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FACULTY OF PHYSICS
UNIVERSITY OF WARSAW

CONTACT GEOMETRY (3)

Journal of Physics A: Mathematical and Theoretical

PAPER

A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory

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Reductions: precontact versus presymplectic

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Contact geometric mechanics: the Tulczyjew triples

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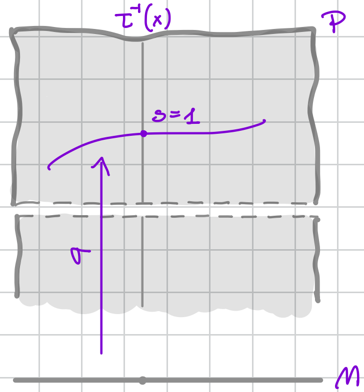
WHAT WE DID YESTERDAY



- WE HAVE DEFINED SYMPLECTIC PRINCIPAL $\mathbb{R}^\times$ BUNDLE
- WE HAVE SHOWN THAT $(P, \tau, M, \mathbb{R}^\times, h, \omega)$ AND $(M, \mathcal{C})$ ARE EQUIVALENT
- WE HAVE FOUND $(P, \tau, M, \mathbb{R}^\times, h, \omega)$ FOR $M = J^1 L^*$

$$(T^* L^\times, \tau, J^1 L^*, \mathbb{R}^\times, h_s, \varpi_{L^\times})$$

CONTACT HAMILTONIAN MECHANICS (BY SYMPLECTIC HOMOGENEOUS TOOLS)

REEB VECTOR FIELD $\quad i_{R_\eta} d\eta = 0, \quad \langle \eta, R_\eta \rangle = 1 \qquad \eta = dz - p_i dq^i \qquad R_\eta = \frac{\partial}{\partial z}$

CONTACT HAMILTONIAN VECTOR FIELD $\quad H : M \longrightarrow \mathbb{R} \qquad i_{X_H^c} d\eta = dH - R_\eta(H)\eta \qquad \langle \eta, X_H^c \rangle = -H$

IN DARBOUX COORDINATES:

$$\eta = dz - p_i dq^i, \quad R_\eta = \frac{\partial}{\partial z}, \quad \mathcal{H}(q^i, p_j, z) \qquad X_{\mathcal{H}}^c = A^i \frac{\partial}{\partial q^i} + B_j \frac{\partial}{\partial p_j} + D \frac{\partial}{\partial z}$$

$$d\eta = dq^i \wedge dp_i \qquad X_{\mathcal{H}}^c \lrcorner\, \eta = D - p_i A^i = -\mathcal{H}$$

$$X_{\mathcal{H}}^c \lrcorner\, d\eta = A^i dp_i - B_j dq^j = \frac{\partial \mathcal{H}}{\partial q^j} dq^j + \frac{\partial \mathcal{H}}{\partial p_i} dp^i + \frac{\partial \mathcal{H}}{\partial z} dz - \left(\frac{\partial \mathcal{H}}{\partial z}\right)(dz - p_j dq^j)$$

$$A^i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad B_j = \frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z} \qquad D = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}$$

$$X_{\mathcal{H}}^c = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} + \left(\frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z}\right) \frac{\partial}{\partial p_j} + \left(p_k \frac{\partial \mathcal{H}}{\partial p_k} - \mathcal{H}\right) \frac{\partial}{\partial z}$$

CONTACT HAMILTONIAN EQUATIONS

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$\dot{p}_j = \frac{\partial \mathcal{H}}{\partial q^j} + p_j \frac{\partial \mathcal{H}}{\partial z}$$
$$\dot{z} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}$$

PROBLEMS

1. IF η IS NOT GLOBAL, THE DEFINITION IS LOCAL
2. WE GET THE SAME VECTOR FIELD FOR (η, H) AND $(f\eta, fH)$
   NOT NICE TO PROVE USING (M, η) TOOLS
3. $\mathcal{L}_{X_H^c} \eta = R_\eta(H)\eta, \quad X_H^c(H) = R_\eta(H) H$ DOES NOT HELP IN NUMERICS

⋮

REEB VECTOR FIELD $\quad i_{R_\eta} d\eta = 0, \quad \langle \eta, R_\eta \rangle = 1 \qquad \eta = dz - p_i dq^i \qquad R_\eta = \frac{\partial}{\partial z}$

CONTACT HAMILTONIAN VECTOR FIELD $\quad H : M \longrightarrow \mathbb{R} \qquad i_{X_H^c} d\eta = dH - R_\eta(H)\eta \qquad \langle \eta, X_H^c \rangle = -H$

$$X_H^c(H) = ?$$

$$X_H^c(H) = i_{X_H^c} dH = i_{X_H^c} \lrcorner \left(i_{X_H^c} d\eta + R_\eta(H)\eta \right) = \underbrace{d\eta(X_H^c, X_H^c)}_{0} + R_\eta(H) \underbrace{i_{X_H^c}\eta}_{-H} =$$

$$= -R_\eta(H)H \quad (\neq 0)$$

$$\mathcal{L}_{X_H}\eta = ?$$

$$\mathcal{L}_{X_H^c}\eta = i_{X_H^c} d\eta + d(i_{X_H^c}\eta) = dH - R_\eta(H)\eta + d(-H) = -R_\eta(H)\eta \quad (\neq 0)$$

$Y \in \mathrm{Sec}\, \mathcal{C} \quad \mathcal{L}_{X_H} Y = ?$ $\qquad \mathcal{L}_{X_H^c} Y \overset{?}{\in} \mathcal{C} \qquad \mathcal{L}_{X_H^c} Y = [X_H^c, Y]$

$$d\eta(X_H^c, Y) = X_H^c(\underbrace{i_Y\eta}_{=0 \text{ SINCE } Y \in \mathcal{C}}) - \underbrace{Y(i_{X_H^c}\eta)}_{Y(-H)} - \eta([X_H^c, Y])$$

$\|$

$$\underbrace{Y \lrcorner dH}_{Y(H)} - R_\eta(H) \underbrace{Y \lrcorner \eta}_{=0}$$

$$\eta([X_H^c, Y]) = 0 \implies [X_H^c, Y] \in \mathcal{C}$$

SINCE (P,ω) IS A SYMPLECTIC MANIFOLD, WE CAN GET VECTOR FIELDS (ON P) FROM FUNCTIONS (ON P). THE POINT IS TO USE HOMOGENEOUS FUNCTIONS

$$\begin{tikzcd} T^*P \arrow[r, "\omega^b"] \arrow[d] & TP \arrow[dd] \\ P \arrow[u, bend left, "d\mathcal{H}"] \arrow[ur, "X_{\mathcal{H}}"] \arrow[d] & \\ M \arrow[r, "X_H^c"] & TM \end{tikzcd}$$

$$\mathcal{H} : P \longrightarrow \mathbb{R} \qquad \mathcal{H}(h_s(p)) = s\mathcal{H}(p)$$

$$d\mathcal{H} = \omega(X_{\mathcal{H}}, \cdot)$$

HOMOGENEOUS

INVARIANT!

HAMILTONIAN VECTOR FIELDS FOR HOMOGENEOUS HAMILTONIANS ARE INVARIANT HENCE PROJECTABLE!

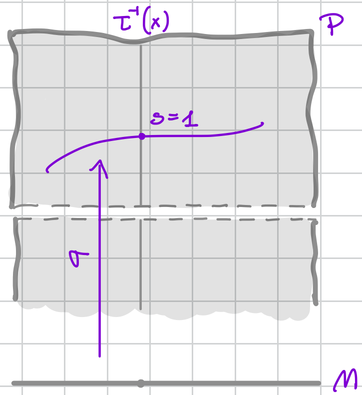
HOW ARE THEY RELATED WITH HAMILTONIAN CONTACT VECTOR FIELDS ON THE BASE?

WE CHOOSE A SECTION $\sigma : M \supset \mathcal{O} \longrightarrow P$

WE GET A VERTICAL COORDINATE s AND A CONTACT FORM η

$$\omega = ds \wedge \eta + s\, d\eta$$

DEFINE $H : \mathcal{O} \longrightarrow \mathbb{R}$ $\quad H(x) = \mathcal{H}(\sigma(x))$, THEN $\mathcal{H}(s,x) = sH(x)$



$$d\mathcal{H}(s,x) = H(x)\,ds + s\,dH(x) = \omega(X_{\mathcal{H}}(s,x), \cdot)$$

$$X_{\mathcal{H}}(s,x) = sF(x)\frac{\partial}{\partial s} + Y(x)$$

$$\omega(X_{\mathcal{H}}(s,x), \cdot) = sF(x)\eta - \langle \eta, Y\rangle ds + s\, i_Y d\eta = H(x)\,ds + s\,dH(x)$$

$$\langle \eta, Y\rangle = -H \qquad i_Y d\eta = dH - F\eta$$

THIS WE CAN CALCULATE CONTRACTING WITH R_η

$$\overbrace{\cancel{d\eta(Y, R_\eta)}}^{0} = \langle dH, R_\eta\rangle - F\overbrace{\langle \eta, R_\eta\rangle}^{1}$$

$$F = R_\eta(H)$$

SUMMARIZING:

THE PROJECTION Y OF $X_{\mathcal{H}}$ ON M SATISFY

$$\langle \eta, Y\rangle = -H \qquad i_Y d\eta = dH - R_\eta(H)\eta$$

$Y = X_H^c$ FOR $H(x) = \mathcal{H}(\sigma(x))$

COMING BACK TO OUR PROBLEMS:

1. IF η IS NOT GLOBAL, THE DEFINITION IS LOCAL

STARTING FROM GLOBAL HOMOGENEOUS FUNCTIONS WE GET GLOBAL CONTACT HAMILTONIAN VECTOR FIELDS

2. WE GET THE SAME VECTOR FIELD FOR (η, H) AND $(f\eta, fH)$

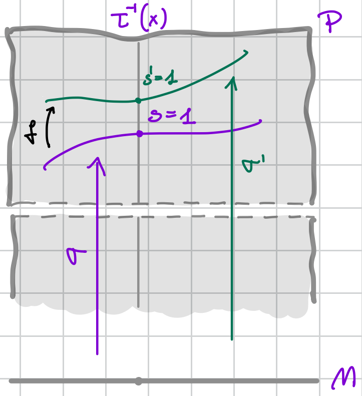
NOT NICE TO PROVE USING (M, η) TOOLS

IT IS EASY TO SEE: LET US FIX A HOMOGENEOUS HAMILTONIAN $\mathcal{H}$ AND TWO SECTIONS σ AND σ'

$$\sigma' = f\sigma \qquad H'(x) = \mathcal{H}(\sigma'(x)) = \mathcal{H}(f(x)\sigma(x)) = f(x)\mathcal{H}(\sigma(x)) = f(x)H(x)$$

$$H' = fH \qquad s = fs' \qquad \eta = \frac{1}{s}\theta = \frac{1}{fs'}\theta = \frac{1}{f}\eta' \qquad \eta' = f\eta$$

$$\eta' = f\eta$$



THREE LEMMAS

Lemat 1. *Niech X_H będzie kontaktowym hamiltonowskim polem wektorowym związanym z hamiltonianem H na rozmaitości kontaktowej (M, η). Wtedy $X_H H = -(\mathcal{R}_\eta H)H$, gdzie $\mathcal{R}_\eta$ jest odpowiadającym jednoformie η polem Reeba.*

Dowód. Zapisując warunek na pole hamiltonowskie

$$X_H \lrcorner \mathrm{d}\eta = \mathrm{d}H - (\mathcal{R}_\eta H)\eta$$

i zwężając obustronnie równanie względem pola X_H, otrzymujemy

$$\begin{gathered} 0 = X_H \lrcorner \mathrm{d}H - X_H \lrcorner (\mathcal{R}_\eta H)\eta \\ 0 = \mathrm{d}H(X_H) - (\mathcal{R}_\eta H)X_H \lrcorner \eta \\ X_H H = -(\mathcal{R}_\eta H)H \end{gathered}$$

□

Lemat 2. *Niech X_f, X_g będą hamiltonowskimi polami wektorowymi na rozmaitości kontaktowej (M, η) zwiazanymi z funkcjami hamiltonowskimi f, g. Wtedy $X_f(g)+X_g(f) = -\mathcal{R}_\eta(fg)$, gdzie $\mathcal{R}_\eta$ jest polem Reeba związanym z jednoformą η.*

Dowód. Pola X_f, X_g są jednoznacznie wyznaczone przez warunki

$$X_f \lrcorner \mathrm{d}\eta = \mathrm{d}f - (\mathcal{R}_\eta f)\eta, \qquad X_g \lrcorner \mathrm{d}\eta = \mathrm{d}g - (\mathcal{R}_\eta g)\eta. \tag{2}$$

Z antysymetryczności zwężenia wynika, że

$$\begin{gathered} X_g \lrcorner (X_f \lrcorner \mathrm{d}\eta) = -X_f \lrcorner (X_g \lrcorner \mathrm{d}\eta) \\ X_g \lrcorner (\mathrm{d}f - (\mathcal{R}_\eta f)\eta) = -X_f \lrcorner (\mathrm{d}g - (\mathcal{R}_\eta g)\eta) \\ X_g(f) + (\mathcal{R}_\eta f)g = -X_f(g) - (\mathcal{R}_\eta g)f \\ X_f(g) + X_g(f) = -\mathcal{R}_\eta(fg) \end{gathered}$$

□

Lemat 3. *Niech f będzie nieznikającą funkcją Hamiltonowską pola wektorowego X na rozmaitości kontatkowej $(M, f\eta)$. Wtedy $X = -\mathcal{R}_\eta$, gdzie $\mathcal{R}_\eta$ jest polem Reeba rozmiatości kontatkowej (M, η).*

Dowód. Hamiltonowskie pole wektorowe X na $(M, f\eta)$ z funkcją hamiltonowską f jest jednoznacznie wyznaczone przez warunki

$$X \lrcorner f\eta = -f, \qquad X \lrcorner \mathrm{d}(f\eta) = \mathrm{d}f - f\mathcal{R}_{f\eta}(f)\eta.$$

Z pierwszego równania wynika, że $X \lrcorner \eta = -1$. Rozpisując drugie równanie

$$X(f)\eta - (X \lrcorner \eta)\mathrm{d}f + fX \lrcorner \mathrm{d}\eta = df - f\mathcal{R}_{f\eta}(f)\eta.$$

Lemat $\boxed{1}$ gwarantuje, że $X(f) = -f\mathcal{R}_{f\eta}(f)$. Dodatkowo $X \lrcorner \eta = -1$, stąd

$$fX \lrcorner \mathrm{d}\eta = 0.$$

Ponieważ f jest nieznikająca, X spełnia te same warunki jak $\mathcal{R}_\eta$ z dokładnością do znaku. Z jednoznaczności obydwu pól wynika, że $X = -\mathcal{R}_\eta$

□

A THEOREM

Stwierdzenie 1. *Niech X_H, X_{fH} będą kontaktowymi hamiltonowskimi polami wektorowymi związanymi odpowiednio z hamiltonianami H, fH na rozmaitości kontaktowej (M, η), gdzie f jest nieznikającą ciągłą funkcją na M. Wtedy $X_H = X_{fH}$.*

CALCULATIONS

Dowód. Pola hamiltonowskie X_H oraz X_{fH} są jednoznacznie zdefiniowane poprzez warunki

$$X_{fH} \lrcorner (f\eta) = -fH, \tag{i}$$
$$X_{fH} \lrcorner \mathrm{d}(f\eta) = \mathrm{d}(fH) - \mathcal{R}_{f\eta}(fH)f\eta, \tag{ii}$$
$$X_H \lrcorner \eta = -H, \tag{iii}$$
$$X_H \lrcorner \mathrm{d}\eta = \mathrm{d}H - \mathcal{R}_\eta(H)\eta. \tag{iv}$$

Należy pokazać, że istnieje pole Y spełniające wszystkie powyższe warunki jednocześnie. Równania $\boxed{\text{i}}$ i $\boxed{\text{iii}}$ są identyczne. Należy więc sprawdzić warunki na zwężenia z pochodnymi jednoform Rozpisując warunek $\boxed{\text{iv}}$, otrzymujemy

$$\begin{gathered} Y \lrcorner \mathrm{d}(f\eta) = \mathrm{d}(fH) - \mathcal{R}_{f\eta}(fH)f\eta \\ Y \lrcorner (\mathrm{d}f \wedge \eta + f\mathrm{d}\eta) = H\mathrm{d}f + f\mathrm{d}h - f\mathcal{R}_{f\eta}(fH)\eta \\ Y(f)\eta - Y \lrcorner \eta\mathrm{d}f + fY \lrcorner \mathrm{d}\eta = H\mathrm{d}f + f\mathrm{d}H - f\mathcal{R}_{f\eta}(fH)\eta. \end{gathered}$$

Korzystając z faktu, że Y spełnia $\boxed{\text{i}}$, po skróceniu otrzymujemy

$$Y(f)\eta + fY \lrcorner \mathrm{d}\eta = f\mathrm{d}H - f\mathcal{R}_{f\eta}(fH)\eta$$

Dodając stronami $R_\eta(H)\eta$, otrzymujemy

$$f[Y \lrcorner \mathrm{d}\eta - \mathrm{d}H + \mathcal{R}_\eta(H)\eta] = f\mathcal{R}_\eta(H)\eta - f\mathcal{R}_{f\eta}(fH)\eta - Y(f)\eta.$$

Ponieważ f jest nieznikająca, Y spełnia równanie $\boxed{\text{iv}}$, jeżeli

$$f\mathcal{R}_\eta(H) - f\mathcal{R}_{f\eta}(fH) - Y(f) = 0. \tag{1}$$

Po rozpisaniu $Y(f)$, korzystając z Lematu $\boxed{2}$, otrzymujemy

$$\begin{aligned} f\mathcal{R}_\eta(H) - f\mathcal{R}_{f\eta}(fH) - Y(f) &= f\mathcal{R}_\eta(H) - f\mathcal{R}_{f\eta}(fH) + Y_f(fH) + \mathcal{R}_{f\eta}(f^2H) \\ &= f\mathcal{R}_\eta(H) - f\mathcal{R}_{f\eta}(fH) + Y_f(fH) + f\mathcal{R}_{f\eta}(fH) + fH\mathcal{R}_{f\eta}(f) \\ &= f\mathcal{R}_\eta(H) + Y_f(fH) + fH\mathcal{R}_{f\eta}(f) && (2) \\ &= f\mathcal{R}_\eta(H) + Y_f(H)f + Y_f(f)H + fH\mathcal{R}_{f\eta}(f), && (3) \end{aligned}$$

gdzie Y_f jest polem hamiltonowskim dla jednoformy $f\eta$ z funkcją hamiltonowską f. Korzystając z Lematu $\boxed{1}$, ostatnie dwa człony upraszczają się, stąd otrzymujemy

$$0 = f[\mathcal{R}_\eta(H) + Y_f(H)]. \tag{4}$$

Na mocy Lematu $\boxed{3}$ wyrażenie w nawiasie znika, więc pole Y spełnia wszystkie warunki $\boxed{\text{i}}$-$\boxed{\text{iv}}$. □

Dynamika kontaktowa na wstędze Moebiusa
Tomasz Sobczak, Dawid Jasiński, Mansur Muzafarov, Marcin Krych
Czerwiec 2024

CONCLUSION (ABOUT CONTACT HAMILTONIAN DYNAMICS): CONTACT HAMILTONIAN VECTOR FIELDS CORRESPOND TO HOMOGENEOUS HAMILTONIANS ON P

WHAT IF WE WANT A GENERATING OBJECT ON M?

WE HAVE ALREADY USED: **FROM A LINE BUNDLE TO PRINCIPAL $\mathbb{R}^\times$ BUNDLE**

A LINE BUNDLE

$$\begin{tikzcd} L \arrow[d, "\rho"'] \\ M \end{tikzcd}$$

THE DUAL

$$\begin{tikzcd} L^* \arrow[d, "\sigma"'] \\ M \end{tikzcd}$$

$\mathbb{R}^\times$- PRINCIPAL

$$\begin{tikzcd} L^\times \arrow[d, "\tau"'] \\ M \end{tikzcd}$$

> FIBERWISE LINEAR FUNCTIONS ON L ARE SECTIONS OF L^* $\quad \ell \longmapsto \langle \sigma(\rho(\ell)), \ell \rangle$
>
> HOMOGENEOUS FUNCTIONS ON $L^\times$ ARE SECTIONS OF L^*

THE OTHER WAY ROUND: **FROM PRINCIPAL $\mathbb{R}^\times$ BUNDLE TO LINE BUNDLE**

AN $\mathbb{R}^\times$-PRINCIPAL BUNDLE

$$\begin{tikzcd} P \arrow[d, "\tau"'] \\ M \end{tikzcd}$$

THE LINE BUNDLE

$$L_P = P \times \mathbb{R} / \mathbb{R}^\times$$

$$(p,t) \sim \left(h_s(p), \tfrac{t}{s}\right)$$

THE DUAL LINE BUNDLE

$$L_P^* = P \times \mathbb{R} / \mathbb{R}^\times$$

$$(p,z) \sim (h_s(p), sz)$$

> FIBERWISE HOMOGENEOUS FUNCTIONS ON P CORRESPOND TO SECTIONS OF L_P^*
>
> $P \simeq L_P^\times$

EXAMPLE: PASSING FROM $\mathcal{H}$ ON P TO H ON M IN CASE $M=J^1L^*$ $\left(M=T^*Q\times\mathbb{R}\right)$

$$
\begin{tikzcd}
T^*P \arrow[r, "\omega^\flat"] \arrow[d] & TP \arrow[dd] \\
P \arrow[u, bend left, "d\mathcal{H}"] \arrow[ur, "X_{\mathcal{H}}"] \arrow[d] & \\
M \arrow[r, "X^c_H"] & TM
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*T^*L^\times \arrow[r, "\omega^\flat_{L^\times}"] \arrow[d] & TT^*L^\times \arrow[dd] \\
T^*L^\times \arrow[u, bend left, "d\mathcal{H}"] \arrow[ur, "X_{\mathcal{H}}"] \arrow[d] & \\
J^1L^* \arrow[r, "X^c_H"] & TJ^1L^\times
\end{tikzcd}
$$

$M = T^*Q\times\mathbb{R}$

$L = Q\times\mathbb{R}$ $\quad (q^i, t)$ $\qquad L^\times = Q\times\mathbb{R}^\times$ $\quad t\neq 0$

$L^* = Q\times\mathbb{R}$ $\quad (q^i, z)$

$T^*L^\times = T^*\left(Q\times\mathbb{R}^\times\right)\simeq T^*Q\times\mathbb{R}^\times\times\mathbb{R}$ $\quad (q^i, \pi_j, t, z)$

$J^1L^*\simeq T^*Q\times\mathbb{R}$ $\quad (q^i, p_j, z)$

$$T^*L^\times \ni (q^i, \pi_j, t, z) \longmapsto \left(q^i, \frac{\pi_j}{t}, z\right)$$

$$
\begin{aligned}
\omega_{L^\times} &= d\pi_i\wedge dq^i + dz\wedge d\tau = \\
&= d(\tau p_i)\wedge dq^i + dz\wedge d\tau = \\
&= \tau\, dp_i\wedge dq^i + p_i\, d\tau\wedge dq^i + dz\wedge d\tau \\
&= \underbrace{(dz - p_i dq^i)}_{\eta}\wedge d\tau + \tau\,\underbrace{dp_i\wedge dq^i}_{d\eta}
\end{aligned}
$$

STARTING FROM $\quad H: T^*Q\times\mathbb{R}\ni(q^i, p_j, z)\longrightarrow H(q^i, p_j, z)\in\mathbb{R}$

CAN DEFINE $\quad \mathcal{H}: T^*Q\times\mathbb{R}\times\mathbb{R}\ni(q^i, \pi_j, \tau, z)\longrightarrow \tau H\left(q^i, \frac{\pi_i}{\tau}, z\right) = \tau\left[H_0\left(q^i, \frac{\pi_i}{\tau}\right) - \lambda z\right]\in\mathbb{R}$

THIS ADDS VISCOSITY-LIKE FORCE TO THE USUAL MECHANICAL HAMILTONIAN SYSTEM

COORDINATE EXPRESSIONS

$$\mathcal{H}(q^i, \tau, \pi_j, z) = \tau H\Big(q^i, \overset{p_j}{\frac{\pi_j}{\tau}}, z\Big)$$

$T^*L^\times = T^*Q \times \mathbb{R}^\times \times \mathbb{R}$ $\qquad$ $T^*Q \times \mathbb{R}$

ON P

$$X_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \tau \frac{\partial H}{\partial z}\frac{\partial}{\partial \tau} - \tau \frac{\partial H}{\partial q^j}\frac{\partial}{\partial \pi_j} + \left(\frac{\pi_k}{\tau}\frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

ON M

$$X^c_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^j} + p_j \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_j} + \left(p_k \frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

PRACTICAL EXAMPLE: VISCOSITY FORCE

$M = T^*Q \times \mathbb{R}$ $\qquad$ $\eta = dz - \theta_Q$ $\qquad$ $H(p, z) = H_0(p) - \lambda z$

$$X^C_H = \underbrace{\frac{\partial H_0}{\partial p_i}}_{\dot q}\frac{\partial}{\partial q^i} - \underbrace{\left(\frac{\partial H_0}{\partial q^i} - \lambda p_i\right)}_{\dot p}\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H_0}{\partial p_i} - H_0 + \lambda z\right)\frac{\partial}{\partial z}$$

$$\dot q^i = \frac{\partial H}{\partial p^i}$$

$$\dot p_j = -\frac{\partial H_0}{\partial q^j} + \lambda p_j$$

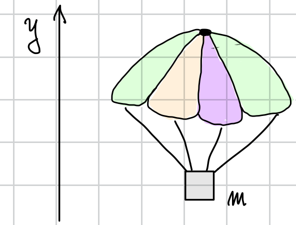
$$\dot z = \underbrace{p_i \frac{\partial H_0}{\partial p_i} - H_0 + \lambda z}_{L_0}$$

PRACTICAL EXAMPLE: PARACHUTE EQUATION

$$\dot y = \frac{1}{m}(p - \gamma z)$$

$$\dot p = -\frac{\partial V}{\partial y} + \frac{\gamma p^2}{m} - \frac{\gamma^2 p z}{m}$$

$$\dot z = \underbrace{\frac{p}{m}(p - \gamma z) - \mathcal{H}}_{L(y, \dot y) = \frac{m}{2}\dot y^2 + \gamma \dot y z - V(y)}$$



$Q = \mathbb{R}$

$M = T^*Q \times \mathbb{R}$

(y, p, z)

$$\boxed{\ddot y - \gamma m \dot y^2 + g = 0}$$

$$\mathcal{H}(y, p, z) = \frac{1}{2m}(p - \gamma z)^2 + V(y) \qquad V(y) = \frac{mg}{\gamma}\left(e^{\gamma y} - 1\right)$$

CONTACT LAGRANGIAN MECHANICS

IN THE LITERATURE ONE CAN FIND CONTACT LAGRANGIAN MECHANICS IN THE CASE OF $M = T^*Q \times \mathbb{R}$

$$T^*Q \times \mathbb{R} \ni (p, z) \longmapsto H(p, z) \in \mathbb{R}$$

$$TQ \times \mathbb{R} \ni (v, z) \longmapsto L(v, z) \in \mathbb{R}$$

THE LEGENDRE TRANSFORMATION HAPPENS ON p AND v ONLY

$$L(v, z) = \langle p, v \rangle - H(p, z)$$

CARTESIAN PRODUCT STRUCTURE IS NEEDED

DIFFERENTIAL CONSEQUENCES OF CONTACT HAMILTONIAN EQUATIONS ARE THE FOLLOWING HERGLOZ EQUATIONS

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^i} \qquad \underbrace{\dot{z} = L(q, \dot{q}, z)}$$

THIS IS AN IMPLICIT DEFINITION OF THE HERGLOZ ACTION FUNCTIONAL $z(\gamma)$ $(\gamma : I \longrightarrow Q)$ WHICH IS MINIMIZED BY SOLUTIONS OF THE HERGLOZ EQUATIONS

HOW ABOUT MORE GENERAL CONTACT MANIFOLDS?

CONTACT HAMILTON-JACOBI THEORY

SYMPLECTIC STUFF FIRST!

IF WE CAN SOLVE THE PDE THEN WE CAN SIMPLIFY ODE (1)

PARTIAL DIFFERENTIAL EQUATION
- OF THE FIRST ORDER
- NOT INVOLVING VALUES OF FUNCTIONS : $H(q^i, \frac{\partial S}{\partial q^i}) = 0$

ORDINARY DIFFERENTIAL EQUATION
HAMILTONIAN EQUATIONS

(2) IF WE CAN SOLVE THE ODE WE CAN RECONSTRUCT SOLUTIONS OF PDE

THEOREM (SYMPLECTIC) LET (P,ω) BE A SYMPLECTIC MANIFOLD AND $L \subset P$ – A LAGRANGIAN SUBMANIFOLD. THE HAMILTONIAN VECTOR FIELD X_H FOR A FUNCTION $H: P \longrightarrow \mathbb{R}$ IS TANGENT TO L IF AND ONLY IF H IS CONSTANT ON L

(1) $$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E$$

PARTIAL DIFFERENTIAL EQUATION
NOT INVOLVING VALUES
OF AN UNKNOWN FUNCTION S

$$H: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

+ DIFFERENTIAL GEOMETRY:

(T^*Q, ω_Q) SYMPLECTIC MANIFOLD

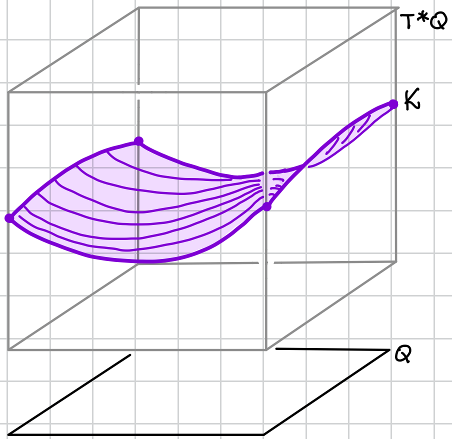
$$H: T^*Q \longrightarrow \mathbb{R}$$
$$(q^i, p_j) \longmapsto H(q^i, p_j)$$
SMOOTH FUNCTION

$K = H^{-1}(E)$ COISOTROPIC SUBMANIFOLD

$dS(Q)$ LAGRANGIAN SUBMANIFOLD

$$dS(Q) \subset K$$

SYMPLECTIC RESULTS:



COISOTROPIC SUBMANIFOLDS ARE ARE EQUIPPED A WITH CHARACTERISTIC DISTRIBUTION

$\chi \subset TK \subset TT^*Q \quad \chi = \{ v \in T_k T^*Q : \ \forall w \in TK \ \ \omega_m(v, w) = 0 \}$

INVOLUTIVE!

PRODUCES A FOLIATION

IN CASE $K = H^{-1}(E)$ χ IS ONE DIMENSIONAL, SPANNED BY X_H

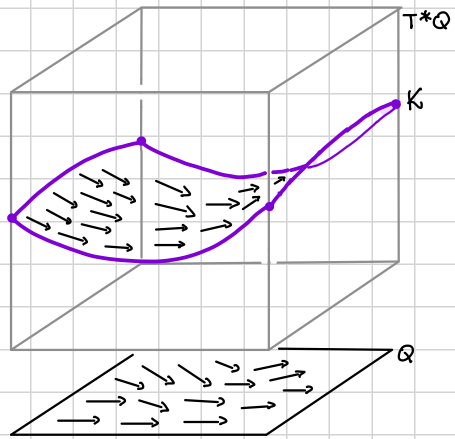
THE FOLIATION IS COMPOSED OF INTEGRAL CURVES OF X_H

COISOTROPIC

IF $L \subset K$ THEN L IS COMPOSED OF LEAVES OF χ I.E. THESE INTEGRAL CURVES

LAGRANGIAN

- SOLVE ODE $\dot{\gamma} = X_H(\gamma)$
- RECONSTRUCT $dS(Q)$
- INTEGRATE dS to S.



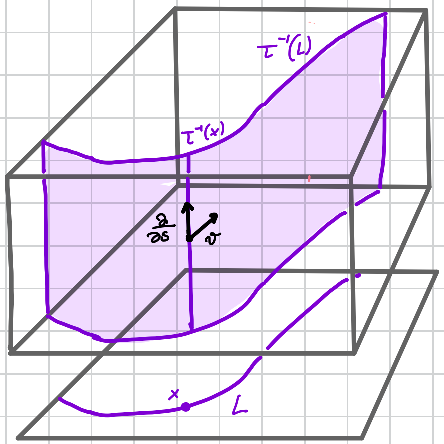
2 **ODE** $\dot{\gamma}(t) = X_H(\gamma(t))$

$\gamma : I \longrightarrow T^*Q$

- SOLVE PDE $H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E$
- RESTRICT X_H $dS(Q)$ AND PROJECT ON Q
- $T\pi_M \left(X_H (dS) \right)$

YOU HAVE A VECTOR FIELD ON Q TO INTEGRAE WITH HALF THE NUMBER OF VARIABLES

NOW CONTACT STUFF



- NO NEED TO DEFINE ISOTROPIC, LAGRANGIAN, COISOTROPIC SUBMANIFOLDS IN A CONTACT SETTING

CONTACT

$N \subset M$ IS ISOTROPIC / LEGENDRIAN / COISOTROPIC IF $\tau^{-1}(N)$ IS ISOTROPIC / LAGRANGIAN / COISOTROPIC

$TN \subset \mathcal{C}$

$TN \subset \mathcal{C}$ AND $\dim N = n$

$T_x N \cap \mathcal{C}_x$ IS COISOTROPIC IN $\mathcal{C}_x$ WITH RESPECT TO ν (OR $d\eta$)

THEOREM (SYMPLECTIC) LET (P, ω) BE A SYMPLECTIC MANIFOLD AND $L \subset P$ – A LAGRANGIAN SUBMANIFOLD. THE HAMILTONIAN VECTOR FIELD X_H FOR A FUNCTION $H : P \longrightarrow \mathbb{R}$ IS TANGENT TO L IF AND ONLY IF H IS CONSTANT ON L

THEOREM (CONTACT) LET $(P, \tau, M, \mathbb{R}^{\times}, h, \omega)$ BE A SYMPLECTIC $\mathbb{R}^{\times}$ BUNDLE AND L BE A HOMOGENEOUS LAGRANGIAN SUBMANIFOLD $L = \tau^{-1}(N)$ THEN THE HAMILTONIAN VECTOR FIELD FOR HOMOGENEOUS HAMILTONIAN IS TANGENT TO L IF AND ONLY IF HAMILTONIAN VANISHES ON L

CONCLUSION (CONTACT) IF $(M, \mathcal{C})$ IS A CONTACT MANIFOLD AND N IS A LEGENDRIAN SUBMANIFOLD THEN X_H^c IS TANGENT TO N IF AND ONLY IF H VANISHES ON N

CONTACT HAMILTON JACOBI THEORY

IF WE CAN SOLVE THE PDE THEN WE CAN SIMPLIFY ODE

1

PARTIAL DIFFERENTIAL EQUATION

- OF THE FIRST ORDER
- ~~NOT~~ INVOLVING VALUES OF FUNCTIONS : $H(q^i, \frac{\partial S}{\partial q^i}, S(q)) = 0$

ORDINARY DIFFERENTIAL EQUATION

CONTACT HAMILTONIAN EQUATIONS

2

IF WE CAN SOLVE THE ODE WE CAN RECONSTRUCT SOLUTIONS OF PDE

METHOD OF CHARACTERISTICS

THANK YOU!