CONTACT MOTES MAGURELE 2024

Notes for the mini course in Bucharest

This minicourse is supposed to be about contact geometry , which is ^a part of dassical differential geometry
Contact structure is a typical example of $\widehat{I^{\dagger}}$ is a very ourse is supposed to be abaet constact geometry, which is a part of dassical differential geometry. Sometry to
down to Earth" subject at least in mudtwemerical context. Contact structure is a typical example of a geometric structure, which usually consists of a manifold together with some distinguished tensor field geometric structure, which usually counts of a manifold together with some distinguished tensor field
c.g. Riemannian nuanifold, Poisson manifold or a nuaeufold with rome operations as a Lie group or geometric structure, which usually cousists of a manifold together with some distinguished tensor field
c.g. Riemannian manifold, Poisson manifold or a maen fold with some operations as a Lie group of ^a special kind

Before we give a formual definition, let us look at few canonical examples of a contact manifold:

1) Let us take any nuomifold Q and consider functions on Q as section of the trivial bundle pr. QXIR->Q First jet of such section at point $q \in \mathbb{Q}$ consists of the differential df(q) and value $f(q)$. In another First jet of ruce section of point $q \in \alpha$ warris of the differential affy, and value $f(q)$. In and
words $J^a(Qx\vert R) \simeq T^*Q \times R$. It is again a bundle over Q with the distinguished set of sections being prolongation of functions , i \overline{H} is again a bundle over Q with the distinguished set of sections being
i.e. sections of the form $q \mapsto (d f(q), f(q))$. Let C denote the distribution proloniquition of funichiais, i.e. sechons of the form q \mapsto (df(q), f(q))⁸ Let E denote the distribution
spanned by vectors tamoent to graphs of prolongations. Let us use coordinates to see what kind of
vectors they a vectors they are and what distribution they span. In T*QxR we will use coordinates (q'ipi,z). Jet of a
function (q')+>f(q') is given by function $(q^{\nu}) \mapsto (q^{\nu})$ is given by prodo agabion of functions, i.e. sections of the form $q \mapsto (d(f), f(q))$ is determined set of sections being
sprodo agabion of functions, i.e. sections of the form $q \mapsto (d(f), f(q))$ let C denote the distribution
squared by recto

 $q^i \longmapsto (q^i, \frac{\partial f}{\partial q^i}, f(q^i))$ then Dal
Pi liting from ^a to TQXID We put ↑

 $X = (p, z)$ dim $\mathcal{C}_X = 2n$ (dim Q = n)

 $dim \mathcal{T}^*\mathcal{Q} \times \mathcal{R} = 2n + 1$

As for the properties of E, dim $T^*QxR = 2n + 1$
As for the properties of C, let us note that $\begin{bmatrix} \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{bmatrix} = -\frac{2}{3}$ of C therefore C is not involutive,
as a nualter of fact it is the very apposite of involutive, but it this i

Before ne pass to the second example, let us notice that $\zeta =$ Kerry for $y \in \Omega^4(M)$ y
Le form fin for $f \neq 0$ even 25 a meatter of fact it is the very opposite of involutive, but thus is something we will dissense in a whole
Before the pass to the second example, let us notice that $\epsilon = \ker \eta$ for $\eta \in \Omega^1(M)$ $\eta = \rho_i dq^i - dz$.
The form $\$ This means that y ⁼ O-dz is not only globally defined, but in this case, also distinguished.

2) The second example is in some sense simmilar to the first one: it is also about the Lets of some sections, but there is one very important difference with respect to the first example.

Let B denote the Möbins band with the infinite vertical direction" in a sense that $B = IR^2/\mathbb{Z}$ where
Z acts on \mathbb{R}^2 according to the formula $k \cdot (x,y) = (x + 2k\pi, (-1)^k y)$ action is linear in y in a seuse that $B = \frac{R^2}{z}$ is
, $\left(\frac{-1}{x}\right)^k y$ action is linear in y

The linearity of action in the second argument means that ^B < St is ^a vector (live) bundle. Probably The linecarity of action in the second argument means that $B \longrightarrow S^4$ is a vector (line) bundle. Probably
everybody tecows that B as a nuarregolot is not orientable. Again we consider $M = J^4(B)$ i.e. jets
of sections. In c .
ج $\frac{B}{2}$ au
 $\frac{2}{3}$ + of action in the second avgument
us that B as a nearer folot is
In coordinate then it will all
 B (q, p, z) are coordinates in
 $p\frac{2}{2}, \frac{2}{2p} >$ dim $C_x = 2$ dim $M =$ lv:
3.

 $C = \langle \frac{2}{3q}, p\frac{2}{3p} \rangle$ dian $C_x = 2$ dian $M - 3$.
To see the difference between this example and the previous one, let us go deeper into the structure. The Mobius band B can be described by the coordinate domains together with appropriate transition.
maps. The same we can say about J1B. Pictures are about B, but we think about J1B.

Using coordinates we can corrie one forms as in example 1. In θ we have $\theta_0 =$
in θ we have η_u = $dz' - p'dq'$. Using coordinate transformation we get that on $\theta_0 =$
we can write
in ε : $\eta_u = \eta_0$ but in $\overline{\theta}$ \sim dz - pdq., Issino coordinates
in A we have η_u = $dz' - p' dq'$. Using coordinate transformation we get that on $\theta - \frac{p}{q}$ $\frac{u_{\text{single coordinate}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{\sqrt{\frac{u_{\text{time}}}{$ \vdots $\gamma_u = \gamma_\theta$ but in $\overline{\tau}$: $\gamma_u = -\gamma_\phi$ $\gamma_u = dz' - p' dq' = -dz + pdq$ η o = dz - pd \emph{q} Simce both forms are nouvanishing and have the raine keomed they have to differ by
multiplication by non-vanishing function. This function must be 1 on ε and -1 on F, moreover JB is
connected. Soucern function that has somewhere on the m-vanishino function. This function must be 1.
Lenction that has value 1 at some point and -1.
was the distribution \mathscr{P} is well defined an the n where on the way. He have the distribution of the some point and -1 on another point has to assume value to
somewhere on the way. He have the reached a contradiction. sonnewhere on the way.
The conclusion is that the
globally defined one form. 3) The fluird example, or rather class of examples is IPT*Q, i.e. projectivized cotangent bundle. Let us first take Q ⁼ Fund Excurple, or
IR, n , for simplicity. $M = (T * 12^n)^x$ $\frac{e\times\infty}{i}$ n \times \Big/ \sim $\Big/$ $\frac{1}{100}$ x $(\mathbb{R}^n \cdot {\mathbb{O}}^1_Y)/\sim$ where $(q_i, p_j) \sim (q_i, \lambda p_j)$ for some $\lambda * 0$. Now we will define C on M woing local contact forms: Let u_k \in M denote an open subset of M given by $u_k = \{\text{[}(q^i, p_j)] : p_k \neq 0\} \subset M$ We have of course $\bigcup_{k=1}^{m} u_k = m$ \bot_n u_k we can introduce local $w_{k} = 1 \cup (q^{l}, p_{j}) \cup p_{k} \neq 0 \subseteq M$ We have of wurse
wordinates $(q^{l}, \overline{w}_{k}^{k}, \overline{w}_{k+1}^{k}, \overline{w}_{k+1}^{k}, \overline{w}_{k}^{k})$ wheve $q^{l}(\Gamma(q, p) \overline{d}) =$ $u_{\kappa} = \left(\sum (q^{i}, p_{j}) \right) p_{\kappa} + o \right) \leq M$ We have of course $\bigcup_{k=1}^{n} u_{k} = M$. In u_{k} we can introduce local
coordinates $(q^{i}, \overline{u}_{k}^{k}, \overline{v}_{k}^{k}, \overline{v}_{k}^{k}, \dots, \overline{v}_{k}^{k})$ where $q^{i} \left(\sum (q, p) \right) = q^{i}$ $\frac{(k)}{\sqrt{i}}$ $\left(q^L, \overline{w}^K_{k+1}, \overline{w}^K_{k+1}, \dots, \overline{w}^K_{k+1}, \dots, \overline{w}^K_{k+1}\right)$ where $q^L(\lfloor (q,p) \rfloor) = q^L$ $\overline{w}^L_{k}(\lfloor (q,p) \rfloor) = \frac{Pe}{P_k}$. In these coordinates
 $\begin{array}{ccc}\n\frac{1}{P_k} & \frac{1}{P_k} & \frac{1}{P_k} & \frac{1}{P_k} & \frac{1}{P_k}\n\end{array}$
 $\begin{array}{$ $Q^2(L)$
On $u_k \wedge u_k$
in part \overline{u} " reads oordinatos

se define

form definition Hence contribute m is an interesting the set $\frac{1}{2}$ is $\frac{1}{2}$ if $\frac{1}{2}$ it $\frac{1}{2}$ is $\frac{1}{2}$ if $\frac{1$ vanish on UK For $\lim_{n \to \infty} \frac{1}{n} \int_{0}^{1} f(x) dx$ when ℓ for some point and -1 at an another point had to assume value 0

the distribution ℓ^2 is not defined a contraction.

The distribution ℓ^2 is not defined on the whole 3 The coording
 $i = \frac{1}{k}$
 $k = \frac{1}{\frac{k}}$
 $\frac{k}{\sqrt{1}}$
 $\frac{k}{\sqrt{1}}$
 $\frac{1}{k}$
 $\frac{1}{k}$

add never or more

add never or more
 $\frac{1}{k}$ (b) $\omega = dq^{\ell} + \sum_{i \neq k} \frac{L}{n_i} dq^i = dq^{\ell} + \sum_{\substack{k \neq k \\ k \neq k}} \frac{L}{n_i} dq^i + \frac{A}{n_i} dq^k = \frac{A}{n_i} (dq^k - \sum_{\substack{k \neq k \\ k \neq k}} \frac{(k)}{n_i} dq^i + \frac{C}{n_i} dq^{\ell} - \frac{A}{n_i} q^{\ell}$

We have got $\frac{u}{\eta} - \frac{1}{\frac{1}{u}u} \frac{1}{\eta} \frac{1}{u}$ on $u_k \cdot u_k$ where $\frac{1}{u} \frac{1}{\ell} \neq 0$.
Sefine a distribution of dimension $2n-2$ at each point Forms " and " have the same kernel. Globally they

define a distribution of dimension an-2 at each point of the nuamifold M of dimension In-1. This distribution
is another example of a concract distribution.
The same can be done for any manifold Q replacing Rⁿ, i.e. we c We have got $\eta = \frac{1}{\sqrt{16}} \eta$ on u_{k} u_{ℓ} where v_{ℓ} +0. To muss η and η have the same kerned.
define a distribution of dimension $4n-2$ at each point of the nuamifold M of dimension $2n-2$.
The same can b Q = For the low dimensional example we can take Q-s². Then every fiber of the projectivized tangent
bundle is a circle. He have then the buide of arcles over s². It will be air task for the futorial to
show that in fact
show that in fact oual example we can take Q-3². Then every fiber of the projectivized the have then the build of uncles over s^2 . It will be air task for the $\mathbb{P}T^*S^2 \simeq S^3$ and what the get heve is a contact structure on the to

 \mathbf{V}

We hopf fibration $S^3 \longrightarrow S^2$
We have seen three examples of the pair (M, \mathcal{Z}) . Each time we had a manifold of odd dimension
and a distribution of codimension one i.e. in particular of even dimension. At least in the fir We have seen three examples of the pair (M, \mathcal{Z}) . Each time we had a manifold of odd dimension
and a distribution of codimension one i.e. in particular of even dimension. At least in two first
cases we have checked that We did not check it in the third example , but the situation is the same . In one of the examples we have checked that the distribution cannot be described as the kermel of a global one-form. Local one forms with this property of course always
exist.

Now is the time for the definition of a contact structure

DEFINITION : A manifold ^M together with ^a regular distribution 2 of colimension 1 which is Now is the time for the definition of a contact structure
DEFINITION: A manifold M together with a regular distri
maximually non-integrable is called a contact meanifold.

varinually non-integrable is called a contact rueaufold.
I marimal non-integrability is in some seuse the oposite of integrability. Our distribution ε is of
paimension 4, therefore taking the quotient TM/c we obtain on maximal non-integrability is in some seuse the goosite of integrability. Our distribution ε is of
codimension 4, therefore taking the quotient TM/c we obtain one dimensional vector bundle over M $\frac{v}{v}$ maximal mon-integrability is in some seuse the oposite of integrability. Our distribution $\frac{v}{c}$ is of
codimension 4, therefore taking the quotient $\tau m/c$ we obtain one dimensional vector bundle over M
(vecto codimension 4, thevefo*re* taking the quotient TM/c we obtain "one disheersional vect
(vector bundles with one-dimensional fibers with be called line bundles). Let us denote b
the projection associated to taking the quotie

 $v: X \times \mathbb{C} \longrightarrow TM/C \qquad V(\bar{v_1}W) = g([V, W](\tau_H w))$ where V and W are any vector fields with call define the following map! S
sheve V and W ave any vector fields with
values in τ and such that $V(\tau(v))$ =v, $W(\tau(w)) - 4$

Let us first check that the definition does not depend on the choice of vector fields V and H, provided tet us first check that the definition does not depend on the
they have correct values v and i. For $f \in \mathcal{C}^{\infty}(m)$ he calculate ^f g([v, w]) -

 $g\left(\left[TV, fW\right]\right) = g\left(f\left[V, W\right] + V(f)W\right) = fg\left(\left[V, W\right]\right) + V(f)g\left(W\right) -$

 g is linear $=$ \circ because W is in ϵ

The above calculation shows that $g([V, W])$ depends only on values of V and W at the point of M - no devivatives $\frac{1}{2}$ involve calculation shows that $\frac{1}{2}$ ($\frac{1}{2}$) depends only on values of $\frac{1}{2}$ and $\frac{1}{2}$ at the point of $M - n\sigma$ devivative involved. The nuap $\frac{1}{2}$ is then an antisymmetric two-form on $\frac{1}{$ The above calculation shows that $g(Lv, w)$ depends only on values of V and H at the point
involved. The nuap v is then an antisymmetric two-form on $\mathcal E$ with vector values. Ma
bility condition means that thus form is n involved. The ruap ν is then an alebisymmetric bo form on ϵ with vector values. Maximual nominited that the dimension of the since it is a bio-form, nondependually

contact forms: we have already stated that locally every contact distribution is given as a kernel of some contact form. Sometimes this contact form can be globally defined , but even in those cases it is not unique because multiplying it by a nonvarishing Innthon gives another, equally good contact some contact form. Sometrines this contact form can be grobally defined, but even in those cases
and unique because multiplying it by a nonvaucistance function gives another, equally good correct the multiplying it by a no

DEFINITION: Let (M,E) be a coulact manifold. Any locally defined one form y on M ruch that

Contact forms are obviacisty non-vaccisicing, since at each joint the kernel is supposed to be an-dimen-
sional. The condition of maximal mornintegrability of ϵ is expressed as follows

 (x) $\eta \wedge (d\eta)^{ \wedge n} + O$ which means that it is a volume form on the domain of η . He has see now that indeed o may not be global : if ^M is not orientable then we do not have ^a global volume form defined on it. The above condition means that dy is mondegenerate on ² . Let us now assume for ^a while that we have chosen an you some open a global volume form defined on it. The above wind hor means that dy is nondegenevate on
E. Let us now assume for a while that we have chosen an in on some open 0 c M. It spans
an anihilator = c T*M that can be vieved as a d'un animation ϵ - 1 in that due be never is a think verme solutive of my the rather rather a Now assume for
lator $\mathcal{E}^{\circ} \subset T^*M$
in of \mathcal{E}° . Using
 $(\overline{v}, \overline{w}) \mapsto \langle \overline{w}, \overline{w} \rangle$
w hand let us
 $V \langle \overline{w}, \overline{w} \rangle - N \langle \overline{w}, \overline{w} \rangle$

 $\mathcal{E} \times_{\mathcal{M}} \mathcal{E} \supseteq (\nu, \mu)$ \longmapsto $\langle \mu, \nu(\nu, \mu) \rangle \in \mathbb{R}$

On the other hand let us take any V , $W \in Sec(C)$ as greviously, while defining y, and calculate O

 $d\eta\ (V,W)$ = $V\langle \eta,W\rangle$ - $W\langle \eta,W\rangle$ - $\langle \eta$, $|$ $\Gamma V,WJ\rangle$ = $\langle \eta$, $V(\vartheta,W)\rangle$ up to a higu d η wincedes with V where we

Trivialize both z^o and TM/2 using y. One can then see that the condition of maximal monintegrability

There are several useful ruotions related to contact geometry that are traditionally defined in the language
of contact forms. Before ne olissenss theme we shoud probably write one important theorem

of whitect formus before he outscuss filed we show modely write one independence fuerrell
Theorem (Davboux fluerrell for confect formus) Let η be a confect formu. For every point $x \in M$ there exists
a neibourhood σ a

As a consequence, there is also a normal form of a contact distribution, which in Davboux coordinates is given as 2 10000 σ and coordin
quence, there is also a
 $\zeta = \langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_i} \rangle$ i.e.

as on jet space.

Once we have a contact form we can define a Reeb vetor field Ry. It is uniquely defined by the following two conditions
 $R_y \cup dy = \theta$ $R_y \cup y = 1$. It is easy to see that in Dauboux coordinates $R_g = \frac{8}{22}$

 R_{ij} \Box $dq = \Theta$ R_{ij} \Box $q = \Delta$. It is easy to see that in Dauboux coordinates $R_{ij} = \frac{\omega}{\partial z}$

Using y and dy we can also define contact Manuillonian vector fields, i.e. associate a vector field to every
function on M: For y defined on OCM and for MEC^{oo}(O) we have X_MEX(O) nun that (CO) we leave $X_{\mathcal{H}} \in \mathcal{X} (O)$ rule that

 $X_{\mathcal{U}} \cup \gamma = -\mathcal{U}$ $X_{\mathcal{U}} \cup \gamma = \gamma \mathcal{U}$
 $X_{\mathcal{U}} \cup \gamma = -\mathcal{U}$ $X_{\mathcal{U}} \cup \gamma = \gamma \mathcal{U}$ $X_{\gamma} \cup \gamma = \gamma \mathcal{U}$

An important property of a symptochic Mamiltonian vector field is that it preserves the structure it comes from. Let us then calculate ϵ_X y to see how things look in our case

 α α γ = di γ γ + α dy = -dll + dll - Ry(H)y = -Ry(H)y

Contact Mauxiltoricau vector field does not cousevie the contact form , however the change is proportional
to the contact form . This shows that the such a field conserves the contact distribution

$$
Y \in \mathcal{S}ec(\mathcal{E}) \quad \mathcal{Z}_{X_{\mathcal{U}}} Y = \left[X_{\mathcal{U}}, Y \right] \in \mathcal{S}ec(\mathcal{E}) \quad \Rightarrow Y(M) \quad \text{or} \quad \mathcal{Z}_{\mathcal{U}} \cup \mathcal{Z}_{\mathcal{U}} \quad \text{and} \quad \mathcal{Z}_{\mathcal{U}} \cup \mathcal{Z}_{\mathcal{U}} \cup \mathcal{Z}_{\mathcal{U}} \times \mathcal{Z}_{\mathcal{U}} \times
$$

The problems we have with the definition of a contact Mamiltonian vector fields are the following!

1) Huis definition is local! what if there are no global contact forms? can us have global comtact vector fields ? (YES!/

2) Do we get the same vector field if we keep the homistonian and change y? (No!) what is the velation between stannithonians and vector fields with different contact forms?

3) He have checked that \mathcal{L}_{X} η \neq 0, but $X_{\mathcal{H}}$ preserves \mathcal{L} . Since contact thamillonian vector fields are symmetries of the function and and not contact form we shold be He have checked that $\pm_{\chi} \eta \neq 0$, but χ_{μ} preseives ϵ . Since contact thousan vector
fields ave symmetries of $\frac{\pi}{\mu}$ the construct distribution and not contact form ne shold be
able to define them romehow wi

Tue able to define them romethow will out unrug any contact forms. Arm we do this? (4ES!)
The way to address these problems will be to look at them from a totally different point of view. To this
end I will introduce now a new

The way to address these problems will be to hook at them from a totally different point of view
end I will improduce now a new concept and point art its relation to contract geometry.
DEFINITION: Let IR^X denote a multi able to define them rounded will be to look at them your a totally different point of view. To this
way to address these problems will be to look at them from a totally different point of view. To this
of I will indroduce

 $P D R^{\times}$ $h : R^{\times} \times P \longrightarrow P$ $h_t : P \longrightarrow P$ $w \in \Omega^2(P)$ w is nondegenerate and closed $\begin{array}{lll} \lambda & h(t,h(s,p)) - h(ts,p) & h_t \cdot h_t = h_{ts} & h_t \cdot \overline{\omega} = t\overline{\omega} & \omega \text{ is homogeneous of } t\ 1 & 1 & 1 & 1 \end{array}$

Example : We have seen au example of confact nuamifold being a projective cotangent bundle. Our xample: We have seen all example of confact manipols being a projective cotangent bundle. C
associated example of a symplectic principal buidle would be $(T^*Q)^\times$ = (T^*Q) { O (Q)} i.e. associated example of a synuplectic principal bundle would be $(T^*\mathbb{Q})^\times$ = $(T^*\mathbb{Q})\setminus\{0, (\mathbb{Q})\}$ i.e.
a cotangent burdle with zero section removed as a bundle over the projective cotangent bundle. -xample: We have seen all example of confact mualified being a projective colanged bundle
associated example of a symplectic principal bundle would be $(T^*Q)^{\times} = (T^*Q) \setminus \{O_Q(Q)\}$
a cohanged bundle with zero section remove homogeneous :

 $\mu_t^* \omega_q$ = $d(tp_t) \wedge dq^t$ = $t dp_t \wedge dq^t$ = $t\omega_q$

Looking at this example you probably can quen what to expect - a symplectic principal bundle structure $u_t * w_q = d(tp_i) \wedge dq^t = t dp_i \wedge dq^t = t w_q$
Looking at this example you probably can quen what to expect - a symplectic principal bundle structure
on P induces a contact structure on the base manifold. There is more - every contact on P^oinduces a coulact stricture on the base miquipal Theve is nume – every coulact struch
a seuse (M, C) has a symplectic principal buide associated. It appears that they are in fact
equivalent notions! He will study th

Every contact (M,2) defines (P, $(M, \tau, \ell_1, \omega)$

The symplectic principal bundle corresponding to a given contact (M, E) can be built will flue incorredients w combact (M, \mathcal{E}) defines $(P, M, \mathcal{E}, \mathcal{L}, \omega)$
The symplectic principal bundle corresponding to a given contact (M, \mathcal{E}) can be built will the impred
which is a previously we of dimension \mathcal{E}_{n+1} and then \math consider $2e^{\circ}$ and denote P = $(2^{\circ})^{\times}$ which is an anchilator of z with removed zelo section. It is a submanifold of T^*M , as a burdle over M it is of rank one, as a submanifold it is of It is a rubiliam fold of I *M, as a buidle over M it is of rank one, as a rubiliam fold it is of
dimension 2n+2. It is also invariant with respect to metholication by non-zero real numbers elements of P awe covectors, therefore they can be mutiplied by numbers.

 $P=(\mathcal{E}^{\circ})^{\times}$ mutiplication. The element we do not have yet is a en et is aus invictive una respect to main patient son sous-sero mai instances.
I P are corectors, therefore they can be much phierd by numbers.
P P R × of To this end we have the following proposition. P are covectors, therefore they can be much phication by non-zero real rich
 P are covectors, therefore they can be much phicat by numbers.
 P P R \star multiplication The element we do not have yet is a symplect To lluis end ne have the following proposition:
PROPOSITION: Pis a symplectic submanifold of of T*M.

 \mathbb{C}^{M} \sim M PROOF : Restriction of com to P cau be vieved as a pull-back PROPOSITION: Pis a symplectic subhinding of of 1 mm.
PROOF: Restriction of compto P can be vieved as a pull-back
by the inclusion. Moreover, locally we can choose a contact PROOF: Restriction of ω_n to P can be vievost as a pull-l
by the inclusion. Moreover, locally we can choose a con We have then $\overline{I}_\gamma: \mathcal{O} \times \mathbb{R}^\times \ni (x, s) \longmapsto$ $\mathcal{S}\eta\in P\subset T^*\mathcal{M}$. Let us exactine \mathcal{I}_{η} * ω_{η} :

in the three $\overline{I}_\eta: \mathbb{R} \times \mathbb{R}^N \ni (x, s) \longmapsto s\eta \in P \subset T^*M$. Let us exactly $\overline{I}_\eta * \omega_H$.
 $\omega_\eta = \overline{I}_\eta^* \omega_H = \overline{I}_\eta^* (d\theta_H) = d(\overline{I}_\eta^* \theta_H) = d((s\eta)^* \theta_H) = d(s\eta) = d(s\eta) + s d\eta$

Huis form is clearly closed as a differe $(\omega_{\gamma})^{\Lambda n+1}$ $=(ds \wedge \eta + sd\eta)^{m+1}$ = $d\sin \eta \wedge (s d\eta)^{n} + S^{n+1}(d\eta)^{n+1} = S^{n} ds \wedge \eta \wedge (d\eta)^{n} + O$ NoteHatit gistoundePet^a

=0 = 0 because y is contact.

We conclude that ω_{η} is a symplectic form on $\theta \times \mathbb{R}^n$. We conclude that ω_η is a symptectic form on $\theta \times \mathbb{R}^*$. It is a local expression of $\omega_{m/p}$ in a minalization provided
If η . This finishes the proof and the whole construction.

Every (P, M, τ, k, ω) defines (M, \mathcal{E})

Now we rhad σ the other way round and look for structure structure associated to a gibeu R^x symplectic principal
bundle. a couta d
 d $\left(\rho\right)$ $\left(\rho\right)$ $\left(\rho\right)$ $\left(\rho\right)$ $\left(\rho\right)$ $\left(\frac{d}{ds}\right)$ $\left(\frac{d}{ds}\right)$ $\left(\frac{d}{ds}\right)$ $\left(\frac{d}{ds}\right)$

Let ∇ be a vectoral vector field

 $\nabla(p) = \frac{d}{ds}\bigg|_{s=1} h_s(p) = \frac{d}{dt}\bigg|_{t=0} h_{et}(p)$

 ∇ is invarriant with respect to h : (P)
 $\nabla(h_t(p)) - \frac{d}{ds} (h_s(h_t(p))) - \frac{d}{ds}|_{s=t} h_t(h_s(p)) \overline{\mathcal{F}_{t}}_{t}$ $\left(\overline{V(p)}\right)$ ∇ is invariant with respect to h . $\nabla (h_t(p)) - \frac{d}{dB} (h_s(h_t(p)) - \frac{d}{dB}|_{s=1} h_t(h_s(p)) = \text{Tr}_t (n(p))$
 $\theta = \dot{u}_R \, \dot{\omega}$ homogeneous

PROPOSITION $\dot{\omega} = d\theta$, i.e. ω is exact.

PROPOSITION $\dot{\omega} = d\theta$, i.e. ω is exact. $\begin{aligned} \mathcal{E}(h_t(p)) &\cdot \frac{d}{dS} \Big|_{S}^2 \\ \omega^2 d\theta, &\text{ i.e. } \omega \\ \omega^2 d\theta^2 d\theta^2, &\text{ i.e. } \omega \\ d\left(i_p \omega\right) + i_p d\omega \\ \omega^2 d\theta &\text{ i.e. } \omega \end{aligned}$

 σ

PROPOSITION $J-d\theta$

$$
\omega - \sigma_p \omega = d(i, \omega) + i \sigma \omega = d(\theta) =
$$

Monicoquieous
Now let us choose a local section of $P \xrightarrow{r} M$. It provides P with a local trivialization $T^-(u) = u \times R^x$ local section of $P \longrightarrow M$. It provides P with a local trivialization $\tau^-(u)$
and a local veotrial coordinate s. Uting this coordinate we law write $\frac{d}{dt}(x)$ and a local vectical coordinate s. Unity this coordinate we can write
 $\frac{d}{dt}(x) = \frac{d}{dt}(x)$ and a local vectical coordinate s. Unity this coordinate we can write 100 let us cluose /(p) = 3 he can also ac_tine a one form 0/5. Thace 0 is homogeneous
we know that 0/5 is invariant. It means that it is a pull-back of some one form from the base: $\eta \in \Omega^1(u)$

$$
\frac{1}{5}\theta = \tau * \gamma, \ \ \eta \in \mathcal{S}^1(u)
$$

 $\frac{1}{5}\theta = \tau * \eta$, $\eta \in \Omega^1(\mathcal{U})$
PROPOSITION η is a local contact form.

 σ gives a local trivialization of P:
M By definition $\overline{L}_{\sigma}*(\theta) = s \gamma$ $\mathcal{I}_{\sigma}: \mathbb{R}^{x} \times \mathcal{U} \ni (s, x) \longmapsto \mathcal{U}_{s}(\sigma(x)) \in \mathcal{P}$

$$
\overline{L}_{\sigma}*(\omega) = \overline{L}_{\sigma}*(d\theta) = d(s_{\theta}) = ds \wedge \eta + sd\eta
$$

Now we use the mondegeneracy augusinent the other way $r(\lambda)$ $#O$ because w is symplectic. I_{σ} is a differ mu succeptusm theorefore $0 + (d s \wedge q + s d \eta)^{\wedge (u+t)} =$ or way notend: ω +0 because w is symplechic. I
is a coufact form.

Our caudidate for C is now kerry,
still need another proposition: but to be sure that the have a well defined global distribution the

PROPOSITION: Let η be defined as above with the use of a section τ kerry does not depend
on the choice of τ .

j

 \longrightarrow M

PROOF: Let us examine the difference between η_r and η_{r} . He have

 $\begin{array}{ccc} \overline{L}^1(x) & & \mathcal{P} & \mathcal{L}^* \mathcal{P} \\ & & \downarrow & \mathcal{P} & \mathcal{V} \end{array}$ = $\frac{1}{5} \theta \quad \tau^* \eta_{\sigma'}$ $\frac{1}{s}$ θ and $\nabla'(x) = f(x) \cdot \nabla(x)$ for some monvanishing POSITION: Let η be definied as above with the use of a section τ : kerry does not depend
the electric of τ .
Let us examine the difference between η_{τ} and η_{τ} . He have
 $\tau'(s)$
 $\tau'(s)$
 $\tau'(s)$
 $\tau'(s)$
 $\tau'($ $\begin{array}{lll} \nabla'(x)=\mathcal{J}(x)\ \nabla(x)&\text{for}\quad \text{some}\quad n\end{array}$
function of defined on the
of the domains of ∇ and ∇' S(p) ⁼ f(t(p))s'(p) - - - - - - -

- - - Since $\frac{1}{5}\Theta = \frac{1}{5} \Theta = \frac{1}{5} \frac{1}{5} \Theta$ we have that $\gamma_s = \frac{1}{5} \gamma_{\sigma'}$ or $f\gamma_{\sigma} = \gamma_{\sigma'}$. It is then

clead that both forms define the came distribution 2. I

He can look at c as a projection of the kernel of O. It contains a vertical direction i.e. $<$ V(p)), moreover, since ne au sur ar c'as a projection of the kemile of 0.11 windows a vention more accuration. I.e. (VCP), marriered,
It is houvergeneous, the kerniel is invariant. It can then be projected to a caustant muck distribution on
Tor

symplectic principal bundle in our $exausples$. He have seem already that in Example 3 we have just $P = (T^*R)^\times$ over $M = (T^*R)^\times$ andle in aux
/~ . Both other examples we hall treat together by showing that there is a canonical contact structure on the total space exautiples we hialt treat together by
of a first jet buidle of a line bundle. α Example 3 we have just $P=(T^*Q)^*$ over $M=(T^*Q)^*$ and H there is a canonical contract structure B and B both belong to this category.

 $J^{\prime}L^{\ast}$

Let mois g: L -> M denote a live bundle, which is a rank one vector bundle over a manifold M. By L* He shall denote Let mois $e: L \rightarrow M$ denote a line bundle, which is a rank one vector bundle over a manifold M. By L^* we shall a
Hie dual line bundle and by $L^* - L$ with zero section removed. Note that L is an R^* paincipal bundle flie dual line bundle and by L^ - L'isifli zero-section removed. Note fliat L'is
with respect to multiplication by reals 'borrowed' from the underslying vector bundle.

Our main object of interest will now be $T^* \mathcal{L}^*$. We will show that it is a principal \mathbb{R}^\times symplectic bundle with Our main object of interest will now be T^*L^{\times} be will show that it is a principal R^{\times} symplechic bundle with
the underlying contact geometry being that of J1L*. To clarify matters we shall need local coordinates.

Let then (xⁱt) be adapted coordinates on L with t being a fibre linear coordinate. The same coordinates
can be used in L^x with the condition t = 0. Then we proceed with constructing coordinates (xⁱt, xⁱ, t) an be used in L with the condition $t \neq 0$. Then he proceed with constructing coordinates (x, t, x, with an action denoted by μ_{ε} , s =0

$$
(x^i, t) \cdot h_s = (x^i, st)
$$

This action can be lifted to TL^{\times} and T^*L^{\times} . The lift to TL^{\times} is just a tangent map:

$$
(x^i, t, x^i, t) \circ \mathcal{T}_{h_s} = (x^i, st, x^i, st)
$$

since h_6 is a diffeomorphism, we can consider T^*h_5 as a map. The adapted coordinates on T^*L^{\times} are (x^{i}, t, p_{j}, z) . He have then x^{i} , $\frac{1}{x^{i}}$

$$
(x^{i},t,p_{j},z)\circ T^{*}A_{s}=(x^{i},\frac{1}{5}t,p_{j},sz)
$$
 and for $\frac{1}{5}(x^{i},t,p_{j},z)\circ T^{*}A_{1}=(x^{i},st,p_{j},\frac{1}{5}z)$

 $(x^{i}+F_{i};z)\circ T^{*}h_{s}=(x^{i},\frac{1}{2}t,p_{i};sz)$ and for $\frac{1}{s}(x^{i},t,p_{i};z)\circ T^{*}h_{1}= (x^{i},st,p_{i};\frac{1}{s}z)$
The action we shall actually need is $T^{*}h_{1}$ cangosed with multiplication by s in the bundle $T^{*}L^{*}\rightarrow L^{*}$. The resulting map will be denoted by $\frac{6}{7}$ d₇x h_s

$$
(x^{i}_{1}+_{1}p_{j}^{2},z)\circ d_{T^{*}}h_{s} = (x^{i}_{1}+_{1}sp_{j}^{2},z) \qquad (d_{T^{*}}h_{s})_{s} = sT^{*}h_{s}
$$

We have now the action $d_{\tau *}$ h on $T^* L^{\times}$ He can easily check, that the canonical symplectic form $\omega_{L^{\times}}$ which is there, de have now the achon $d_{\tau *}$ h on τ^* L^x. He van easily check, that the canonical symptectic form wix which is there, because
it is a cotangent bundle is actualy homogeneous with respect to this action. The easiest may is to check it in conimates :

$$
\omega_{\mu} = dp_i \wedge dq^i + dz \wedge dt \quad (df_i \wedge h_s)^* \omega_{\mu} = d(sp_i) \wedge dq^i + dz \wedge d(st) = s(qp_i \wedge dq^i + dz \wedge dt) = sw_i \wedge dz \wedge dt
$$

The same thing may of carse be done globaly: we can use the definition of the Lionville form θ_{l} x and show that The raise fluing may of carrie be dome globaly: He can use the definition of the Liouville form θ_1 x and the
it is homogeneous. Then ω_{1x} = d θ_1 x is also homogeneous. Now we have the following imgredients:

t is homogeneous Then ω_{l} = d θ_{l} is absortion of the value of the following ingredients :
 $(\top^* \bot, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, \frac{7}{2})$. His homogeneous symplectic form . What is missing is a base manifold T^*L^* , ?, ?, $d_{\tau^*}h$, ω_{ι^*}) the homogeneous symplectic form. What is mussimp is a base manifold and projection the total space the R^x action the R^\times action h are the set of n and n and n argue that μ blue to the space the R^x action the distribution of a base the space the space the space that space the space that f is the space of t

 $PROPOSITION: T^{*1*}$ $\sqrt{R^x} \approx J/L$

M

 ON L^*

Let us cousider the matural correspondence between sections of L^* and homogeneous functions on L^{\times} :

ON L^{\times} And T^*L^{\times}

 L^* to the total definition of \mathcal{T}_σ every (local) section of L^{*}>M there corresponds a homogenears $\int \int \tau$ function fr om L^*

$$
\forall_{\sigma}(\ell) = \langle \sigma(\tau^*(\ell)), \ell \rangle
$$

 $J^1\! L^*$: If the two sections $\nabla_{\mathbf{z}}$, $\nabla_{\mathbf{z}}$ have the same jet at x*e* m it
meaus that they differ by the same jet at $x \in M$ it
means that they differ by $\nabla_4(y) = \lambda(y) \nabla_2(y)$ $\qquad \nabla_3(z) = \langle \nabla_4(y), k \rangle = \langle \lambda(y) \nabla_4(y), k \rangle = \lambda(y) \frac{1}{2} \cdot (e)$ $multiplication \left(\begin{array}{ccc} b_1 & a & b_2 \end{array} \right)$ $multiplication \rightarrow R$ such that $d\lambda(x) = 0$,
 $\lambda: M \longrightarrow R$ such that $d\lambda(x) = 0$, $\lambda(x)$ = 1 For

*

 $j^{\prime}\nabla_{4}(x) = j_{4}\nabla_{2}(x) \iff \nabla_{2}(y) = \lambda(y)\nabla_{2}(y)$ for y avound x.

<mark>(2)</mark> every homogeneous function corresponds there is a to some section contains

- $\sqrt[4]{\sigma_1}$ $\sqrt[3]{\sigma_2}$ any I over we get $(l) = \alpha \left(\lambda f_{\overline{s}},l)\right)$ = $f_{\gamma_{2}}(e) g_{\lambda(x)} + \lambda(x) d f_{\gamma_{2}}(e) =$
 $\frac{f_{\gamma_{2}}(e) g_{\lambda(x)} + \lambda(x) d f_{\gamma_{2}}(e)}{4}$ $df_{\tau_{\epsilon}}(e)$
- Moreover $\partial_{T^*} h_s (df_\tau(\ell)) = df_\tau (h_s(\epsilon))$ which means that
there is a map from $J^1{}^*$ to $T^*{}^{\!\!\!\!\!\!\perp\!\!\!\!\!\!\sim\!\!\!\!\sim}$.
- This ruap is one-to-one, because (1) every covector a ET*L* is a differential of ^a homogeneous function :

$$
\alpha(x_0, t_0) = \alpha_0 d x^i + \alpha d t \qquad \oint (x_1^i t) = t \left(\frac{\alpha_0^i}{t_0} (x^i - x_0^i) + \alpha \right)
$$

$$
\alpha(x_0, t_0) = d f(x_0, t_0)
$$

CONTACT HAMILTONIAN MECHANICS

The idea of using confact structure in meclearuics goes back to Gustav Hergloz (1881-1953) and was recently extensively
developed by the Spanish geometric mechanics group. It was devised to deal with dissipative systems wh The idea of using coulact structure in mechanics goes back to Gustav Hergloz (1881-1953) and was recently extensively
developed by the spanish geometric mechanics array. It was devised to deal with dissipative systems whic therefore, if Maericharian represents the energy, there cannot be Hamiltonian niechaeuis have niam advantages: the fact that a certain differential equation has and (1881-1953) and ras recently extended to dissipative systems which
to deal with dissipative systems which
the dissipation. On the direct an traje
and a certain differential equation h symplectic Mamiltonian origin allows for gaining some knowledge about the solutions even if we are not fit into the scheme of symplectic mechanism
therefore, if Mamichanian represents the erred
Mamiltonian mechanian have manny at
not able to solve the equations explicitly
mimerially salve equations with bound Ect able to solve the equations expucitly. One may also use the so called synecytechic integrators to
numerically solve equations with bounded numerical error even in long time solutions. The idea of Icoking for another geometric structure that can be appropriate for wider class of system is not so the loca of receiving for incohes germierin structure had a appropriate for which causs of system to be

As re have meutioned before, contact Maunitonian mechanics is expressed in ternes of contact forms
and Reeb vector fields. Let us spell art the equations and properties once more and then, as usual, hook for probleces :

for problems:
Let (M, E) be a contact manifold with a contact form y . He can then define a Reel vector field Ry
associated to y by the following conditions: se à courair ridde prop what is

RyJdy ⁼ ⁰, Rydy ⁼ 1 . As you can see Reel vector field does not belong to C. The definition strongly depends on y One can even find, that for every As you iau see "Reeb vector field" does not belong to C. The definition is
depends on y. One iam even find, that for every vector ve TM, v & E
we can fined a local contact form y such that R, (x)=v. Changing we can fined a local coulact form y such that "R, (x)= 0. Changing y
Means dramatically changing the Reed vector field. If we use Darboux coordinates for y, i-e y= dz $i.e.$ $y = dz' - p_i dq^i$

1 oz
Using y and Ry ne can define the Manihanian vector field X for every smooth function M on M: $M: M \longrightarrow R$, $X_M \in \mathcal{X}(M)$: $X_M \cup dq = dH - R_q(H) \gamma$ $X_M \cup \gamma = -dG$ y and R_{η} we can define the Mannihonian vector field X_{η}^c for ζ
i $M \rightarrow R$, $X_{\eta}^c \in \chi(M)$: $X_{\eta}^c \rightarrow \alpha \eta = dH - R_{\eta}(H)\eta$ $X_{\eta}^c \rightarrow \eta = -d$
calculate the Mannihonian contact vector field X_{η}^c for a given
no

Let us calculate the Mannibruan comfact vector field X^c_{tt} for a given Mannihronian function in Darboux coordinates :

$$
\eta = dz - p_i dq^i, \quad R_{\eta} = \frac{\partial}{\partial z}, \quad \mathcal{M}(q^i, p_{j}^{\cdot}, z) \qquad X_{\mu}^c = A' \frac{\partial}{\partial q^i} + B_j \frac{\partial}{\partial p_j^{\cdot}} + D \frac{\partial}{\partial z}
$$

 γ = $dz-p_i dq^i$, $R_p = \frac{\partial}{\partial z}$, $M(q^i, p^i, z)$ $X^c_{u} = A \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial p^i} + D \frac{\partial}{\partial z}$
dqⁱ Δdp_i $X^c_{u} = A \frac{\partial}{\partial q^i} + D \frac{\partial}{\partial z}$

 $\partial \phi = dq^{\iota} \wedge dp_{i}$ $dq^{i} \wedge dp_{i}$ $X_{\mathcal{H}}^{c} \vee y = D-p_{i}A^{i} = -$

 X_{H}^{c} J dy= $A^i = -\mathcal{U}$
 $A^i = -\mathcal{U}$
 $A^i d\rho_i - B_j d\varphi^j = \frac{\partial \mathcal{U}}{\partial q^j} d\varphi^j + \frac{\partial \mathcal{U}}{\partial \rho_i} d\rho^i + \frac{\partial \mathcal{U}}{\partial z} dz - (\frac{\partial \mathcal{U}}{\partial z}) (dz - \rho_j dq^j)$

 $A^i d\rho_i - B_j dq = \frac{B}{\partial \rho_i}$
 $A^i = \frac{\partial H}{\partial \rho_i}$ $B_j = \frac{\partial H}{\partial q_j} +$ $p_i \frac{\partial H}{\partial z}$ $D = p_i \frac{\partial H}{\partial p_i}$ $\overline{\mathcal{H}}$

 x_{di} + dy
 $x_{di}^c = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} +$ $P_i A^i = -\alpha L$
 $A^i dP_i - B_j dq^j = \frac{\partial H}{\partial q^j} dq^j + \frac{\partial H}{\partial p_i}$
 $A^i = \frac{\partial H}{\partial p_i}$ $B_j = \frac{\partial H}{\partial q^j} + P_j \frac{\partial H}{\partial z}$
 $\left(\frac{\partial H}{\partial q^j} + P_j \frac{\partial H}{\partial z}\right) \frac{\partial}{\partial p_j} + (P_k \frac{\partial H}{\partial p_k} - P_k \frac{\partial}{\partial z})$

Repend on z, we get usual $D = p_i \frac{\partial \mathcal{M}}{\partial p_i} - \mathcal{H}$
 $\mathcal{H} = \frac{\partial \mathcal{M}}{\partial p_i}$
 $\mathcal{H} = \frac{\partial \mathcal{M}}{\partial p_i}$ $\frac{\partial H}{\partial z}\bigg)$ (elz - p.dg^d)
 $\dot{q} = \frac{\partial H}{\partial p_i}$
 $\dot{p}_j = \frac{\partial H}{\partial q_j} + p_j \frac{\partial H}{\partial z}$
 $\dot{z} = p_i \frac{\partial H}{\partial p_i} - H$

 $X_{di}^c = \frac{\partial M}{\partial p_i} \cdot \frac{\partial}{\partial q_i} + \left(\frac{\partial M}{\partial q_i} + p_i \cdot \frac{\partial M}{\partial z}\right)$
As we can see, if it does not depend on z
equations for q and p plus something , we get usual Mamilton 2 = oqs 1
p: <u>2H</u>
pi opi -<u>H</u> equations for q and p plus something re van solve for z provided the previous two are salved. For $M = T^*Q \times \mathbb{R}$ this

z' provided the frevious to are salved. For $M = T^*Q \times \mathbb{R}$ this
gives a resonable generalization of the symplectic case. In this case we have the projection from gives a resourcist generalization of the squipterfic three. In this aise we have the physical of the equations.
M to T*Q , therefore we can get back the physical dynamics once we have solved the equations. Real mediamical applications always live on T*QxIR, at least to my knowledge. Since y is in this Lecit medicial appulations buddys reve on I colone, at reast to hay raintage. Since of is in ruis niau contact dynamics :

(1) Manuitorian contact vector fields are local (as defined here) (2) There is difficult relationship between Manuillonicrus and vector fields: if we want to keep the field fixed but change the form by muchiplying not be seen as a probecce but (2a) on non-ovientable nuaculformal by the same poichon. It near the
not be seen as a probecce but (2a) on non-ovientable nuaculfolds there ruay not be a global home. To viau for ^a given recor field that is locally Hamiltonian (26) Try to prove (2) by direct calculations and you forniau coulact vector fields are local (as defined here) (2) There is difficult relationship be
inclus and vector fields : if he want to keep the field fixed but channel the formula
convanishing fenction, we have to multi and you with be buried under meets of paper. This is because the nice formula is by acrect calculation
and you with be buried under meets of paper. This is because the nice formula is have for Al depend
an Darboux coordina on Dobtoux coordinates. If he change y, we have to change coordinates and everything gets messy. He
can avoid using coordinates and do it, but still in traditional language it is an unpleasant job that you would rather delegate to students then do by yourself. (3) From the point of view of mumerical methods Huis whole job of contract Mounifornian mechanics does not seem useful, because mechael Mamiltonian
nor contact form are not conserved along the bajectory, therefore on the first sight he have nothing nor contact form are not conserved along the trajectory, therefore on the first right we have mothing
to build our geometric integrator on Geometric integrators are usually based on the idea that some rior coutact fo*rmi* air not cousevred alonip-flic trajectory, flievefore on the first right He have
to bieild our geometric integrator a Geometric integrators ave unially based on the idea flia
geometric structure do not graniemic structure de not change alonge the solution (4) Contact Moduiltonian vector fields de preserve

do preserve a contact structure, which can be seen in the following way: locally z = kery. Let us then take Ye Sec $({\cal C})$, then $\langle y,Y\rangle$ = 0 therefore $\alpha f_{X_{H}}< g_{H}Y> =0$ \overline{O} = E
 E), then $\langle y, Y \rangle = 0$ there for $\langle x, Y \rangle = 0$ there for $\langle x, Z \rangle$, $\langle y, Y \rangle = 0$ $\langle \gamma, \sum_{\mu}^{N} \langle \gamma \rangle \rangle = \langle \gamma, \sum_{\mu}^{N} \langle \gamma \rangle \rangle = \sum_{\mu}^{N} \langle \gamma \rangle \epsilon \rangle$ $\frac{1}{\sqrt{2}}$ \sim y \mapsto = 0

This diows that coutact Mannihonian vector fields are proper symmetries of the coutact Amiture therefore their dioceled be
a possibility to generate them sometion uring global objects related to C. We shall how approach co a possibility to generate them sometion using global objects related to E. We
tonian vector fields from the point of view of principal symplectic IR*-bundles.

Since (P,w) is a sympletic manifold there is there of course the procedure of generaling Manikhanian rector fields
Hiere . The point is now to use homogeneous Mannikanians.

 $T^*P \xrightarrow{\omega^*} TP$ $M: \mathcal{P} \longrightarrow \mathbb{R}$ homogeneous, which means M ($h_o(p)$) = sM(p)

 $d\mathcal{U}$ $\chi_{\pmb{\mathcal{H}}}$ 7 $\frac{dJ(X_{\mathcal{H}},\cdot)=dN}{dV}$ Maurine ave in Mauritorian vector fields for homogemedes Manuikorians are invariant with respect to h_s , therefore they are projectable honioquiedes on M. Let us check how they are related to Marine
projectable to their contact vector fields on M. invacriaus, i.e. τ the #

 $\begin{array}{ccc} \n\sqrt{2} & \downarrow & \downarrow \ \n\mathsf{M} & \longrightarrow & \mathsf{M} \n\end{array}$

For luat ne meet a gicture ne have already
associated vectical coordinate s. He then expressed as had know , which that is the a symplectic local section form $u \nabla$ of ω P can and be

 $\hat{\omega}$ dsey + say

Any honogeneous Mamiltonian can luce be written as M (s,x) = sH(x) for H(x) = Any honogeneous Mamiltonian can then be written as $M(s,x)$ = $sH(x)$ for $H(x)$ - $M(\sigma(x))$
Let us now look for X_n in this setting:
 $dM = H(x)ds + s dH(x) = w(X_n, \cdot)$ X_n is invariant, therefore it is of the form $X_n = sF(x) \frac{dx}{dx}$

 $dA = H(x)ds + s\theta H(x) = \omega(X_{\mathcal{H}}, \cdot)$ X $_{\mathcal{H}}$ is invariant, luevefore it is of the form $X_{\mathcal{H}} = sF(x) \frac{\partial}{\partial s} + Y(x)$
a function on M

a function on M

1

 $\begin{array}{c|c} \hline \begin{array}{|c} \hline \end{array} & \mathcal{M} \end{array}$

$dM = H(x)dx + SdH = \omega(M_{H_1} \cdot) = SF(x)y - \langle y, y \rangle ds + S \dot{i}_y dy = -\langle y, y \rangle ds + S(F(x)y + \dot{i}_y dy)$

L

comparing terms with the same colour we get the following < $\langle \gamma, Y \rangle = -H$, $\dot{\gamma}_y d\gamma = dH - F(x) \gamma$. Function F can be found by contracting both sides with Reeb vector field Ry: α_1 y) = - H, α_2 dy = aH - F'(x)y. Funchon + can
dy(Y, R_g) = R_g(H) - F'(x)<g, R_g> => F(x) = Rg(H)

Summarizing, conditions for Y are $\langle y, y \rangle = -H$, $i_y \lrcorner d\eta = dH - R(H) \eta$ comparing remission with the same colore are get the sonorming
found by contracting both tides with Reeb vector field R_p : $d\eta(Y_iR_p) = R_p(H) - F(X) \langle \eta, R_p \rangle$ => $F(x) = R_q(H)$
Summarizing, contributions for Y are $\langle \eta, Y \rangle = -H$, i Comfact Mauuiltonian vector fields ave reerefore projections of usual Mauuiltonian vector fields. for lionnogeneous Mc
miltonians In terms of doing numerical calculation it is a very good menage - it means that he have to fue cautact Manuiltonian problèm to a apugnectic one, uperairs", solve the equation urino air favourite symplectic
integrators and then just forget the universiancy part of the rotution. There are also ways to write fluis variational way and he variational integrators. Let us now spend a far nuonients on the subject of generaling
objects of contact hamiltonian vector fields

If ne ave happy enough with a function which is defined on an R*-principal bundle as a qenevating object for a hammionian contact vector field we may leave it at that. For the purposes of building a stagrangian approach to contact dynamics we might want to know a generafield we may leave it at fliat. For the purposes of building a shippaugian approach to comfact depudences we puight want to kno
timp driect which "lives" down on the contact manifold itself. We have already disscussed the principal bundles that the the following.

positional bundles that has the followship.
Having from a lime bunde 1,8 we build an \mathbb{R}^{\times} -primaipal bundle by removing the zero section: $\int\limits_{M}^{P=L^{\times}}$ He need also the dual bundle $\int\limits_{L}^{L^{\ast}}$

because the we can have a correspondence between homogeneous functions on P= L^x zero sections: \int . He need also the dual bundle
and sections of $L^* \longrightarrow M$. Our generative object for for a Maniltonian contact vertor field wantal then be a section of the appropriate live bundle. It would be than reseful to for a Mauulfouwau coulact vedor field wauld lieu be a sechou of lie appropriale live bundle. It woud be liai reseful to
know liow to build flus bundle, provided ne know P, i.e. "how to stick in lie nuissing zero. The answe one has to use the associated budle construction :

From P->M we pass to Lp = PxR/Rx using the following action $s \cdot (\rho,\pi)$ = $(h_{s}(\rho), \frac{\pi}{s})$. The deval Lp* can also be vieved From P->M He pass to Lp = Pxk/_Rx librity the jollowing achon S.(p, r) = (h_s(p), S). The deel Lp can also be vieved
as an associated bundle by the action s(p, z) = (h_s(p), sz). There is a canonical identification of The invarge of the zero section is composed of equivalence classes of the form $L(p, o) = \{ (p', o) : L(p') = \tau(p) \}$. From P->M He pass to Lp=Pxk/px Uting the polowing achon S.(p,n)=(h,(p), $\frac{1}{S}$). The dual Lp' can also be vieved
as an associal ed bundle by the action S.(p,z)=(h,(p),sz). There is a canonical identification of Lp* wit The each equivalence class different from the zero section there is a representative with second element equal to
He have then the map $P \ni p \mapsto [p, n] \in L_p^{\infty}$. The action of \mathbb{R}^{\times} on L_p^{∞} reads $s \cdot \mathbb{E}(p, n) =$ $reads$ $s \cdot \Gamma(p, n) = \Gamma(h_s(p), n)$]= We have then the units and $P \Rightarrow L(\rho, n) \Rightarrow L(\rho, n) = \rho^{\times}$. The action of \mathbb{R}^{\times} on $L(\rho^{\times})$ reads $\Rightarrow L(\rho, n) = L(\rho, \rho)$, $\Rightarrow L(\rho, s \land \rho)$.
Let now ρ, q be elements of P over the same point $x \in M$. Then there exists \Rightarrow $L(p,r)$] and $L(q,z)$] reads then: $\langle E(q, z)J, E(p, r)J \rangle = -z\pi s_0$ o arow special a fax accounted some the subject
framework hundle as a quievaling offer for a hast
mangine approach to contract dependence from the substrate
have aloready discussed the passage from the
functions on $P-L^{\times$ $\left(\overline{\mathcal{M}}_{t}\left(q\right)\right)\left\{ \begin{array}{cc} \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \ & \sqrt{2} & \sqrt{$

 \mathcal{A}

\langle [(q,z)], s [(p,m)] \rangle = \langle [(q,z)], [(p,sm)] \rangle = zsmso = s(zmso).

EXAMPLE: Let us now analyze the gassage from homogeneous Manikonian and its vector field and a contact Mannihonian Example: Let is now avalyze the passage from homogeneous Manultonian and its vector field and a contact Manultonna
and a contact Manuiltonnan vector field in case of a nuclianical craneple, i.e. M = J1L* or even nuore preu $M = T*$ ϵ : Let us now analyze the passage from home
contact Mannihomian vector field in case of
 $Q \times R$, which is always a local picture of J^2L^*

 $T^*T^*L^*$ \longrightarrow $\overrightarrow{LT^*L^*}$ $T^*P \longrightarrow \text{TP}$ The general picture, specified for 3^{12} is the $T^*T^*L^* \longrightarrow T^*L^*$ $\frac{1}{2}$ in the quieties picture, specified for $\frac{1}{2}$ is the $\frac{1}{2}$ $T^*P \longrightarrow TP$ The general picture of $3^{\prime}2^{\prime}$ is the T^*T
dM $\begin{pmatrix} 1 & x \end{pmatrix}$ \uparrow \uparrow 7 dH $\left(\begin{matrix} x & x \\ y & x \end{matrix}\right)$ $\begin{array}{ccc} \n\vee & \vee & \vee \ \n\mathsf{P} & \downarrow = \mathsf{Q} \times \mathsf{IR} \end{array}$. In the following we shall introduce coordinates, R^x action ^saud all the elements with your v J^qL^k $AMPLE:$ $AMPLE:$ $AMPLE:$ $AMPLE:$ $B \times R$ $B \times$ $\begin{array}{ccc}\n\mathsf{c} & \mathsf{c} & \mathsf{c} \\
\mathsf{c} & \mathsf{c} & \mathsf{c} \\
\mathsf{d} & \mathsf{d} & \mathsf{d} & \mathsf{d}\n\end{array}$ where $\begin{array}{ccc}\n\mathsf{c} & \mathsf{c} & \mathsf{c} \\
\mathsf{c} & \mathsf{c} & \mathsf{c} \\
\mathsf{c} & \mathsf{c} & \mathsf{c} \\
\mathsf{c} & \mathsf{d} & \mathsf{d}\n\end{array}$ where $\begin{array}{ccc}\n\mathsf{c} & \mathsf{c} &$ $L = Q \times R$ (q^{i}, t) L^{*} = $M \times R^{\times}$ (q^{ι}, t) $t \neq 0$ $L^* = Q \times 12^*$ (q^i, z) $T^*L^x = T^*(Q \times \mathbb{R}^x) \approx T^*Q \times \mathbb{R}^x \times \mathbb{R}^*$ $(q^{\widetilde t},\overline{\mathbb J}_j,\ t$ $(\sigma_x = dq^i \Delta d\overline{u}_i + dt \Delta dz = dq^i \Delta d(t_p_i) + dt \Delta dz =$ (qi, t)0hj ⁼ (gi , st) = $dq^{i} \wedge (p_{i}dt + tdp_{i}) + dt \wedge dz =$ $(q^{i}, \bar{J}_{j}, t, z) \cdot d_{T} * h_{s} = (q^{i}, s \bar{J}_{j}, st, z)$ = $dt \wedge (dz - p_i dq^i) + t_o q^i \wedge dp_i$ $J¹L[*] \approx T*Q \times R$ $(qⁱ, p_j, z)$ q $J^1\ell^* \simeq T^*\mathbb{Q} \times \mathbb{R}$ (g^i, p_i, z)
projection $T^*\ell^* \xrightarrow{\mathcal{I}} J^1\ell^*$ reads in coordinates $(g^i, p_i, z) \circ \tau = (g^i, \frac{\bar{J}_i}{\bar{z}}, z)$ The relation between the homogeneous Namitarian and the Hamiltonian On M : ϵ P_i $0a T^*Q \times R^* \times R$) π_{ij} , π_{ij} , π_{ij} , π_{ij} + , z)
 $\sigma_{L} = \frac{dq^{i} \cdot d\pi_{i} + dt \cdot dz - dq^{i} \cdot d(t \cdot p_{i}) + dt \cdot dz}{dt}$
 $= \frac{dq^{i} \cdot (p_{i} dt + t d p_{i}) + dt \cdot dz}{dt}$
 $= \frac{dt \cdot (dz - p_{i} dq^{i}) + t \frac{dq^{i} \cdot dp_{i}}{dt}$
 $= \frac{dq^{i}}{dt}$
 $= \frac{dp^{i}}{dt}$
 $= \frac{2p^{i}}{dt}$
 $=$ $\mathcal{H}\left(q, \frac{1}{L}, \frac{1}{d}, z\right) = \pm \left(q, \frac{1}{L}, \frac{1}{L}, z\right)$ $T^*L^x = T^*Q \times R^x \times R$ $T^*Q \times R$ t
 t $\frac{t}{t}$ $\$ (q^i , \overline{v}_j , t , z) $\overline{\omega}_{L^x} = dq^i \times d\overline{v}_i + dt \times dz = dq^i \times d(e_i) + dt \times dz =$
 $= dq^i \times (p_i dt + t d\rho_i) + dt \times dz =$
 $= dt \times (dz - p_i dq^i) + t d\rho_i dz$
 $= dt \times (dz - p_i dq^i) + t d\rho_i dz$
 $= q_i \times (q_i \wedge q_i) + \frac{q_i}{q_i} + z$
 $= \frac{q_i}{q_i} \frac{z}{z} + \frac{z}{z}$
 $= \frac{z}{z}$

There are two practical examples we may want to play with: PRACTICAL EXAMPLE: VISCOSITY FORCE PRACTICAL EXAMPLE: PARACHUTE EQUATION $\dot{y} = \frac{\lambda}{\mu} (p - \delta z)$ $M = T^*Q \times \mathbb{R}$ $\gamma = dz - \theta_Q$ $H(p_1z) = H_0(p) - \lambda z$ These are two practical examples we may exact to p

PRACTICAL EXAMPLE: VISCOSITY FORCE PRACTICAL
 $M = T^*Q \times R$ $\gamma = dz - \theta_Q$ $H(p_1z) = H_0(p) - \lambda z$ is
 $X_+^C = \frac{\partial H_0}{\partial p_i} \frac{\partial}{\partial q_i} - (\frac{\partial H_0}{\partial q_i} - \lambda p_i) \frac{\partial}{\partial p_i} + (p_i \frac{\partial H_0}{\partial p_i}$ p $7 - y(y_1 \dot{y}) = \frac{w}{x} \dot{y}^2 + 8\dot{y}z - v(y)$ $=\frac{p}{m}(p-8z)-8l$ $\frac{1}{4u} (p-8z)$
= $-\frac{2V}{8y} + \frac{8p^2}{4u} - \frac{8p^2}{4u}$
= $P (x - x) = 1$ TICAL EXA

1 (p-82)

- 2 (p-82)

- 2 m

- 2 m
 CHUTE EQUATION
 $\begin{matrix}\n\frac{Q-R}{T+T+Q\times R} \\
\frac{Q-R}{T+T+Q\times R} \\
\frac{(q_1R+Z)}{T+Q+Q+Q}\n\end{matrix}$ $\frac{\partial H_0}{\partial p_i}$ $\frac{\partial}{\partial q_i} - \left(\frac{\partial H_0}{\partial q_i} - \lambda p_i\right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H_0}{\partial p_i} - H_0 + \lambda z\right) \frac{\partial}{\partial z}$
 $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial H_0}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ $\frac{\partial}{\partial p_i}$ γ *m* is $\gamma^2 + g = 0$ $\dot{q} = \frac{\partial \mu}{\partial p^i}$
 $\dot{p}_q = -\frac{\partial \mu_q}{\partial q^j} + \lambda p_j$ \dot{z} = $p_i \frac{\partial H_o}{\partial p_i} - H_o + \lambda z$ The Hergloz Lagrangian $\mathcal{M}(y_1 p_1 z) = \frac{1}{2m} (p - \gamma z)^2 +$ $V(y) = \frac{mg}{\gamma} (e^{3y}-1)$ $\frac{p_i \frac{\partial p_i}{\partial p_i} - H_c}{L_o}$ Gustav Hergla 1881 - 1953

CONTACT HAMILTON-JACOBI THEORY

Hamilton-Jacobi equation is ^a classical part of symplectic mechanics. Before we pass to the contact case , I would Hamilton-Jacobi equation is a classical part of symplectic nuediacics. Before we pass to the contact case, I would
like us to review the symplectic version from the geometric point of view. We may look at the relation betw Mauritan-Jacobi equation and a Mauritan equation in two warp : 1 He may want to solve a partial differential equation of the first order by means of a certain ordinary differential equation on the other way round : ral deffevents
(2) we may of the first order by means of a certain ordinary differential equation or the other way round: (2) we may want to
solve a difficult ordinary differential equation using possibly simupler partial differential equation. Dis solve a difficult ordinary differential equation unitip possibly sinupler partial differential equation. Disregarding
Fedinical difficulty for a while, we niall look at the georuety of both problems.

1 From ^a PDE to a lamiltonian ODE

Urino a symplectic structure we can deal with a spenial kind of a gavtial diffevential equations of a first arder-mannely
Hose that do not involve values of unknown function. In traditional notation such a PDE for one func those that do not involve values of unknown function. In traditional notation such a PDE for one function u of
several variables (q^s...q") can be written as F(qⁱ, u;)=0 where u; denotes a partial derivative Iu/jqi. In geometric language re would understand this equation as a condition for the differential of a function. This can be formulated for any manifold Q. We define a submanifold $K\subset T^*Q$ by the condition $\kappa'=\Gamma^{-1}(0)$ quere duplicity in the world a monotron of the equation is a monotron for the expression of a function.
This can be for mulated for any mean fold α . We define a subsurant fold $\kappa \in T^*Q$ by the condition $\kappa' = F^{-1}(Q)$ Note that K, as a submanifold of codimension 1 (there are conditions for F of course) is a coisotropic submanifold of T*Q. This means that it carries a one-dimensional dravactemitic distribution. Characteristic destinbutions of coisotropic submanifolds are involutive. Here it is of cause automatic since the distribution is one dimensional. This means that K foliated by one dimensional momanifolds called characteristics. The characteristic distribution is spanned by the Mamiltonian vector field XF of a function defining L. Now, if n is a solution of the equation then the innage du (a) is ^a submanifold in K . Moreover, it is ^a Lagrangian submanifold of T* ^Q , therefore it must be composed of is a rubanacu fold in K. Moneover, it is a Lagrangian submanifold of T*Q, hierefore it and be composed of
leaves of the characteristic foliation. Assuming the can find the trajectories of XF ne may solve the PDE in the following Hay: -puritive equation is a dissimile part of equapirie mechanics. Before we gave to the case of the square interval of the society of the society of the state of the state of the society of the society of the society of the

Our boundary conditions should commits of value of n on a submanifold NCQ of codimension 4 and a value of differentials at points of J. Then we can calculate characteristics starting from boundary values and get the dagrangian rubinamifald being the graph of du Function n itself can be

2) From Hamiltonian ODE to PDE.
Now we are looking for the solutions of Hamiltonian equations for a given Manultonian M: T*Q -> R. Let us assume that re can find a solution of the Mannifon-Jacobi equation M (dS) = E. In coordinates we have assume that he can prod a rolution of the Manuiton-paobi equation M(els)=E. In coordinates he have
M(qⁱ, $\frac{\partial S}{\partial q^i}$)=E. The invage dS(Q) is a dagrangian subpraneigold contained in a coisohopic submanifold M⁻¹(E)
In In particular the Mamitranian vector field is targent to dS(Q). Knowing as we can consider ^a vector field In paracular ne natulal sunacul venta per is raupelle to ast

 $TdS(X_{Q}(q))$ = $X_{x}(elS(q))$ This Hay we have to integrate a vector field X_{Q} with half the variables we This Hay ne liave to integrate a vector beld XQ with hialf the vaerables H
initially had. The rolutions may then be lifted to T*Q by meaces of dS.

Both procedures are based on symplectic Hamiton-Jacobi theorem that can be formulated as follows:

oth procedures ave based on symplech'c Alamibon-Jacobi Heeorem that can be formulated as follows:
THEOREM Let (P,w) be a symplechic mamifold Al:P -> R be a spuoald function and LcP be a Lagrangian submanifold. Then X_H is tangent to L if and only if H is (locally) constant on L.
The procedures s. and 2 from the Manistron-Jacobi story are just clever applications of this quite simple symplectic theorem.

The symplectic constructions repeated above may be generalized to the case when our partial differential equation depends on values of The symplectic constructions repeated above may be genevalused to the case when our partial differential equation depends on values of
function. The equation itself is then a nubmanifold in J¹(QXR)=T*QXR, howally given a function. The equation itself is nieu a monique form in a (axik) = 1.3xk, haddy given as a rever set of some function. In coordinate isotropic, coisotropic and Lagrangiam submanifold business can be queralized to the contact context. For example we would call a submanifold $K \subset M$ coisotropic when T_K is a coisotropic subspace of \mathcal{L}_K with respect to the tro form Y (or dry if we have chosen the contact form.) The same about dagrangian and isotropic. He may also thow that the level set of a regular function is a coisotropic subweau fold of M while, in case M-T*QxR or more general M-J12*, a jet prolongation of a section (function) is a Legendre submanifold of M. Lequare submanifold is ^a wisotropic name for Lagralgiam submanifold in sympledic case, i . e. maximal isotropic.

Now we can repeat the mode procedure of passing from a partial differential equation of the form $F(q^i, u, u_j)$ = 0 to the ordinary differential equation of the form... this is the point where wenight want to do some calculations in coordinates to see what is Now we can repeat the nhote procedure of passing from .
differential equation of the form ... this is the point is
what : The geometric background is as it was before : F what The geometric background is as it was before $F^{\prec}(O)$ is a coisotropic submursurfold therefore it has a characteristic foliation what The geometric background is as it was before : F⁻¹(0) is a coisobropic submusurfold therefo*re* it has a characteristic foli
by one dimensional rubmanifolds called characteristics. Any degendre submanifold confained by one dimensional rushwampoids called characteristics. Any degendre rushwamp.
characteristics. Let us now find an equation for characteristics of $K = F^{-1}(0)$.
An element $\alpha^i \frac{\partial}{\partial q_i} + \beta_i \frac{\partial}{\partial \beta_i} + \frac{\partial}{\partial z}$ belongs t

An element $a^2\frac{\partial}{\partial q^2}+b^2\frac{\partial}{\partial p^2}+c\frac{\partial}{\partial z}$ belongs to TKAE if
 $a^2\frac{\partial F}{\partial q^2}+b^2\frac{\partial F}{\partial p^2}+c\frac{\partial F}{\partial z}=0$ and $c=p_ia^2$ The symplectic amily later in ϵ is then spanned by vectors of the form

 $\frac{\partial F}{\partial p_i}$ a $-\left(\frac{\partial F}{\partial q_i}+p_i\frac{\partial F}{\partial z}\right)\frac{\partial}{\partial p_i}+p_k\frac{\partial F}{\partial p_k}\frac{\partial}{\partial z}$ this is a value of the contact Manitorian vector fields at points F=0

Let us now formulate the contact Mamilton Jacobi theory in the language of symplectic IR principal bundles. The origi nat definition of degendre subsuantifold that says \angle is degendre when it is maximal and isotropic, i.e. when it is of dimension n and $TL\subset\mathcal{C}$ can be replaced by the following:

PROPOSITION $L \subset M$ is a Legendre submanifold if and only if $\tau^{-1}(L) \subset P$ is a dagrangian submanifold of (P, ϖ) PROOF : if the dimension of M is 2n+1, then the dimension of $\tau^-(L)$ must be n+1 if $\tau^-(P)$ is Lagrangian. Since $\tau^-(L)$ RooF: if the dimension of M is 2n+1, then the dimension of L (L) must be n+1 if "L"(P) is Lagrangian. mode T"(L)
is a nuion of fibres , L must be of dimension on which is a correct dimension for L. If we choose a vertical the symplectic form can be written as ω = dsny + sdy with y being a local contact form. Let $v \epsilon T \tau'(r)$. The vartical vector $\frac{2}{\sigma s}$ also belongs \star TT(L). We have then \int_{α}^{R} *M* is
 MILK be
 ω riHeu as
 $\omega(\frac{2}{25}, v)$ =

 $0 = \omega(\frac{3}{25}, v) = \langle \eta, v \rangle$ It meaus that $\tau\tau(v) \in \mathcal{E}$, therefore L is isotropic.

 $\begin{array}{c}\n\begin{array}{c}\n\text{DSTIOM} & \text{L} & \text{L} \\
\text{DFTIOM} & \text{L} \\
\text{DFTIOM} & \text{L} \\
\text{DFTIOM} & \text{L} \\
\text{DFTIOM} & \text{L} \\
\text{DFTIUM} & \text{L} \\
\text{DFTIUM} & \text{L} \\
\text{DFTIUM} & \text{L} \\
\end{array}\n\end{array}$

Since P is symplectic we can use the symplectic property and state that Xn is taugent to as ω = ds.ng + sdg with y being a local

v) = <m,v> It meaus that $Tr(v) \in \mathcal{E}$, the

Since \mathcal{F} is symplectic the can use the syn
 $\tau^1(L)$ if and only if M is constant on τ^1
 $\tau^1(L)$ is supposed to be minure eplech's property-aud state that Xx is taugent
(L) But H is supposed to be homogeneous while $\tau^{\text{-}}$ (L) if and only if M is carstant on $\tau^{\text{-}}$ (L). But It is supposed to be homogeneous wh:
 $\tau^{\text{-}}$ (L) is supposed to be union of fibers of R* action. This means that this carstant must be 0. $L^1(L)$ is supposed to be union of fibers of R^x action. This nueaus that this carstacle must be o .
But then also X_p^c is tangent to Land H vanishes on L. Sime T is symptotic He case use the symptotic frequent to did that the last X_n is tongue to $\tau^*(t)$ if and only if M is caused on $\tau^*(t)$. But H is Ω apposed to be homogeneous while
 $T^*(t)$ is typosed to be u

2 10 Vo He have then the contact Mannihon pacobi theorem for free:
THEOREM: Let L be a Lequidre submiquitold of M. Then X_{μ}^C is tangent to L if and only if H vanitues on L.

Let us now look at this in neone , invectioninal" way, i.e. explore the theorem for the case of $M = T^*Q \times IR$ and $M = J^4L^*$

 $M = T^*Q \times \mathbb{R}$ = (p,z) $L = j^1S(Q) = \{(dS(q), S(q)) : q \in Q\}$ $\mathcal{P} = T^*(\mathbb{Q} \times \mathbb{R}^{\times}) = T^*\mathbb{Q} \times \mathbb{R}^{\times} \times \mathbb{R} \text{ is } (\pi, \tau, z)$
 $\mathcal{T} : (\pi, \tau, z) \longmapsto (\frac{\mathcal{F}}{\tau}, z)$ $M: P \longrightarrow \mathbb{R}$ reads $M(\overline{x}, \overline{c}, z) = \mathcal{I} H(\frac{\overline{x}}{\overline{c}}, z)$ for $H: M \longrightarrow \mathbb{R}$

Contact Maunition-Jacobi equation is $H(dS(q), S(q)$ = 0 in coordinates $H(q^i, \frac{\partial S}{\partial q^i}, s(q))$ = 0 (*)

Let us deucte b By r the projection $N = \frac{1}{2} \pi \sqrt{2 \times 1} R \rightarrow Q$ is a solution of Marinal Jacobi equation (*) then trajectories of X_p^C with Let us denote by a the projection α = $\pi_{\alpha} \circ p \pi_{\pm} : T^* \Omega \times \mathbb{R} \longrightarrow \Omega$. If S is a solution of thannulou-Jacob equation (*) then projectories of X_{μ} with
innihal conditions on L_s can be obtained from trajectori degrees of freedom by half.

$M = J^{\frac{d}{2}}L^*$

What will be the ingredents of a Manniton-Jacobi theorem? We start with a line bundle $L\longrightarrow Q$ and its duol L^* <Q. The volution of a $M = J^4 L^*$
What will be the impredents of a Mannifon-Jacobi theorem? We start with a line bundle $L \longrightarrow Q$ and its dual $L^* \longrightarrow Q$. The value symplectic principal bundle which we determined to be T^*L^{\times} . What is then the corresponding contact Hamitorian? It is symplectic porincipal bundle which we deferencied to be T^*L^\times . We
a section of Lp*. The question is now what is Lp* for P = T*L*?

PROPOSITION: For $P = T^*L^*$ We get $L_P \cong J^4L^* \times L$, $L_P^* \cong J^4L^* \times L^*$

PROPOSITION: For P-T*L* He get $L_p \approx J^2 L^* \times L$, $L_p^* \approx J^1 L^* \times L^*$
A section of $L_p^* \longrightarrow J^* L^*$ can then be identified with the map $\tau : J^2 L^* \longrightarrow L^*$ covering the identity on Q. The Manniton - Jarobi
equation now reads σ equa*tion wow* reads $\sigma(j^eS)$ =0. If S is a solution of tleis equation then trajectories of X_v with
can be obtained from trajectories of a vector field on Q which is a projection of X_{v [j} sca)