these ingredients at hand, one introduces

$$\begin{cases} T^a := De^a \equiv de^a + \omega^a{}_b \wedge e^b \\ R^a{}_b := D\omega^a{}_b \equiv d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \end{cases} \tag{HP2}$$

For any $\phi = (\phi^a{}_b) \in \mathcal{F}(M; SO(1,D-1))$

$$e^a \to e'^a = \phi^a{}_b \, e^b \tag{HP3}$$

preserves (HP1)

Asking for T^a to transform as previously, one obtains the transformation law for affine spin-connection

$$\omega^a{}_b \to \omega'^a{}_b = \phi^a{}_c \left(\omega^c{}_d \, \bar{\phi}^d{}_b + d\bar{\phi}^c{}_b \right) \tag{HP4}$$

Putting together (HP3) and (HP4), by direct computation one obtains the transformation law

$$R^a{}_b \to R'^a{}_b = \phi^a{}_m \, R^m{}_n \, \bar{\phi}^n{}_b \tag{HP5}$$

Putting all these together with

$$\phi^{a_1}{}_{b_1} \cdots \phi^{a_D}{}_{b_D} \, \varepsilon_{a_1 \dots a_D} = \varepsilon_{b_1 \dots b_D}$$

it results that the D-form

$$\overset{[D]}{\lambda} := R^{ab}([\omega], e) \wedge *(e_a \wedge e_b) \qquad (HP6)$$

is invariant under (HP3)-(HP4)

At the same time, as (HP6) involves only geometric objects, it results that

$$\mathcal{L}_X \overset{[D]}{\lambda} = d\left(L_X \overset{[D]}{\lambda}\right) \qquad (HP7)$$

Infinitesimal version of (HP3) - (HP4) is

$$\begin{cases} \delta_\theta e^a_\mu = \theta^a{}_b e^b{}_\mu \\ \delta_\theta \omega^{ab}_\mu = \theta^a{}_m \omega_\mu{}^{mb} - \omega_\mu^{am} \theta_m{}^b - \partial_\mu \theta^{ab} \end{cases} \qquad \theta \in \mathcal{F}(M, so(1, D-1)) \quad (HP8)$$

while the infinitesimal version of diffeomorphisms reads

$$\begin{cases} \delta_\varepsilon e^a{}_\mu = \varepsilon^\rho \partial_\rho e^a{}_\mu + e^a{}_\rho \partial_\mu \varepsilon^\rho \\ \delta_\varepsilon \omega_\mu{}^{ab} = \varepsilon^\rho \partial_\rho \omega^{ab}_\mu + \omega_\rho^{ab} \partial_\mu \varepsilon^\rho \end{cases} \qquad (HP9)$$

By performing the integration of (HP6) one displays the Lagrangian action

$$S^{HP}[e,\omega] = -\int d^Dx \; e\left[e_a{}^\mu e_b{}^\nu\left(\partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_\mu{}^a{}_c\,\omega_\nu{}^{cb} - \omega^a_{\nu c}\,\omega_\mu{}^{cb}\right) - 2\Lambda\right] \qquad \text{(HP10)}$$

with $\quad e^a{}_\mu\, e_b{}^\mu = \delta^a_b \quad , \quad e = \det e^a_\mu$.

By construction (HP10) is manifestly gauge invariant under (HP8)-(HP9)

In addition, in (HP10) ω / $e^a{}_\mu$ can play the role of auxiliary fields as $\Gamma^\lambda_{\mu\nu}$ / $g_{\mu\nu}$ do.

Indeed, by direct computation one obtains

$$\frac{\delta S^{HP}}{\delta\omega_{\mu ab}} \equiv \partial_\nu\left(e\left(e_{a\mu}\, e_b{}^\nu - e_a{}^\nu e_{b\mu}\right)\right) - e\left(e_{a\mu}\, e^{m\nu} - e_a{}^\nu e^m{}_\mu\right)\omega_{\nu bm}$$

$$+\, e\left(e_{b\mu}\, e^{m\nu} - e_a{}^\nu e^m{}_\mu\right)\omega_{\nu am} = 0 \qquad \text{(HP11)}$$

To solve (HP11) w.r.t $\omega_{\mu ab}$, one first "homogenize" the indices of affine spin connection

$$\omega_{a|bc} := e_a{}^\mu\, \omega_{\mu|bc} \qquad \text{(HP12)}$$

The procedure casts (HP11) into

$$e\left(e^m{}_\mu\, \omega_{[a|b]m} + \omega_{[a}\, e_{b]\mu}\right) + \partial_\nu\left(e\left(e_{a\mu}\, e_b{}^\nu - e_a{}^\nu\, e_{b\mu}\right)\right) = 0$$

with

$$\omega_a := \eta^{pq}\, \omega_{p|aq}$$

$$\Leftrightarrow \quad \omega_{[a|b]c} - \eta_{c[a}\, \omega_{b]} = -e^{-1} e_c{}^\mu\, \partial_\nu\left(e\left(e_{a\mu}\, e_b{}^\nu - e_a{}^\nu\, e_{b\mu}\right)\right)$$

$$\Downarrow$$

$$\omega_b = e_b{}^\nu\, e_m{}^\rho\, \partial_\nu e^m{}_\rho$$

$$\Downarrow$$

$$\Omega_{abc} := \omega_{[a|b]c} = -e_{[a}{}^\nu\, e_{b]\mu}\, \partial_\nu e_c{}^\mu \qquad \text{(HP13)}$$

$$\Downarrow$$

$$\omega_{a|bc} = \frac{1}{2}\left(\Omega_{abc} - \Omega_{bca} + \Omega_{cab}\right) \qquad \text{(HP14)}$$

$$\Updownarrow$$

$$T^a = 0$$

By direct computation it can check

At this point one can conclude that

$$\bar{S}^{HP}[e] = \int d^D x \; e \left(R(\omega(e)) - 2\Lambda \right) = S^{H}[g(e)] \qquad (HP15)$$

To cover the full HE action, the gauge parameters should be fixed via a partial gauge-fixing condition

$$\eta_{a[\mu} e^{a}{}_{\nu]} = 0 \qquad (HP16)$$