CERN – SEENET-MTP – ICTP PhD Training Program

Bucharest 2024 Minischool

"Mathematical methods in Gravitation and Cosmology"

November 13-17, 2024, Bucharest-Magurele

FACULTY OF PHYSICS

UNIVERSITY OF WARSAW

CONTACT GEOMETRY (2)

Journal of Physics A: Mathematical and Theoretical

PAPER

A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory

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Published 2 November 2022 • © 2022 IOP Publishing Ltd
Journal of Physics A: Mathematical and Theoretical, Volume 55, Number 43
Citation Katarzyna Grabowska and Janusz Grabowski 2022 *J. Phys. A: Math. Theor.* **55** 435204
DOI 10.1088/1751-8121/ac9adb

Annali di Matematica Pura ed Applicata (1923 -)
https://doi.org/10.1007/s10231-023-01341-y

Reductions: precontact versus presymplectic

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Received: 9 February 2023 / Accepted: 28 April 2023

International Press
Publishers of scholarly mathematical and scientific journals and books

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Contact geometric mechanics: the Tulczyjew triples

pp. 599-654
Volume 28 (2024) Number 2
https://dx.doi.org/10.4310/ATMP.240914022224

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FEW OBSERVATIONS

- DEFINITION OF A CONTACT HAMILTONIAN VECTOR FIELD IS LOCAL! QUESTION: CAN WE HAVE GLOBAL HAMILTONIAN VECTOR FIELDS IF THERE IS NO GLOBAL η? (YES!)
- DO WE GET THE SAME VECTOR FIELD IF WE KEEP H AND CHANGE η? (NO!, EASY) IF NOT, HOW SHOULD WE CHANGE H? (EASY TO GUESS, NOT EASY TO PROVE!)
- X_H^c PRESERVES $\mathcal{C}$ I.E. CONTACT HAMILTONIAN VECTOR FIELDS ARE SYMMETRIES OF $\mathcal{C}$ (AND NOT SYMMETRIES OF η). CAN WE DEFINE X_H^c NOT USING η? (YES $\longrightarrow$ DIFFERENT GEOMETRIC LANGUAGE NEEDED!)

DEFINITION: LET $\mathbb{R}^\times$ DENOTE THE MULTIPLICATIVE GROUP OF NON-ZERO REALS $(\mathbb{R}^\times = \mathbb{R}\setminus\{0\}, \cdot)$.
A SYMPLECTIC PRINCIPAL BUNDLE IS AN $\mathbb{R}^\times$ PRINCIPAL BUNDLE P TOGETHER WITH A HOMOGENEOUS SYMPLECTIC FORM ω.

$$(\underbrace{P, M, \tau}, h, \omega)$$

$$h_s^*\omega = s\omega \qquad \omega \in \Omega^2(P)$$

NONDEGENERATE AND CLOSED

$$P \circlearrowleft \mathbb{R}^\times$$
$$\downarrow$$
$$M$$

$$h : \mathbb{R}^\times \times P \longrightarrow P$$
$$h(s, h(t, p)) = h(st, p)$$
$$h_s : P \ni p \longmapsto h(s, p) \in P$$
$$h_s \circ h_t = h_{st}$$

EXAMPLE: $(T^*Q)^\times \longrightarrow \mathbb{P}T^*Q$ $\quad \omega = \omega_Q\big|_{(T^*Q)^\times}$ $\quad \omega_Q = dp_i \wedge dq^i$ $\quad (q^i, p_j)\circ h_s = (q^i, s p_j)$

$$h_s^*\omega_Q = d(sp_i)\wedge dq^i = s\,dp_i\wedge dq^i = s\omega_Q$$

THEOREM: CONTACT MANIFOLDS AND SYMPLECTIC PRINCIPAL BUNDLES ARE EQUIVALENT NOTIONS!

(M, C) DEFINES (P, M, τ, h, ω)

- $P = (C^\circ)^\times$ $\quad T^*M \supset C^\circ$ – A LINE BUNDLE OVER M
  REMOVING 0-SECTION WE GET AN $\mathbb{R}^\times$-PRINCIPAL BUNDLE WITH ACTION „BORROWED"
  FROM A VECTOR BUNDLE.

  $\tau = \pi_M\big|_{(C^\circ)^\times}$

- **PROPOSITION:** P IS A SYMPLECTIC SUBMANIFOLD OF T^*M

  PROOF:
  WE HAVE TO SHOW THAT $i^*\omega_M$ IS A SYMPLECTIC FORM IF $i : (C^\circ)^\times \longrightarrow T^*M$ IS THE INCLUSION MAP.

  CLOSED – OBVIOUS, NONDEGENERATE – TO PROVE

  LET US LOCALLY CHOOSE A CONTACT FORM ON $\mathcal{O} \subset M$ AND DEFINE THE MAP

  $$I_\eta : \mathcal{O} \times \mathbb{R}^\times \longrightarrow (C^\circ)^\times \subset T^*M \qquad \text{(A LOCAL TRIVIALIZATION OF } P = (C^\circ)^\times \longrightarrow M\text{)}$$
  $$(x, s) \longmapsto s\eta$$

  $$I_\eta^* i^* \omega_M = (i \circ I_\eta)^* d(\theta_M) = d\left((i\circ I_\eta)^* \theta_M\right) = d\left(s\,\eta^* \theta_M\right) = d(s\eta) = ds\wedge\eta + s\,d\eta$$

  $$(ds\wedge\eta + s\,d\eta)^{\wedge(n+1)} = \underbrace{s^{n+1}(d\eta)^{\wedge(n+1)}}_{0} + \underbrace{(n+1)s^{n+1}\, ds\wedge\eta\wedge(d\eta)^{\wedge n}}_{\neq 0} + \underbrace{\ldots}_{0} \neq 0$$

  0: $2n+2$ FORM ON A MANIFOLD OF DIMENSION $2n+1$

  $\neq 0$ BECAUSE η IS CONTACT

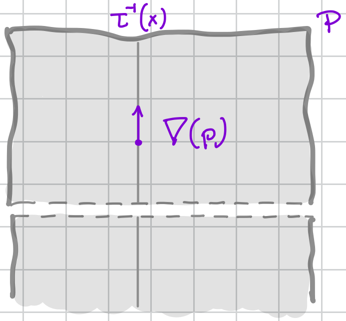
  0 BECAUSE ds AND η APPEAR MORE THAN ONCE

  I.E. $(i \circ I_\eta)^* \omega_M$ IS NONDEGENERATE AND $i^*\omega_M$ IS NONDEGENERATE ∎

- $\omega = i^*\omega_M$ IS HOMOGENEOUS BECAUSE ω_M IS HOMOGENEOUS

REMARK: IF (M, η) IS CONTACT (GLOBAL!) THEN $P_0 = M \times \mathbb{R}$ WITH $\omega = d(e^t\eta) = e^t dt\wedge\eta + e^t d\eta$ IS A SYMPLECTIC MANIFOLD.
P_0 IS ISOMORPHIC TO THE „POSITIVE" PART OF P. IN THE LITERATURE P_0 IS CALLED A SYMPLECTISATION OF (M, η).
WE HAVE CONSTRUCTED A SYMPLECTISATION OF GENERAL (M, C).

(P, M, τ, h, ω) DEFINES (M, ξ)



- $\nabla \in \mathfrak{X}(P)$ $\qquad \nabla(p) = \frac{d}{ds}\Big|_{s=1} h_s(p) = \frac{d}{dt}\Big|_{t=0} h_{e^t}(p)$

- ∇ IS INVARIANT

$$Th_t(\nabla(p)) = \frac{d}{ds}\Big|_{s=1} h_t(h_s(p)) = \frac{d}{ds}\Big|_{s=1} h_s(h_t(p)) = \nabla(h_t(p))$$

HOMOGENEOUS $\qquad$ HOMOGENEOUS

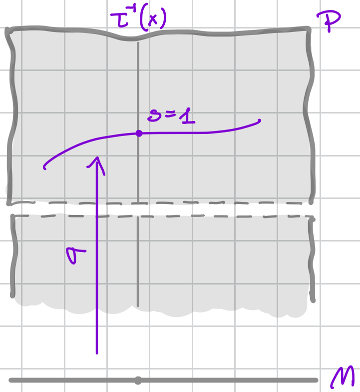
- $\theta = i_\nabla \omega$

INVARIANT

PROPOSITION $\quad \omega = d\theta$

PROOF: SINCE $h_s^*\omega = s\omega$ THEN $\mathcal{L}_\nabla \omega = \omega$

$$\omega = \mathcal{L}_\nabla \omega = d i_\nabla \omega + i_\nabla \underbrace{\cancel{d\omega}}_{0} = d\theta \quad \blacksquare$$



LOCAL VERTICAL COORDINATE

- σ – A LOCAL SECTION OF P OVER $\mathcal{O}$, $\quad \mathcal{O} \times \mathbb{R}^\times \ni (x, s) \xrightarrow{I_\sigma} h_s(\sigma(x)) \in P$ $\quad$ A LOCAL TRIVIALISATION

$\nabla = \frac{\partial}{\partial s} \qquad \frac{1}{s}\theta = \tau^*\eta$

INVARIANT
I.E. A PULL-BACK

PROPOSITION: η IS A LOCAL CONTACT FORM

PROOF:

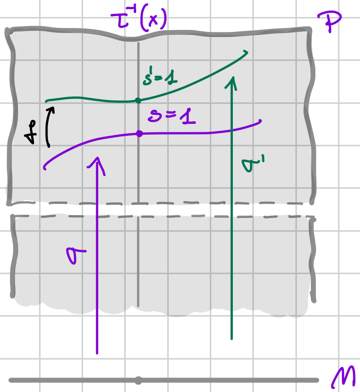
$$I_\sigma^*\theta = s\eta$$

$$I_\sigma^*\omega = I_\sigma^*(d\theta) = d(s\eta) = ds \wedge \eta + s\,d\eta$$

SINCE ω IS SYMPLECTIC, SO IS $I_\sigma^*\omega$, THEREFORE

$$0 \neq (I_\sigma^*\omega)^{\wedge n+1} = (ds\wedge\eta + s\,d\eta)^{\wedge n+1} = (n+1)s^n\, ds \wedge \underbrace{\eta \wedge (d\eta)^{\wedge n}}_{\neq 0}$$

I.E. η IS A CONTACT FORM



- **PROPOSITION:** $\ker\eta$ DOES NOT DEPEND ON THE CHOICE OF A SECTION σ

PROOF:

$\sigma : \mathcal{O} \longrightarrow P$
$\sigma' : \mathcal{U} \longrightarrow P$
$\mathcal{U} \cap \mathcal{O} \neq \emptyset$

ON $\mathcal{U} \cap \mathcal{O}$ WE HAVE $\sigma'(x) = f(x)\sigma(x)$ FOR $f \in C^\infty(\mathcal{O} \cap \mathcal{U})$ $f \neq 0$

$$s(p) = f(\tau(p))\, s'(p)$$

$$\eta = \frac{1}{s}\theta = \frac{1}{f s'}\theta = \frac{1}{f}\eta'$$

$$f\eta = \eta' \Rightarrow \ker\eta = \ker\eta' \quad \blacksquare$$

CHOOSING AN OPEN COVER OF M AND LOCAL CONTACT FORMS ASSOCIATED WITH LOCAL SECTIONS WE GET GLOBALLY DEFINED $\mathcal{C} = \ker\eta$

REMARK: NOTE THAT θ IS A NON-VANISHING FORM ON P, THEREFORE $\ker\eta \subset TP$ IS A REGULAR DISTRIBUTION ON P. IT CONTAINS A VERTICAL DIRECTION, SINCE $\langle\theta, \nabla\rangle = \langle i_\nabla \omega, \nabla\rangle = \omega(\nabla, \nabla) = 0$. $\mathcal{C}$ IS A PROJECTION ON M OF $\ker\theta$.

EXAMPLE: WHAT IS P FOR $M = J^1L^*$

WE HAVE SEEN A PRINCIPAL SYMPLECTIC BUNDLE FOR $M = \mathbb{P}T^*Q$ (EXAMPLE 3). OTHER EXAMPLES (EXAMPLE 1, EXAMPLE 2) ARE SPECIAL CASES OF THE FIRST JET BUNDLE. WE SHALL THEN CONSIDER THE GENERAL CASE

$$L \xrightarrow{\tau} Q$$ A LINE BUNDLE, I.E. A VECTOR BUNDLE WITH ONE DIMENSIONAL FIBRE

$$L^\times \longrightarrow Q$$ AFTER REMOVING ZERO SECTION WE HAVE A $\mathbb{R}^\times$-PRINCIPAL BUNDLE

$$L^* \xrightarrow{\tau} Q$$ WE SHALL CONSIDER JETS OF THE DUAL LINE BUNDLE J^1L^*

LET US LOOK AT $T^*L^\times$

$T^*L^\times$ IS AN $\mathbb{R}^\times$ PRINCIPAL BUNDLE:

- $\mathbb{R}^\times$ ACTS ON $L^\times$ BY MULTIPLICATION $\quad (q^i, t)\circ h_s = (q^i, st)$
- THE ACTION CAN BE LIFTED TO $TL^\times$ $\quad (d_T h)_s = Th_s \qquad (q^i, t, \dot q^j, \dot t)\circ (d_T h_s) = (q^i, st, \dot q^j, s\dot t)$
- DUALIZING WE GET $\qquad (q^i, t, p_j, z)\circ T^*h_s = (q^i, \tfrac{1}{s}t, p_j, sz)$

$$(q^i, t, p_j, z)\circ T^* h_{\frac{1}{s}} = (q^i, st, p_j, \tfrac{1}{s}z)$$

- COVECTORS CAN BE MULTIPLIED BY NUMBERS $\quad d_{T^*}h_s = sT^*h_s \qquad (q^i, t, p_j, z)\circ d_{T^*}h_s = (q^i, st, sp_j, z)$

THE LIFT OF $\mathbb{R}^\times$ ACTION TO $T^*L^\times$

THIS IS A PRINCIPAL ACTION. COORDINATES ON THE BASE SHOULD BE

$$(q^i, t, p_j, z) \longmapsto (q^i, p_j/t, z)$$

BUT WHAT IS THE BASE?

NOTE THAT $\omega_{L^\times}$ IS HOMOGENEOUS: $(d_{T^*}h_s)^*(dp_i\wedge dq^i + dz\wedge dt) = d(sp_i)\wedge dq^i + dz\wedge d(st) = s(dp_i\wedge dq^i + dz\wedge dt) = s\omega_{L^\times}$

WE HAVE THEN $\left(T^*L^\times, ?, ?, d_{T^*}h, \omega_{L^\times}\right)$ A SYMPLECTIC PRINCIPAL BUNDLE OVER WHO-KNOWS-WHAT!

THINGS THAT HAPPEN ON L^* AND J^1L^*	THINGS THAT HAPPEN ON $L^\times$ AND $T^*L^\times$

THINGS THAT HAPPEN ON L^* AND J^1L^*

$$\begin{tikzcd} L^* \arrow[d, shift right, "\rho"'] \\ Q \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}$$

$j^1\sigma'(q) = j^1\sigma(q)$ MEANS THAT $\sigma' = \lambda\sigma$

WHERE $\lambda\in C^\infty(Q)$ $\lambda(q)=1$, $d\lambda(q)=0$

THINGS THAT HAPPEN ON $L^\times$ AND $T^*L^\times$

f_σ A HOMOGENEOUS FUNCTION ON $L^\times$

$$f_\sigma(\ell) = \langle \sigma(\rho(\ell)), \ell\rangle$$

FUNCTIONS DIFFER BY THE SAME λ:

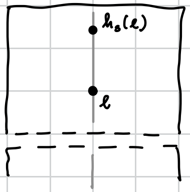
$$f_{\sigma'}(\ell) = \langle\sigma'(\rho(\ell)), \ell\rangle = \langle\lambda(\rho(\ell))\sigma(\rho(\ell)), \ell\rangle = \lambda(\rho(\ell))f_\sigma(\ell)$$

EVERY HOMOGENEOUS $f : L^\times \longrightarrow \mathbb{R}$
CORRESPONDS TO A SECTION
(HOMOGENEOUS ON $L^\times$ IS LINEAR ON L)

DIFFERENTIALS ARE EQUAL:

$$df_{\sigma'}(e) = d(\lambda f_{\sigma'})(e) = f_\sigma(e)\, \overset{0}{\cancel{d\lambda(\rho(e))}} + \overset{=1}{\lambda(\rho(e))}\, df_\sigma(e) =$$

$$= df_\sigma(e)$$



$$df_\sigma(h_s(e)) = d_{T^*}h_s(df_\sigma(e))$$

THIS MEANS THAT THERE IS A MAP FROM J^1L^* TO $T^*L^\times/_{\mathbb{R}^\times}$

THE MAP IS ONE-TO-ONE BECAUSE EVERY COVECTOR IS A DIFFERENTIAL OF SOME HOMOGENEOUS FUNCTION

(q^i_0, t_0) $\quad \alpha = \alpha_i dq^i + \alpha_t dt \quad$ $f(q^i, t) = t\left(\frac{\alpha_i}{t_0}(q^i - q^i_0) + \alpha_t\right)$

SUMMARIZING: WE HAVE RECOGNIZED $T^*L^\times/_{\mathbb{R}^\times}$ AS J^1L^*. THE CANONICAL CONTACT STRUCTURE ON THE FIRST JET BUNDLE COMES FROM THE HOMOGENEOUS SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE $T^*L^\times$