CONTACT NOTES
MAGURELE 2024

\+ ×

− ÷

Notes for the mini course in Bucharest

This minicourse is supposed to be about contact geometry, which is a part of classical differential geometry. It is a very „down to Earth" subject at least in mathematical context. Contact structure is a typical example of a geometric structure, which usually consists of a manifold together with some distinguished tensor field e.g. Riemannian manifold, Poisson manifold or a manifold with some operations as a Lie group or a Lie groupoid. This time we shall study a manifold together with a distinguished distribution of a special kind.

Before we give a formal definition, let us look at few canonical examples of a contact manifold:

1. Let us take any manifold Q and consider functions on Q as section of the trivial bundle $pr_1 : Q\times\mathbb{R}\longrightarrow Q$. First jet of such section at point $q\in Q$ consists of the differential $df(q)$ and value $f(q)$. In another words $J^1(Q\times\mathbb{R})\simeq T^*Q\times\mathbb{R}$. It is again a bundle over Q with the distinguished set of sections being prolongation of functions, i.e. sections of the form $q\longmapsto (df(q), f(q))$. Let $\mathcal{C}$ denote the distribution spanned by vectors tangent to graphs of prolongations. Let us use coordinates to see what kind of vectors they are and what distribution they span. In $T^*Q\times\mathbb{R}$ we will use coordinates (q^i, p_j, z). Jet of a function $(q^i)\longmapsto f(q^i)$ is given by

$$q^i \longmapsto \Big(q^i, \underset{\overset{\shortparallel}{p_j}}{\frac{\partial f}{\partial q^j}}, f(q^i)\Big)$$ then lifting $\frac{\partial}{\partial q^i}$ from Q to $T^*Q\times\mathbb{R}$ we get $$\frac{\partial}{\partial q^i} + \frac{\partial^2 f}{\partial q^i\partial p^j}\frac{\partial}{\partial p_j} + \underset{\overset{\shortparallel}{p_i}}{\frac{\partial f}{\partial q^i}}\frac{\partial}{\partial z} = \underbrace{\frac{\partial}{\partial q^i} + \overset{\text{arbitrary}}{a_j}\frac{\partial}{\partial p_j} + p_i\frac{\partial}{\partial z}}_{\text{tangent at point } (q^i, p_j, z)}$$

changing f we can make it arbitrary

Summarizing, $\mathcal{C} = \left\langle \frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial z}, \frac{\partial}{\partial p_j}\right\rangle$

$$x=(p,z) \quad \dim \mathcal{C}_x = 2n \quad (\dim Q = n)$$

$$\dim T^*Q\times\mathbb{R} = 2n+1$$

As for the properties of $\mathcal{C}$, let us note that $\left[\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial z}, \frac{\partial}{\partial p_j}\right] = -\frac{\partial}{\partial z}\notin\mathcal{C}$ therefore $\mathcal{C}$ is not involutive, as a matter of fact it is the very opposite of involutive, but this is something we will disscuss in a while.

Before we pass to the second example, let us notice that $\mathcal{C} = \ker\eta$ for $\eta\in\Omega^1(M)$ $\eta = p_i dq^i - dz$. The form η is globally defined on M. Of course any other one form $f\cdot\eta$ for $f\neq 0$ everywhere is also o.k. in a sense that $\mathcal{C} = \ker f\eta$, but η is distinguished due to the Liouville form $\theta = p_i dq^i$

This means that $\eta = \theta - dz$ is not only globally defined, but in this case, also distinguished.

(2) The second example is in some sense simmilar to the first one: it is also about the jets of some sections, but there is one very important difference with respect to the first example.

Let B denote the Möbius band with the infinite „vertical direction" in a sense that $B = \mathbb{R}^2/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}^2$ according to the formula

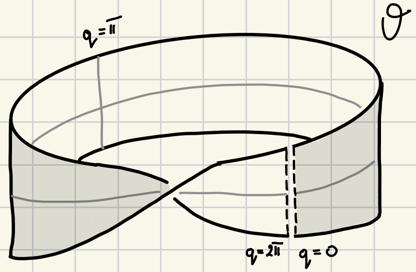
$$k \cdot (x, y) = (x + 2k\pi, (-1)^k y)$$

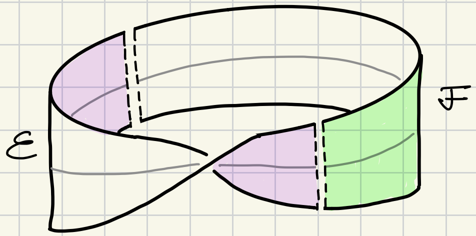
action is linear in y

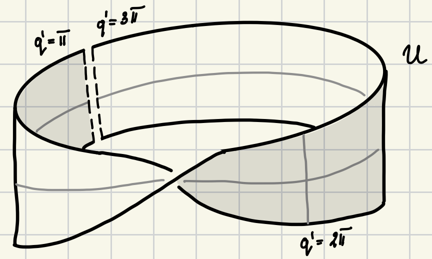
The linearity of action in the second argument means that $B \longrightarrow S^1$ is a vector (line) bundle. Probably everybody knows that B as a manifold is not orientable. Again we consider $M = J^1(B)$ i.e. jets of sections. In coordinates then it will all look like in the first example, namely if (q, z) are coordinates in B and (q, p, z) are coordinates in $J^1 B$ then

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle \quad \dim \mathcal{C}_x = 2 \quad \dim M = 3.$$

To see the difference between this example and the previous one, let us go deeper into the structure. The Möbius band B can be described by two coordinate domains together with appropriate transition maps. The same we can say about $J^1 B$. Pictures are about B, but we think about $J^1 B$.







$(q, p, z) \quad q \in \,]0, 2\pi[$

$\mathcal{O} \cap \mathcal{U} = \mathcal{E} \cup \mathcal{F}$

$(q', p', z') \quad q' \in \,]\pi, 3\pi[$

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle$$

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, \frac{\partial}{\partial p'} \right\rangle$$

In $\mathcal{E}$

$$q' = q, \quad p' = p, \quad z' = z$$

In $\mathcal{F}$

$$q' = q + 2\pi \quad p' = -p \quad z' = -z$$

In A the transformation is identity so of course $\mathcal{C}$ is well defined

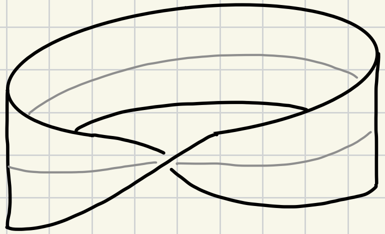
$$\frac{\partial}{\partial q} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial p'}$$

$$\frac{\partial}{\partial q} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial q'} + (-p')\left(-\frac{\partial}{\partial z'}\right) \quad \frac{\partial}{\partial p} = -\frac{\partial}{\partial p'}$$

$$\left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, -\frac{\partial}{\partial p'} \right\rangle = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, \frac{\partial}{\partial p'} \right\rangle$$

In B the transformation is not identity but still $\mathcal{C}$ (given in coordinates) is well defined



Using coordinates we can write one forms as in example 1. In $\mathcal{O}$ we have $\eta_{\mathcal{O}} = dz - pdq$, in $\mathcal{U}$ we have $\eta_{\mathcal{U}} = dz' - p'dq'$. Using coordinate transformation we get that on $\mathcal{O} \cap \mathcal{U}$ we can write

in $\mathcal{E}$: $\eta_{\mathcal{U}} = \eta_{\mathcal{O}}$ but in $\mathcal{F}$: $\eta_{\mathcal{U}} = -\eta_{\mathcal{O}}$

$$\eta_{\mathcal{U}} = dz' - p'dq' = -dz + pdq$$
$$\eta_{\mathcal{O}} = dz - pdq$$

Since both forms are nonvanishing and have the same kernel they have to differ by multiplication by non-vanishing function. This function must be 1 on $\mathcal{E}$ and -1 on $\mathcal{F}$, moreover J^1B is connected. Smooth function that has value 1 at some point and -1 on another point has to assume value 0 somewhere on the way. We have therefore reached a contradiction.
The conclusion is that the distribution $\mathcal{C}$ is well defined on the whole J^1B, but it is not the kernel of any globally defined one form.

③ The third example, or rather class of examples is $\mathbb{P}T^*Q$, i.e. projectivized cotangent bundle. Let us first take $Q = \mathbb{R}^n$, for simplicity.

$$M = (T^*\mathbb{R}^n)^{\times}/_{\sim} = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})/_{\sim} \quad \text{where} \quad (q_i, p_j) \sim (q_i, \lambda p_j) \quad \text{for some } \lambda \neq 0.$$

Now we will define $\mathcal{C}$ on M using local contact forms: Let $\mathcal{U}_k \subset M$ denote an open subset of M given by the following condition

$$\mathcal{U}_k = \{ [(q^i, p_j)] : p_k \neq 0 \} \subset M$$

We have of course $\bigcup_{k=1}^{n} \mathcal{U}_k = M$. In $\mathcal{U}_k$ we can introduce local coordinates $(q^i, \pi^k_1, \dots, \pi^k_{k-1}, \pi^k_{k+1}, \dots, \pi^k_n)$ where $q^i([(q,p)]) = q^i$ $\quad \overset{(k)}{\pi}_\ell([(q,p)]) = \frac{p_\ell}{p_k}$. In these coordinates we define

$$\overset{(k)}{\eta} = dq^k + \sum_{i \neq k} \overset{(k)}{\pi}_i \, dq^i$$

this form does not vanish on $\mathcal{U}_k$

On $\mathcal{U}_k \cap \mathcal{U}_\ell$ we have $p_k \neq 0$ and $p_\ell = 0$. The coordinate change in part „π" reads

$$\begin{matrix} i \neq k \\ i \neq \ell \end{matrix} \quad \overset{(\ell)}{\pi}_i = \frac{p_i}{p_\ell} \qquad \overset{(k)}{\pi}_i = \frac{p_i}{p_k} \qquad \overset{(\ell)}{\pi}_i = \frac{\overset{(k)}{\pi}_i}{\overset{(k)}{\pi}_\ell}, \qquad \overset{(\ell)}{\pi}_k = \frac{1}{\overset{(k)}{\pi}_\ell}$$

$$\overset{(\ell)}{\eta} = dq^\ell + \sum_{i \neq \ell} \overset{(\ell)}{\pi}_i \, dq^i = dq^\ell + \sum_{\substack{i \neq \ell \\ i \neq k}} \frac{\overset{(k)}{\pi}_i}{\overset{(k)}{\pi}_\ell} dq^i + \frac{1}{\overset{(k)}{\pi}_\ell} dq^k = \frac{1}{\overset{(k)}{\pi}_\ell}\Big(dq^k + \sum_{\substack{i \neq k \\ i \neq \ell}} \overset{(k)}{\pi}_i \, dq^i + \underbrace{\overset{(k)}{\pi}_\ell \, dq^\ell}_{} \Big) = \frac{1}{\overset{(k)}{\pi}_\ell} \overset{(k)}{\eta}$$

add here removing $i \neq \ell$

We have got $\eta^{(l)} = \frac{1}{\Pi^{(k)}_l}\eta^{(k)}$ on $U_k \cap U_l$ where $\Pi^{(k)}_l \neq 0$. Forms $\eta^{(l)}$ and $\eta^{(k)}$ have the same kernel. Globally they define a distribution of dimension $2n-2$ at each point of the manifold M of dimension $2n-1$. This distribution is another example of a contact distribution.
The same can be done for any manifold Q replacing $\mathbb{R}^n$, i.e. we can have $M = \mathbb{P}T^*Q = (T^*Q)^\times/\sim$
For the low dimensional example we can take $Q = S^2$. Then every fiber of the projectivized tangent bundle is a circle. We have then the bundle of circles over S^2. It will be our task for the tutorial to show that in fact

$\mathbb{P}T^*S^2 \simeq S^3$ and what we get here is a contact structure on the total space of the Hopf fibration $S^3 \longrightarrow S^2$

We have seen three examples of the pair $(M, \mathcal{C})$. Each time we had a manifold of odd dimension and a distribution of codimension one i.e. in particular of even dimension. At least in two first cases we have checked that this distribution was not integrable. We did not check it in the third example, but the situation is the same. In one of the examples we have checked that the distribution cannot be described as the kernel of a global one-form. Local one forms with this property of course always exist.

Now is the time for the definition of a contact structure

DEFINITION: A manifold M together with a regular distribution $\mathcal{C}$ of codimension 1 which is maximally non-integrable is called a contact manifold.

maximal non-integrability is in some sense the oposite of integrability. Our distribution $\mathcal{C}$ is of codimension 1, therefore taking the quotient TM/C we obtain one dimensional vector bundle over M (vector bundles with one dimensional fibers will be called line bundles). Let us denote by $\rho: TM \longrightarrow TM/C$ the projection associated to taking the quotient. Now I can define the following map:

$\nu: \mathcal{C} \times_M \mathcal{C} \longrightarrow TM/C \qquad \nu(v, w) = \rho\left([V, W](\tau_M(v))\right)$ where V and W are any vector fields with values in $\mathcal{C}$ and such that $V(\tau(v)) = v$, $W(\tau(w)) = w$

Let us first check that the definition does not depend on the choice of vector fields V and W, provided they have correct values v and w. For $f \in \mathcal{C}^\infty(M)$ we calculate

$$\varrho([V, fW]) = \varrho(f[V,W] + V(f)W) \underset{\varrho \text{ is linear}}{=} f\varrho([V,W]) + V(f)\underbrace{\varrho(W)}_{=0 \text{ because } W \text{ is in } \mathcal{C}} - f\varrho([V,W])$$

The above calculation shows that $\varrho([V,W])$ depends only on values of V and W at the point of M — no derivatives involved. The map ν is then an antisymmetric two form on $\mathcal{C}$ with vector values. Maximal nonintegrability condition means that this form is nondegenerate. Note, that since it is a two-form, nondegeneracy means that $\mathcal{C}$ must be of even dimension. This in turn means that M itself must be of odd dimension.

Contact forms: we have already stated that locally every contact distribution is given as a kernel of some contact form. Sometimes this contact form can be globally defined, but even in those cases it is not unique because multiplying it by a nonvanishing function gives another, equally good contact form. Nevertheless contact forms are very usefull in practice, so let us look at their properties.

DEFINITION: Let $(M, \mathcal{C})$ be a contact manifold. Any locally defined one form η on M such that $\mathcal{C} = \ker \eta$ wherever η is defined, is called a contact form.

Contact forms are obviously non-vanishing, since at each point the kernel is supposed to be $2n$-dimensional. The condition of maximal nonintegrability of $\mathcal{C}$ is expressed as follows

(*) $\eta \wedge (d\eta)^{\wedge n} \neq 0$ which means that it is a volume form on the domain of η. We can see now that indeed η may not be global: if M is not orientable then we do not have a global volume form defined on it. The above condition means that $d\eta$ is nondegenerate on $\mathcal{C}$. Let us now assume for a while that we have chosen an η on some open $\mathcal{O} \subset M$. It spans an anihilator $\mathcal{C}^\circ \subset T^*M$ that can be viewed as a dual vector bundle of $TM/\mathcal{C}$. We leave then a local section of $\mathcal{C}^\circ$. Using this section we define a two form on $\mathcal{C}$ by the following formula

$$\mathcal{C} \times_M \mathcal{C} \ni (v, w) \longmapsto \langle \eta, \nu(v,w) \rangle \in \mathbb{R}$$

On the other hand let us take any $V, W \in Sec(\mathcal{C})$ as previously, while defining η, and calculate

$d\eta(V,W) = V\cancelto{0}{\langle \eta, W\rangle} - W\cancelto{0}{\langle \eta, V\rangle} - \langle \eta, [V,W]\rangle = \langle \eta, \nu(v,w)\rangle$ up to a sign $d\eta$ coincides with ν when we

Trivialize both $\mathcal{C}^0$ and $TM/\mathcal{C}$ using η. One can then see that the condition of maximal nonintegrability coincides with the condition of $d\eta$ being nondegenerate on $\mathcal{C}$ which can encoded indeed in $(*)$

There are several useful notions related to contact geometry that are traditionally defined in the language of contact forms. Before we discuss them we should probably write one important theorem

THEOREM (Darboux theorem for contact forms) Let η be a contact form. For every point $x \in M$ there exists a neibourhood $\mathcal{O}$ and coordinates (q^i, p_j, z) in $\mathcal{O}$ such that $\eta = dz - p_i dq^i$.

As a consequence, there is also a normal form of a contact distribution, which in Darboux coordinates is given as

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_j} \right\rangle \quad \text{i.e. as on jet space.}$$

Once we have a contact form we can define a Reeb vetor field R_η. It is uniquely defined by the following two conditions

$$R_\eta \lrcorner\, d\eta = 0 \qquad R_\eta \lrcorner\, \eta = 1.$$ It is easy to see that in Darboux coordinates $R_\eta = \frac{\partial}{\partial z}$

Using η and $d\eta$ we can also define contact Hamiltonian vector fields, i.e. associate a vector field to every function on M. For η defined on $\mathcal{O} \subset M$ and for $H \in C^\infty(\mathcal{O})$ we have $X_H \in \mathfrak{X}(\mathcal{O})$ such that

$$X_H \lrcorner\, \eta = -H \qquad X_H \lrcorner\, d\eta = dH - R_\eta(H)\eta$$

An important property of a symplectic Hamiltonian vector field is that it preserves the structure it comes from. Let us then calculate $\mathcal{L}_{X_H}\eta$ to see how things look in our case

$$\mathcal{L}_{X_H}\eta = d\, i_{X_H}\eta + i_{X_H} d\eta = -dH + dH - R_\eta(H)\eta = -R_\eta(H)\eta$$

Contact Hamiltonian vector field does not conserve the contact form, however the change is proportional to the contact form. This shows that the such a field conserves the contact distribution

$$Y \in Sec(\mathcal{C}) \qquad \mathcal{L}_{X_H} Y = [X_H, Y] \in Sec(\mathcal{C})$$

$$d\eta(X_H, Y) = X_H \underbrace{\cancel{\langle \eta, Y\rangle}}_{0} - Y\langle \eta, X_H\rangle - \eta([X_H, Y]) \qquad \langle \eta, X_H\rangle \to Y(H)$$

$$\| \quad \langle dH, Y\rangle - R_\eta(H)\underbrace{\cancel{\langle \eta, Y\rangle}}_{0}$$

$$\cancel{Y(H)} = \cancel{Y(H)} - \langle \eta, [X_H, Y]\rangle \Rightarrow \langle \eta, [X_H, Y]\rangle = 0$$

The problems we have with the definition of a contact Hamiltonian vector fields are the following:

1. this definition is local! what if there are no global contact forms? can we have global contact vector fields? (YES!)

2. Do we get the same vector field if we keep the Hamiltonian and change η? (NO!) what is the relation between Hamiltonians and vector fields with different contact forms?

3. We have checked that $\mathcal{L}_{X_H}\eta \neq 0$, but X_H preserves $\mathcal{C}$. Since contact Hamiltonian vector fields are symmetries of the contact distribution and not contact form we should be able to define them somehow without using any contact forms. Can we do this? (YES!)

The way to address these problems will be to look at them from a totally different point of view. To this end I will introduce now a new concept and point out its relation to contact geometry.

DEFINITION: Let $\mathbb{R}^\times$ denote a multiplicative group of non-zero reals, i.e $\mathbb{R}^\times = (\mathbb{R}\setminus\{0\}, \cdot)$.
A symplectic principal bundle is an $\mathbb{R}^\times$-principal bundle together with a homogeneous symplectic form defined on a total space of the total space of the bundle

Notation:

$$\begin{array}{c} P \circlearrowleft \mathbb{R}^\times \\ \tau \downarrow \\ M \end{array} \qquad h: \mathbb{R}^\times \times P \longrightarrow P \qquad h_t : P \to P \qquad \omega \in \Omega^2(P) \ \ \omega \text{ is nondegenerate and closed}$$

$$h(t, h(s,p)) = h(ts, p) \qquad h_t \circ h_s = h_{ts} \qquad h_t^* \omega = t\omega$$

ω is homogeneous of order 1.

Example: We have seen an example of contact manifold being a projective cotangent bundle. Our associated example of a symplectic principal bundle would be $(T^*Q)^\times = (T^*Q)\setminus\{0_Q(Q)\}$ i.e. a cotangent bundle with zero section removed as a bundle over the projective cotangent bundle. As $\mathbb{R}^\times$ action we take just multiplication by non-zero reals. We can easily check that ω_Q is homogeneous:

$$h_t^* \omega_Q = d(tp_i) \wedge dq^i = t\, dp_i \wedge dq^i = t\omega_Q$$

Looking at this example you probably can guess what to expect - a symplectic principal bundle structure on P induces a contact structure on the base manifold. There is more - every contact structure in a sense $(M, \mathcal{C})$ has a symplectic principal bundle associated. It appears that they are in fact equivalent notions! We will study this correspondence now.

Every contact $(M, \mathcal{C})$ defines (P, M, τ, h, ω)

The symplectic principal bundle corresponding to a given contact $(M, \mathcal{C})$ can be built with the ingredients we already used. Let then M be of dimension $2n+1$ and then $\mathcal{C}$ is of rank $2n$. As previously we consider $\mathcal{C}^\circ$ and denote $P = (\mathcal{C}^\circ)^\times$ which is an anihilator of $\mathcal{C}$ with removed zero section. It is a submanifold of T^*M, as a bundle over M it is of rank one, as a submanifold it is of dimension $2n+2$. It is also invariant with respect to multiplication by non-zero real numbers; elements of P are covectors, therefore they can be multiplied by numbers.

$P = (\mathcal{C}^\circ)^\times$

$$\begin{tikzcd} P \arrow[d, "\tau"'] \arrow[r, phantom, "\circlearrowleft"] & \mathbb{R}^\times \\ M & \end{tikzcd}$$

$\pi_M|_P$

multiplication of covectors

The element we do not have yet is a symplectic structure. To this end we have the following proposition:

PROPOSITION: P is a symplectic submanifold of of T^*M.

PROOF: Restriction of ω_M to P can be viewed as a pull-back by the inclusion. Moreover, locally we can choose a contact form η and trivialize P over some open subset $\mathcal{O}$ of M.

We have then $I_\eta : \mathcal{O} \times \mathbb{R}^\times \ni (x, s) \longmapsto s\eta \in P \subset T^*M$. Let us examine $I_\eta{}^* \omega_M$:

$$\omega_\eta = I_\eta^* \omega_M = I_\eta^* (d\theta_M) = d(I_\eta^* \theta_M) = d\left((s\eta)^* \theta_M\right) = d(s\eta) = ds \wedge \eta + s\, d\eta$$

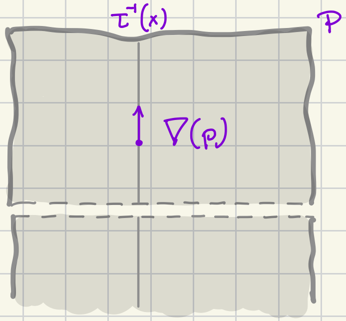
this form is clearly closed as a differential of something. To check nondegeneracy we calculate

$$(\omega_\eta)^{\wedge n+1} = (ds \wedge \eta + s\,d\eta)^{n+1} = ds \wedge \eta \wedge (s\,d\eta)^{\wedge n} + \underbrace{s^{n+1} (d\eta)^{\wedge n+1}}_{=0} = s^n ds \wedge \underbrace{\eta \wedge (d\eta)^{\wedge n}}_{\neq 0 \text{ because } \eta \text{ is contact}} \neq 0$$

We conclude that ω_η is a symplectic form on $\mathcal{O} \times \mathbb{R}^\times$. It is a local expression of $\omega_M|_P$ in a trivialization provided by η. This finishes the proof and the whole construction.

Note that if η is global it is customary to consider $P_0 = M \times \mathbb{R} \ni (x, t)$ $\quad \omega = d(e^t \eta) = e^t d\eta + e^t dt \wedge \eta$
P_0 is symplectic and isomorphic to „a positive part" of $P = M \times \mathbb{R}^\times$, i.e. for $s > 0$ via $t \longmapsto s = e^t$.

Every (P, M, τ, h, ω) defines $(M, \mathcal{C})$



Now we shall go the other way round and look for a contact structure associated to a given $\mathbb{R}^\times$ symplectic principal bundle.

Let ∇ be a vertical vector field

$$\nabla(p) = \frac{d}{ds}\Big|_{s=1} h_s(p) = \frac{d}{dt}\Big|_{t=0} h_{e^t}(p)$$

∇ is invariant with respect to h: $\nabla(h_t(p)) = \frac{d}{ds}\Big|_{s=1}\left(h_s(h_t(p))\right) = \frac{d}{ds}\Big|_{s=1} h_t(h_s(p)) = Th_t(\nabla(p))$

homogeneous

$\theta = i_\nabla \omega$

invariant

homogeneous

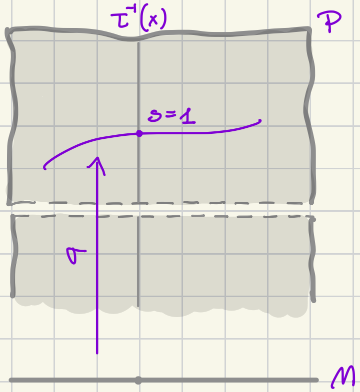
PROPOSITION $\omega = d\theta$, i.e. ω is exact.

PROOF We know that ω is homogeneous, which means $h_s^*\omega = s\omega$ $\mathcal{L}_\nabla \omega = \omega$

$$\omega = \mathcal{L}_\nabla \omega = d(i_\nabla \omega) + \cancel{i_\nabla d\omega} = d(\theta) \quad \square$$

Now let us choose a local section of $P \xrightarrow{\tau} M$. It provides P with a local trivialization $\tau^{-1}(\mathcal{U}) = \mathcal{U} \times \mathbb{R}^\times$ and a local vertical coordinate s. Using this coordinate we can write $\nabla(p) = \frac{\partial}{\partial s}$. We can also define a one form θ / s. Since θ is homogeneous we know that θ/s is invariant. It means that it is a pull-back of some one form from the base:

$$\frac{1}{s}\theta = \tau^*\eta, \quad \eta \in \Omega^1(\mathcal{U})$$



PROPOSITION η is a local contact form.

PROOF:

σ gives a local trivialization of P: $I_\sigma : \mathbb{R}^\times \times \mathcal{U} \ni (s, x) \longmapsto h_s(\sigma(x)) \in P$

By definition $I_\sigma^*(\theta) = s\cdot\eta$

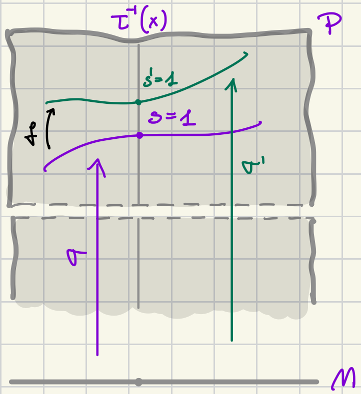
$$I_\sigma^*(\omega) = I_\sigma^*(d\theta) = d(s\eta) = ds \wedge \eta + s\, d\eta$$

Now we use the nondegeneracy argument the other way round: $\omega^{\wedge(n+1)} \neq 0$ because ω is symplectic. I_σ is a diffeomorphism therefore $0 \neq (ds\wedge\eta + s\,d\eta)^{\wedge(n+1)} = s^n\, \underbrace{ds \wedge \eta \wedge (d\eta)^{\wedge n}}_{\neq 0}$ therefore η is a contact form.

Our candidate for $\mathcal{C}$ is now $\ker\eta$, but to be sure that we have a well defined global distribution we still need another proposition:

PROPOSITION: Let η_σ be defined as above with the use of a section σ. $\ker\eta$ does not depend on the choice of σ.

PROOF: Let us examine the difference between η_σ and $\eta_{\sigma'}$. We have



$\tau^*\eta_\sigma = \frac{1}{s}\theta \qquad \tau^*\eta_{\sigma'} = \frac{1}{s'}\theta$ and $\sigma'(x) = f(x)\sigma(x)$ for some nonvanishing function f defined on the intersection of the domains of σ and σ'

$$s(p) = f(\tau(p))\, s'(p)$$

Since $\frac{1}{s}\theta = \frac{1}{s'f}\theta = \frac{1}{f}\frac{1}{s'}\theta$ we have that $\eta_\sigma = \frac{1}{f}\eta_{\sigma'}$ or $f\eta_\sigma = \eta_{\sigma'}$. It is then clear that both forms define the same distribution $\mathcal{C}$. □

We can look at $\mathcal{C}$ as a projection of the kernel of θ. It contains a vertical direction i.e. $\langle \nabla(p)\rangle$, moreover, since θ is homogeneous, the kernel is invariant. It can then be projected to a constant rank distribution on M. The fact that this distribution is contact follows from the local considerations above.

For the completness of our presentation we should now look for the $\mathbb{R}^\times$ symplectic principal bundle in our examples. We have seen already that in Example ③ we have just $P = (T^*Q)^\times$ over $M = (T^*Q)^\times/\sim$. Both other examples we shall treat together by showing that there is a canonical contact structure on the total space of a first jet bundle of a line bundle. Examples ① and ② both belong to this category.

J^1L^*

Let now $\varrho: L \to M$ denote a line bundle, which is a rank one vector bundle over a manifold M. By L^* we shall denote the dual line bundle and by $L^\times$ – L with zero section removed. Note that L is an $\mathbb{R}^\times$ principal bundle with respect to multiplication by reals 'borrowed' from the underlying vector bundle.

Our main object of interest will now be $T^*L^\times$. We will show that it is a principal $\mathbb{R}^\times$ symplectic bundle with the underlying contact geometry being that of J^1L^*. To clarify matters we shall need local coordinates.

Let then (x^i, t) be adapted coordinates on L with t being a fibre linear coordinate. The same coordinates can be used in $L^\times$ with the condition $t \neq 0$. Then we proceed with constructing coordinates $(x^i, t, \dot{x}^i, \dot{t})$ in $TL^\times$. If we remove zero section from L we obtain, instead of a line bundle, an $\mathbb{R}^\times$ principal bundle with an action denoted by h_s, $s \neq 0$

$$(x^i, t) \circ h_s = (x^i, st)$$

This action can be lifted to $TL^\times$ and $T^*L^\times$. The lift to $TL^\times$ is just a tangent map:

$$(x^i, t, \dot{x}^i, \dot{t}) \circ Th_s = (x^i, st, \dot{x}^i\, s\dot{t})$$

since h_s is a diffeomorphism, we can consider T^*h_s as a map. The adapted coordinates on $T^*L^\times$ are (x^i, t, p_j, z). We have then

$$(x^i, t, p_j, z) \circ T^*h_s = (x^i, \tfrac{1}{s}t, p_j, sz) \quad \text{and for } \tfrac{1}{s} \quad (x^i, t, p_j, z) \circ T^*h_{\frac{1}{s}} = (x^i, st, p_j, \tfrac{1}{s}z)$$

The action we shall actually need is $T^*h_{\frac{1}{s}}$ composed with multiplication by s in the bundle $T^*L^\times \longrightarrow L^\times$. The resulting map will be denoted by $d_{T^*}h_s$

$$(x^i, t, p_j, z) \circ d_{T^*}h_s = (x^i, st, sp_j, z) \qquad (d_{T^*}h)_s = sT^*h_{\frac{1}{s}}$$

We have now the action $d_{T^*}h$ on $T^*L^\times$. We can easily check, that the canonical symplectic form $\omega_{L^\times}$ which is there, because it is a cotangent bundle is actualy homogeneous with respect to this action. The easiest way is to check it in coordinates:

$$\omega_{L^\times} = dp_i \wedge dq^i + dz \wedge dt \qquad (d_{T^*}h_s)^* \omega_{L^\times} = d(sp_i) \wedge dq^i + dz \wedge d(st) = s(dp_i \wedge dq^i + dz \wedge dt) = s\omega_{L^\times}$$

The same thing may of course be done globaly: we can use the definition of the Liouville form $\theta_{L^\times}$ and show that it is homogeneous. Then $\omega_{L^\times} = d\theta_{L^\times}$ is also homogeneous. Now we have the following ingredients:

$$(T^*L^\times, ?, ?, d_{T^*}h, \omega_{L^\times})$$

the homogeneous symplectic form

the total space of a bundle

the $\mathbb{R}^\times$ action

What is missing is a base manifold and projection. Since the title of this paragraph is J^1L^* we shall now argue that $T^*L^\times/\mathbb{R}^\times \simeq J^1L^*$

PROPOSITION: $T^*L^\times/\mathbb{R}^\times \simeq J^1L^*$

Let us consider the natural correspondence between sections of L^* and homogeneous functions on $L^\times$:

ON L^*

$$\begin{tikzcd} L^* \arrow[d] \\ M \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

J^1L^*: If the two sections σ_1, σ_2 have the same jet at $x \in M$ it means that they differ by multiplication by a function $\lambda : M \longrightarrow \mathbb{R}$ such that $d\lambda(x) = 0$, $\lambda(x) = 1$

$$j^1\sigma_1(x) = j_1\sigma_2(x) \iff \sigma_1(y) = \lambda(y)\sigma_2(y)$$

for y around x.

(2) every homogeneous function corresponds to some section

ON $L^\times$ AND $T^*L^\times$

To every (local) section of $L^* \to M$ there corresponds a homogeneous function f_σ on $L^\times$

$$f_\sigma(\ell) = \langle \sigma(\tau^*(\ell)), \ell \rangle$$

$$\sigma_1(y) = \lambda(y)\sigma_2(y) \qquad f_{\sigma_1}(\ell) = \langle \sigma_1(y), \ell \rangle = \langle \lambda(y)\sigma_2(y), \ell \rangle = \lambda(y) f_{\sigma_2}(\ell)$$

$$f_{\sigma_1} = \lambda f_{\sigma_2}$$

For any ℓ over x we get

$$df_{\sigma_1}(\ell) = d(\lambda f_{\sigma_2})(\lambda) = f_{\sigma_2}(\ell)\underbrace{d\lambda(x)}_{=0} + \underbrace{\lambda(x)}_{1} df_{\sigma_2}(\ell) = df_{\sigma_2}(\ell)$$

Moreover $d_{T^*}h_s(df_\sigma(\ell)) = df_\sigma(h_s(\ell))$ which means that there is a map from J^1L^* to $T^*L^\times/\sim$.

This map is one-to-one, because (1) every covector $\alpha \in T^*L^\times$ is a differential of a homogeneous function:

$$\alpha(x_0^i, t_0) = \alpha_i dx^i + \alpha dt \qquad f(x^i, t) = t\left(\frac{\alpha_i}{t_0}(x^i - x_0^i) + \alpha\right)$$

$$\alpha(x_0, t_0) = df(x_0, t_0)$$

$\square$

CONTACT HAMILTONIAN MECHANICS

The idea of using contact structure in mechanics goes back to Gustav Herglotz (1881-1953) and was recently extensively developed by the Spanish geometric mechanics group. It was devised to deal with dissipative systems which does not fit into the scheme of symplectic mechanics where a Hamiltonian is allways constant on trajectories, therefore, if Hamiltonian represents the energy, there cannot be any dissipation. On the other hand, Hamiltonian mechanics have many advantages: the fact that a certain differential equation has symplectic Hamiltonian origin allows for gaining some knowledge about the solutions even if we are not able to solve the equations explicity. One may also use the so called symplectic integrators to numerically solve equations with bounded numerical error even in long time solutions.
The idea of looking for another geometric structure that can be appropriate for wider class of system is not so stupid then, even if in principle we can write down equations using the concept of an external force.

As we have mentioned before, contact Hamiltonian mechanics is expressed in terms of contact forms and Reeb vector fields. Let us spell out the equations and properties once more and then, as usual, look for problems:

Let $(M, \mathcal{C})$ be a contact manifold with a contact form η. We can then define a Reeb vector field R_η associated to η by the following conditions:

$R_\eta \lrcorner d\eta = 0, \; R_\eta \lrcorner \eta = 1$. As you can see Reeb vector field does not belong to $\mathcal{C}$. The definition strongly depends on η. One can even find, that for every vector $v \in T_xM$, $v \notin \mathcal{C}$ we can find a local contact form η such that $R_\eta(x) = v$. Changing η means dramatically changing the Reeb vector field. If we use Darboux coordinates for η, i.e. $\eta = dz - p_i dq^i$ then $R_\eta = \frac{\partial}{\partial z}$.

Using η and R_η we can define the Hamiltonian vector field X^c_H for every smooth function H on M:

$$H : M \longrightarrow \mathbb{R}, \quad X^c_H \in \mathfrak{X}(M): \quad X^c_H \lrcorner d\eta = dH - R_\eta(H)\eta \quad X^c_H \lrcorner \eta = -H$$

Let us calculate the Hamiltonian contact vector field X^c_H for a given Hamiltonian function in Darboux coordinates:

$$\eta = dz - p_i dq^i, \quad R_\eta = \frac{\partial}{\partial z}, \quad H(q^i, p_j, z) \qquad X^c_H = A^i \frac{\partial}{\partial q^i} + B_j \frac{\partial}{\partial p_j} + D\frac{\partial}{\partial z}$$

$\eta = dz - p_i dq^i$, $R_\eta = \frac{\partial}{\partial z}$, $\mathcal{H}(q^i, p_j, z)$ $\qquad X^c_{\mathcal{H}} = A^i \frac{\partial}{\partial q^i} + B_j \frac{\partial}{\partial p_j} + D\frac{\partial}{\partial z}$

$d\eta = dq^i \wedge dp_i$ $\qquad X^c_{\mathcal{H}} \lrcorner \eta = D - p_i A^i = -\mathcal{H}$

$$X^c_{\mathcal{H}} \lrcorner d\eta = A^i dp_i - B_j dq^j = \frac{\partial \mathcal{H}}{\partial q^j} dq^j + \frac{\partial \mathcal{H}}{\partial p_i} dp^i + \frac{\partial \mathcal{H}}{\partial z} dz - \left(\frac{\partial \mathcal{H}}{\partial z}\right)(dz - p_j dq^j)$$

$$A^i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad B_j = \frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z} \qquad D = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}$$

$$X^c_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} + \left(\frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z}\right) \frac{\partial}{\partial p_j} + \left(p_k \frac{\partial \mathcal{H}}{\partial p_k} - \mathcal{H}\right) \frac{\partial}{\partial z}$$

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$\dot{p}_j = \frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z}$$
$$\dot{z} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}$$

As we can see, if $\mathcal{H}$ does not depend on z, we get usual Hamilton equations for q and p plus something we can solve for z provided the previous two are solved. For $M = T^*Q \times \mathbb{R}$ this gives a resonable generalization of the symplectic case. In this case we have the projection from M to T^*Q, therefore we can get back the physical dynamics once we have solved the equations. Real mechanical applications always live on $T^*Q \times \mathbb{R}$, at least to my knowledge. Since η is in this case global, then also $X^c_{\mathcal{H}}$ is global. In general however we have the following problems with Hamiltonian contact dynamics:

(1) Hamiltonian contact vector fields are local (as defined here) (2) There is difficult relationship between Hamiltonians and vector fields: if we want to keep the field fixed but change the form by multiplying it by a nonvanishing function, we have to multiply the Hamiltonian by the same function. It may not be seen as a problem but (2a) on non-orientable manifolds there may not be a global Hamiltonian for a given vector field that is locally Hamiltonian (2b) Try to prove (2) by direct calculations and you will be buried under sheets of paper. This is because the nice formula we have for $\mathcal{H}$ depend on Darboux coordinates. If we change η, we have to change coordinates and everything gets messy. We can avoid using coordinates and do it, but still in traditional language it is an unpleasant job that you would rather delegate to students then do by yourself. (3) From the point of view of numerical methods this whole job of contact Hamiltonian mechanics does not seem useful, because neither Hamiltonian nor contact form are not conserved along the trajectory, therefore on the first sight we have nothing to build our geometric integrator on. Geometric integrators are usually based on the idea that some geometric structure do not change along the solution. (4) Contact Hamiltonian vector fields do preserve

do preserve a contact structure, which can be seen in the following way: locally $\mathcal{C} = \ker \eta$. Let us then take $Y \in Sec(\mathcal{C})$, then $\langle \eta, Y \rangle = 0$ therefore $\mathcal{L}_{X_H} \langle \eta, Y \rangle = 0$

$$0 = \mathcal{L}_{X^c_H} \langle \eta, Y \rangle = \underbrace{\langle \underbrace{\mathcal{L}_{X^c_H} \eta}_{\sim \eta}, Y \rangle}_{=0} + \langle \eta, [X^c_H, Y] \rangle = \langle \eta, [X^c_H, Y] \rangle \Rightarrow [X^c_H, Y] \in \mathcal{C}$$

This shows that contact Hamiltonian vector fields are proper symmetries of the contact structure therefore there should be a possibility to generate them somehow using global objects related to $\mathcal{C}$. We shall now approach contact Hamiltonian vector fields from the point of view of principal symplectic $\mathbb{R}^\times$-bundles.

Since (P, ω) is a symplectic manifold there is there of course the procedure of generating Hamiltonian vector fields there. The point is now to use homogeneous Hamiltonians.

$$\begin{tikzcd}
T^*P \arrow[rr, "\omega^\flat"] \arrow[d] & & TP \arrow[dd, "T\tau"] \\
P \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_\mathcal{H}"] \arrow[d, "\tau"] & & \\
M \arrow[rr, "X^c_H"] & & TM
\end{tikzcd}$$

$\mathcal{H}: P \longrightarrow \mathbb{R}$ homogeneous, which means $\mathcal{H}(h_s(p)) = s\mathcal{H}(p)$

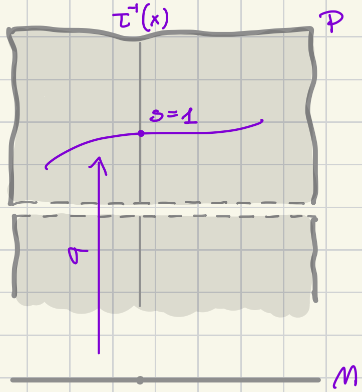
$$\omega(X_\mathcal{H}, \cdot) = d\mathcal{H}$$

homogeneous

invariant, i.e. projectable

homogeneous

Hamiltonian vector fields for homogeneous Hamiltonians are invariant with respect to h_s, therefore they are projectable on M. Let us check how they are related to Hamiltonian contact vector fields on M.



For that we need a picture we have already had, which is a local section σ of P and the associated vertical coordinate s. We then know that the symplectic form ω can be expressed as

$$\omega = ds \wedge \eta + s d\eta$$

Any homogeneous Hamiltonian can then be written as $\mathcal{H}(s,x) = sH(x)$ for $H(x) = \mathcal{H}(\sigma(x))$.
Let us now look for $X_\mathcal{H}$ in this setting:

$$d\mathcal{H} = H(x)ds + s dH(x) = \omega(X_\mathcal{H}, \cdot)$$

$X_\mathcal{H}$ is invariant, therefore it is of the form $X_\mathcal{H} = sF(x)\frac{\partial}{\partial s} + Y(x)$

a function on M

a vector field on M

$$d\mathcal{H} = H(x)ds + s\,dH = \omega(\mathcal{H}_{H},\cdot) = sF(x)\eta - \langle\eta, Y\rangle ds + s\, i_Y d\eta = -\langle\eta,Y\rangle ds + s\left(F(x)\eta + i_Y d\eta\right)$$

Comparing terms with the same colour we get the following $\langle\eta,Y\rangle = -H$, $i_Y d\eta = dH - F(x)\eta$. Function F can be found by contracting both sides with Reeb vector field R_η:

$$\underbrace{d\eta(Y,R_\eta)}_{=0} = R_\eta(H) - F(x)\underbrace{\langle\eta,R_\eta\rangle}_{=1} \Rightarrow F(x) = R_\eta(H)$$

Summarizing, conditions for Y are $\langle\eta,Y\rangle = -H$, $i_Y \lrcorner d\eta = dH - R(H)\eta$ which are precisely the conditions for the contact vector field for Hamiltonian $H(x) = \mathcal{H}(\tau(x))$ defined for the form η. Contact Hamiltonian vector fields are therefore projections of usual Hamiltonian vector fields for homogeneous Hamiltonians. In terms of doing numerical calculation it is a very good message – it means that we have to transform the contact Hamiltonian problem to a symplectic one „upstairs", solve the equation using our favourite symplectic integrators and then just forget the unnecessary part of the solution. There are also ways to write this problem in a variational way and use variational integrators. Let us now spend a few moments on the subject of generating objects of contact Hamiltonian vector fields.

If we are happy enough with a function which is defined on an $\mathbb{R}^*$-principal bundle as a generating object for a hamiltonian contact vector field we may leave it at that. For the purposes of building a Lagrangian approach to contact dynamics we might want to know a generating object which „lives" down on the contact manifold itself. We have already dissussed the passage from line bundles to principal bundles that has the following.

Starting from a line bunde $L \xrightarrow{\tau} M$ we build an $\mathbb{R}^\times$-principal bundle by removing the zero section: $P = L^\times \to M$. We need also the dual bundle $L^* \to M$ because then we can have a correspondence between homogeneous functions on $P = L^\times$ and sections of $L^* \longrightarrow M$. Our generating object for for a Hamiltonian contact vector field would then be a section of the appropriate line bundle. It woud be then useful to know how to build this bundle, provided we know P, i.e. „how to stick in the missing zero. The answer is relatively simple: one has to use the associated bundle construction:

From $P \to M$ we pass to $L_P = P\times\mathbb{R}/_{\mathbb{R}^\times}$ using the following action $s\cdot(p,r) = (h_s(p), \frac{r}{s})$. The dual L_P^* can also be viewed as an associated bundle by the action $s\cdot(p,z) = (h_s(p), sz)$. There is a canonical identification of $L_P^\times$ with P. The image of the zero section is composed of equivalence classes of the form $[(p,0)] = \{(p',0) : \tau(p') = \tau(p)\}$. In each equivalence class different from the zero section there is a representative with second element equal to 1. We have then the map $P \ni p \longmapsto [(p,1)] \in L_P^\times$. The action of $\mathbb{R}^\times$ on $L_P^\times$ reads $s\cdot[(p,r)] = [(h_s(p), r)] = [(p, sr)]$. Let now p, q be elements of P over the same point $x \in M$. Then there exists s_0 such that $p = h_{s_0}(q)$. The evaluation between $[(p,r)]$ and $[(q,z)]$ reads then:

$$\langle [(q,z)], [(p,r)]\rangle = z r s_0 \qquad [(q,z)] = [h_t(q), tz] \quad p = h_{s_0} h_{\frac{1}{t}} h_t(q) = h_{\frac{s_0}{t}}(h_t(q))$$

$$\langle [(h_t(q), tz)], [p,r]\rangle = \cancel{t} z r \frac{s_0}{\cancel{t}} = z r s_0$$

WELL DEFINED

$$\langle [(q,z)], s[(p,r)] \rangle = \langle [(q,z)], [(p,sr)] \rangle = zsrs_0 = s(zrs_0).$$

EXAMPLE: Let us now analyze the passage from homogeneous Hamiltonian and its vector field and a contact Hamiltonian and a contact Hamiltonian vector field in case of a mechanical example, i.e. $M = J^1L^*$ or even more precisely $M = T^*Q \times \mathbb{R}$, which is always a local picture of J^1L^*

$$\begin{tikzcd}
T^*P \arrow[r, "\omega^b"] \arrow[d] & TP \arrow[dd] \\
P \arrow[u, bend left, "dH"] \arrow[ur, "X_{\mathcal{H}}"] \arrow[d] & \\
M \arrow[r, "X_H^c"] & TM
\end{tikzcd}$$

The general picture, specified for J^1L^* is the following.

We shall work with the assumption that $L = Q \times \mathbb{R}$. In the following we shall introduce coordinates, $\mathbb{R}^\times$ action and all the elements of the structure:

$$\begin{tikzcd}
T^*T^*L^\times \arrow[r, "\omega^b_{L^\times}"] \arrow[d] & TT^*L^\times \arrow[dd] \\
T^*L^\times \arrow[u, bend left, "d\mathcal{H}"] \arrow[ur, "X_{\mathcal{H}}"] \arrow[d] & \\
J^1L^* \arrow[r, "X_H^c"] & TJ^1L^\times
\end{tikzcd}$$

$L = Q \times \mathbb{R} \quad (q^i, t) \qquad L^\times = M \times \mathbb{R}^\times \quad (q^i, t) \; t \neq 0$

$L^* = Q \times \mathbb{R}^* \quad (q^i, z)$

$T^*L^\times = T^*(Q \times \mathbb{R}^\times) \simeq T^*Q \times \mathbb{R}^\times \times \mathbb{R}^* \quad (q^i, \pi_j, t, z)$

$(q^i, t) \circ h_s = (q^i, st)$

$(q^i, \pi_j, t, z) \circ d_{T^*} h_s = (q^i, s\pi_j, st, z)$

$J^1L^* \simeq T^*Q \times \mathbb{R} \quad (q^i, p_j, z)$

$$\omega_{L^\times} = dq^i \wedge d\pi_i + dt \wedge dz = dq^i \wedge d(tp_i) + dt \wedge dz =$$
$$= dq^i \wedge (p_i dt + t dp_i) + dt \wedge dz =$$
$$= dt \wedge \underbrace{(dz - p_i dq^i)}_{\eta} + t \underbrace{dq^i \wedge dp_i}_{d\eta}$$

projection $T^*L^\times \xrightarrow{\tau} J^1L^*$ reads in coordinates $(q^i, p_j, z) \circ \tau = (q^i, \frac{\pi_j}{t}, z)$

The relation between the homogeneous Hamiltonian and the Hamiltonian on M:

$$\mathcal{H}(q^i, \tau, \pi_j, z) = \tau H(q^i, \underbrace{\frac{\pi_j}{\tau}}_{p_j}, z)$$

$T^*L^\times = T^*Q \times \mathbb{R}^\times \times \mathbb{R} \qquad T^*Q \times \mathbb{R}$

On $T^*Q \times \mathbb{R}^\times \times \mathbb{R}$
$$X_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \tau \frac{\partial H}{\partial z}\frac{\partial}{\partial \tau} - \tau \frac{\partial H}{\partial q^j}\frac{\partial}{\partial \pi_j} + \left(\frac{\pi_k}{\tau}\frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

On $T^*Q \times \mathbb{R}$
$$X^c_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^j} + p_j \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_j} + \left(p_k \frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

There are two practical examples we may want to play with:

PRACTICAL EXAMPLE: VISCOSITY FORCE

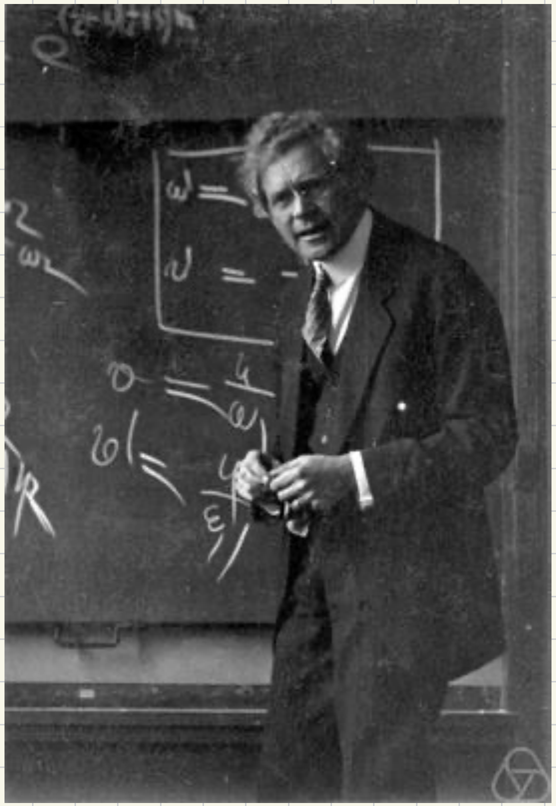
$M = T^*Q \times \mathbb{R}$ $\quad \eta = dz - \theta_Q \quad H(p,z) = H_0(p) - \lambda z$

$$X_H^C = \underbrace{\frac{\partial H_0}{\partial p_i}}_{\dot q}\frac{\partial}{\partial q^i} - \underbrace{\left(\frac{\partial H_0}{\partial q^i} - \lambda p_i\right)}_{\dot p}\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H_0}{\partial p_i} - H_0 + \lambda z\right)\frac{\partial}{\partial z}$$

$$\dot q^i = \frac{\partial H}{\partial p^i}$$

$$\dot p_j = -\frac{\partial H_0}{\partial q^j} + \lambda p_j$$

$$\dot z = \underbrace{p_i \frac{\partial H_0}{\partial p_i} - H_0}_{L_0} + \lambda z$$



Gustav Herglotz
1881 - 1953

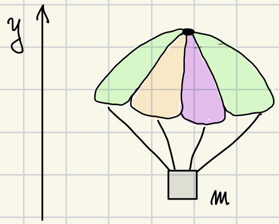
PRACTICAL EXAMPLE: PARACHUTE EQUATION

$$\dot y = \frac{1}{m}(p - \gamma z)$$

$$\dot p = -\frac{\partial V}{\partial y} + \frac{\gamma p^2}{m} - \frac{\gamma^2 p z}{m}$$

$$\dot z = \underbrace{\frac{p}{m}(p - \gamma z) - \mathcal{H}}_{L(y,\dot y) = \frac{m}{2}\dot y^2 + \gamma \dot y z - V(y)}$$

The Herglotz Lagrangian



$Q = \mathbb{R}$
$M = T^*Q \times \mathbb{R}$
(y, p, z)

$$\boxed{\ddot y - \gamma m \dot y^2 + g = 0}$$

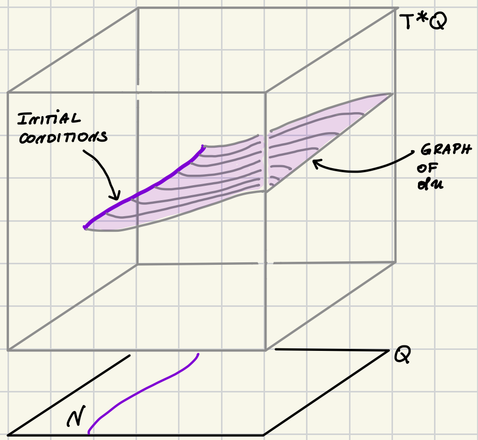
$$\mathcal{H}(y,p,z) = \frac{1}{2m}(p - \gamma z)^2 + V(y) \qquad V(y) = \frac{mg}{\gamma}\left(e^{\gamma y} - 1\right)$$

CONTACT HAMILTON-JACOBI THEORY

Hamilton-Jacobi equation is a classical part of symplectic mechanics. Before we pass to the contact case, I would like us to review the symplectic version from the geometric point of view. We may look at the relation between a Hamilton-Jacobi equation and a Hamilton equation in two ways: (1) We may want to solve a partial differential equation of the first order by means of a certain ordinary differential equation or the other way round: (2) we may want to solve a difficult ordinary differential equation using possibly simpler partial differential equation. Disregarding technical difficulty for a while, we shall look at the geometry of both problems.

(1) From a PDE to a Hamiltonian ODE

Using a symplectic structure we can deal with a special kind of a partial differential equations of a first order - namely those that do not involve values of unknown function. In traditional notation such a PDE for one function u of several variables $(q^1 \dots q^n)$ can be written as $F(q^i, u_i)=0$ where u_i denotes a partial derivative $\partial u/\partial q^i$. In more geometric language we would understand this equation as a condition for the differential of a function.
This can be formulated for any manifold Q. We define a submanifold $K \subset T^*Q$ by the condition $K = F^{-1}(0)$ for a certain $F: T^*Q \longrightarrow \mathbb{R}$ and say that $u \in C^1(Q)$ is a solution of the equation if at each point $du(q) \in K$.
Note that K, as a submanifold of codimension 1 (there are conditions for F of course) is a coisotropic submanifold of T^*Q. This means that it carries a one-dimensional characteristic distribution. Characteristic distributions of coisotropic submanifolds are involutive. Here it is of course automatic since the distribution is one dimensional. This means that K foliated by one dimensional submanifolds called characteristics. The characteristic distribution is spanned by the Hamiltonian vector field X_F of a function defining K. Now, if u is a solution of the equation then the image $du(Q)$ is a submanifold in K. Moreover, it is a Lagrangian submanifold of T^*Q, therefore it must be composed of leaves of the characteristic foliation. Assuming we can find the trajectories of X_F we may solve the PDE in the following way:



Our boundary conditions should consists of value of u on a submanifold $N \subset Q$ of codimension 1 and a value of differentials at points of N. Then we can calculate characteristics starting from boundary values and get the Lagrangian submanifold being the graph of du. Function u itself can be found by integrating $\theta_M = p_i dq^i$ along the characteristics.

② From Hamiltonian ODE to PDE.

Now we are looking for the solutions of Hamiltonian equations for a given Hamiltonian $H: T^*Q \longrightarrow \mathbb{R}$. Let us assume that we can find a solution of the Hamilton-Jacobi equation $H(dS) = E$. In coordinates we have $H(q^i, \frac{\partial S}{\partial q^i}) = E$. The image $dS(Q)$ is a Lagrangian submanifold contained in a coisotropic submanifold $H^{-1}(E)$. In particular the Hamiltonian vector field is tangent to $dS(Q)$. Knowing dS we can consider a vector field X_Q defined on Q and such that it is dS-related with X_H, i.e.

$$T\,dS(X_Q(q)) = X_H(dS(q))$$ This way we have to integrate a vector field X_Q with half the variables we initially had. The solutions may then be lifted to T^*Q by means of dS.

Both procedures are based on symplectic Hamilton-Jacobi theorem that can be formulated as follows:

THEOREM Let (P, ω) be a symplectic manifold $H: P \longrightarrow \mathbb{R}$ be a smooth function and $L \subset P$ be a Lagrangian submanifold. Then X_H is tangent to L if and only if H is (locally) constant on L.

The procedures 1. and 2 from the Hamilton-Jacobi story are just clever applications of this quite simple symplectic theorem.

The symplectic constructions repeated above may be generalized to the case when our partial differential equation depends on values of function. The equation itself is then a submanifold in $J^1(Q\times\mathbb{R}) = T^*Q\times\mathbb{R}$, usually given as a level set of some function. In coordinates or just in case we have $Q = \mathbb{R}^n$ we have $F(q^i, u, u_j) = 0$. $T^*Q\times\mathbb{R}$ is a contact manifold with global contact form. The whole isotropic, coisotropic and Lagrangian submanifold business can be generalized to the contact context. For example we would call a submanifold $K \subset M$ coisotropic when $T_xK \cap \mathcal{C}_x$ is a coisotropic subspace of $\mathcal{C}_x$ with respect to the two form γ (or $d\eta$ if we have chosen the contact form.) The same about Lagrangian and isotropic. We may also show that the level set of a regular function is a coisotropic submanifold of M while, in case $M = T^*Q\times\mathbb{R}$ or more general $M = J^1L^*$, a jet prolongation of a section (function) is a Legendre submanifold of M. Legendre submanifold is a coisotropic name for Lagrangian submanifold in symplectic case, i.e. maximal isotropic.

Now we can repeat the whole procedure of passing from a partial differential equation of the form $F(q^i, u, u_j) = 0$ to the ordinary differential equation of the form... this is the point when we might want to do some calculations in coordinates to see what is what: The geometric background is as it was before: $F^{-1}(0)$ is a coisotropic submanifold therefore it has a characteristic foliation by one dimensional submanifolds called characteristics. Any Legendre submanifold contained in it is always composed of these characteristics. Let us now find an equation for characteristics of $K = F^{-1}(0)$. We will work in coordinates (q, p, z).
An element $a^i\frac{\partial}{\partial q^i} + b_j\frac{\partial}{\partial p_j} + c\frac{\partial}{\partial z}$ belongs to $TK \cap \mathcal{C}$ if

$$a^i\frac{\partial F}{\partial q^i} + b_j\frac{\partial F}{\partial p_j} + c\frac{\partial F}{\partial z} = 0 \quad \text{and} \quad c = p_i a^i$$ The symplectic annihilator in $\mathcal{C}$ is then spanned by vectors of the form

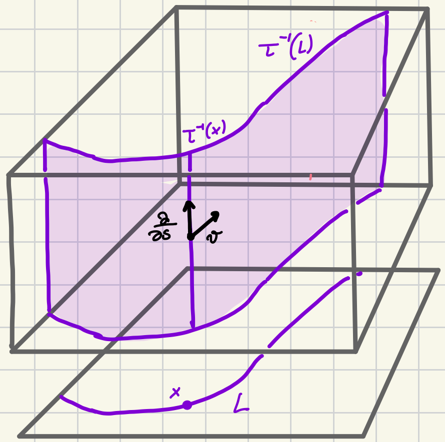
$\frac{\partial F}{\partial p_j}\frac{\partial}{\partial q^j} - \left(\frac{\partial F}{\partial q^i} + p_i \frac{\partial F}{\partial z}\right)\frac{\partial}{\partial p_i} + p_k \frac{\partial F}{\partial p_k}\frac{\partial}{\partial z}$ this is a value of the contact Hamiltonian vector fields at points $F=0$

Let us now formulate the contact Hamilton Jacobi theory in the language of symplectic $\mathbb{R}^\times$ principal bundles. The original definition of Legendre submanifold that says L is Legendre when it is maximal and isotropic, i.e. when it is of dimension n and $TL \subset \mathcal{C}$ can be replaced by the following:

PROPOSITION $L \subset M$ is a Legendre submanifold if and only if $\tau^{-1}(L) \subset P$ is a Lagrangian submanifold of (P, ω)

PROOF: if the dimension of M is $2n+1$, then the dimension of $\tau^{-1}(L)$ must be $n+1$ if $\tau^{-1}(P)$ is Lagrangian. Since $\tau^{-1}(L)$ is a union of fibres, L must be of dimension n, which is a correct dimension for L. If we choose a vertical coordinate s then the symplectic form can be written as $\omega = ds \wedge \eta + s d\eta$ with η being a local contact form. Let $v \in T\tau^{-1}(L)$. The vertical vector $\frac{\partial}{\partial s}$ also belongs to $T\tau^{-1}(L)$. We have then

$$0 = \omega\left(\frac{\partial}{\partial s}, v\right) = \langle \eta, v \rangle$$ It means that $T\tau(v) \in \mathcal{C}$, therefore L is isotropic.



Since P is symplectic we can use the symplectic property and state that $X_{\mathcal{H}}$ is tangent to $\tau^{-1}(L)$ if and only if $\mathcal{H}$ is constant on $\tau^{-1}(L)$. But $\mathcal{H}$ is supposed to be homogeneous while $\tau^{-1}(L)$ is supposed to be union of fibers of $\mathbb{R}^\times$ action. This means that this constant must be 0. But then also X_H^c i tangent to L and H vanishes on L.

We have then the contact Hamilton Jacobi theorem for free:

THEOREM: Let L be a Legendre submanifold of M. Then X_H^c is tangent to L if and only if H vanishes on L.

Let us now look at this in more „mechanical" way, i.e. explore the theorem for the case of $M = T^*Q \times \mathbb{R}$ and $M = J^1L^*$

$M = T^*Q \times \mathbb{R} \ni (p, z)$ $\quad L_S = j^1S(Q) = \{(dS(q), S(q)) : q \in Q\}$

$P = T^*(Q \times \mathbb{R}^\times) = T^*Q \times \mathbb{R}^\times \times \mathbb{R} \ni (\pi, \tau, z)$

$\tau : (\pi, \tau, z) \longmapsto (\frac{\pi}{\tau}, z)$

$\mathcal{H} : P \longrightarrow \mathbb{R}$ reads $\mathcal{H}(\pi, \tau, z) = \tau H(\frac{\pi}{\tau}, z)$ for $H : M \longrightarrow \mathbb{R}$

Contact Hamilton-Jacobi equation is $H(dS(q), S(q)) = 0$ in coordinates $H(q^i, \frac{\partial S}{\partial q^i}, S(q)) = 0$ (*)

Let us denote by r the projection $r = \pi_Q \circ pr_1 : T^*Q \times \mathbb{R} \longrightarrow Q$. If S is a solution of Hamilton-Jacobi equation $(*)$ then trajectories of X_H^c with initial conditions on L_S can be obtained from trajectories of the vector field $r_*(X_H^c|_{j^1S(Q)})$, which again cuts the number of degrees of freedom by half.

$M = J^1L^*$

What will be the ingredients of a Hamilton-Jacobi theorem? We start with a line bundle $L \longrightarrow Q$ and its dual $L^* \longrightarrow Q$. The solution of a Hamilton-Jacobi theorem is now a section S of $L^* \longrightarrow Q$. The homogeneous Hamiltonian lives on an appropriate symplectic principal bundle which we determined to be $T^*L^\times$. What is then the corresponding contact Hamiltonian? It is a section of $L_P{}^*$. The question is now what is $L_P{}^*$ for $P = T^*L^\times$?

PROPOSITION: For $P = T^*L^\times$ we get $L_P \simeq J^1L^* \times_Q L$, $L_P{}^* \simeq J^1L^* \times L^*$

A section of $L_P{}^* \longrightarrow J^1L^*$ can then be identified with the map $\sigma : J^1L^* \longrightarrow L^*$ covering the identity on Q. The Hamilton-Jacobi equation now reads $\sigma(j^1S) = 0$. If S is a solution of this equation then trajectories of X_σ with initial conditions on $j^1S(Q)$ can be obtained from trajectories of a vector field on Q which is a projection of $X_\sigma|_{j^1S(Q)}$ on Q.