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FACULTY OF PHYSICS
UNIVERSITY OF WARSAW

CONTACT GEOMETRY (1)

Journal of Physics A: Mathematical and Theoretical

PAPER

A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory

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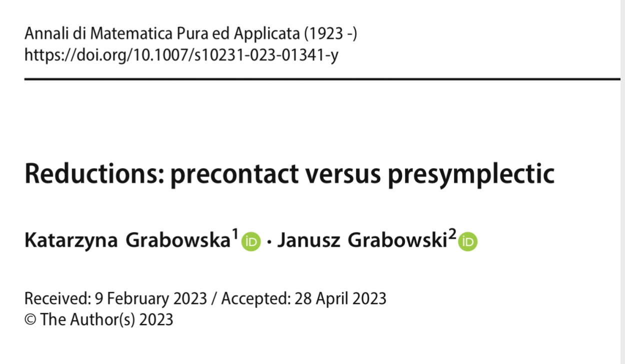

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Reductions: precontact versus presymplectic

Katarzyna Grabowska[1] · Janusz Grabowski[2]

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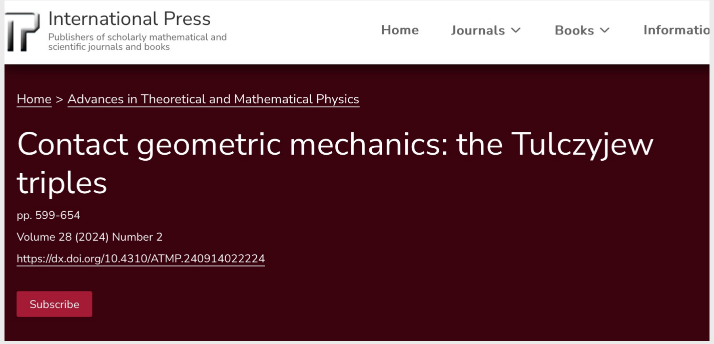

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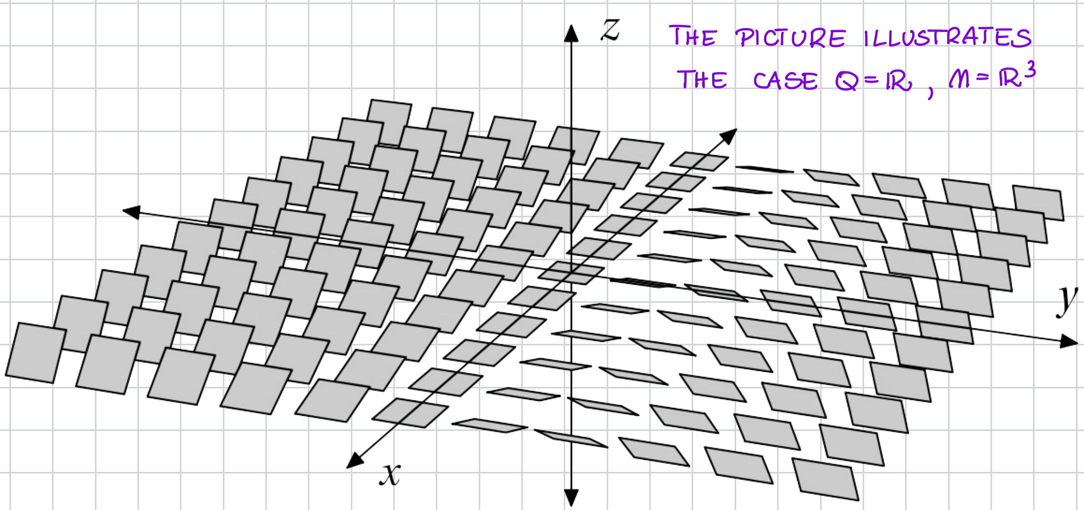


THE PICTURE ILLUSTRATES THE CASE $Q=\mathbb{R}$, $M=\mathbb{R}^3$

A CONTACT MANIFOLD IS A MANIFOLD M TOGETHER WITH A CERTAIN DISTRIBUTION $\mathcal{C}$.

EXAMPLE 1. CONTACT STRUCTURE ON $M=T^*Q\times\mathbb{R}$

$$\begin{tikzcd} Q\times\mathbb{R} \arrow[d, "pr_1"'] \\ Q \arrow[u, bend right, "f"'] \end{tikzcd} \qquad f\in C^\infty(Q)$$

$J^1(Q\times\mathbb{R})$

$$\begin{tikzcd} T^*Q\times\mathbb{R}\ni\big(df(q),f(q)\big) \arrow[d] \\ Q \arrow[u, bend right, "j^1f"'] \end{tikzcd}$$

$\left(q^i, p_j=\frac{\partial f}{\partial q^i}, z=f(q)\right)$

CONTACT DISTRIBUTION $\mathcal{C}$ IS SPANNED BY ALL VECTORS TANGENT TO FIRST JET PROLONGATIONS OF FUNCTIONS

$$\dot q^i\frac{\partial}{\partial q^i}\longmapsto \dot q^i\Big(\frac{\partial}{\partial q^i}+\underbrace{\frac{\partial^2 f}{\partial q^j\partial q^i}}_{\text{VARYING } f \text{ WE CAN CHANGE THIS ARBITRARILY}}\frac{\partial}{\partial p_j}+\underbrace{\frac{\partial f}{\partial q^i}}_{p_i}\frac{\partial}{\partial z}\Big)=$$

$$=\overset{\text{ARBITRARY}}{\dot q^i}\Big(\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial z}\Big)+\overset{\text{ARBITRARY}}{a_j}\frac{\partial}{\partial p_j}\ \Big\}\ \text{TANGENT AT } (q^i,p_i,z)$$

$$\mathcal{C}=\Big\langle\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial z},\frac{\partial}{\partial p_j}\Big\rangle$$

A CARTAN DISTRIBUTION

- $\dim Q=n$, $\dim T^*Q\times\mathbb{R}=2n+1$ $\quad\dim\mathcal{C}_x=2n$ $\quad x=(p,z)$
- $\Big[\frac{\partial}{\partial p_i},\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial z}\Big]=-\frac{\partial}{\partial z}\notin\mathcal{C}$
- $\mathcal{C}=\ker\eta\quad\Omega^1(M)\ni\eta=dz-p_i\,dq^i=dz-\theta_Q$

A LIOUVILLE FORM ON T^*Q

$$\boxed{\begin{aligned}&(M,\mathcal{C})\\ &M=T^*Q\times\mathbb{R}\\ &\mathcal{C}=\Big\langle\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial z},\frac{\partial}{\partial p_j}\Big\rangle\end{aligned}}$$

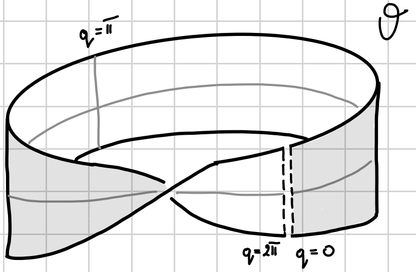
EXAMPLE 2. THE MÖBIUS BAND CASE

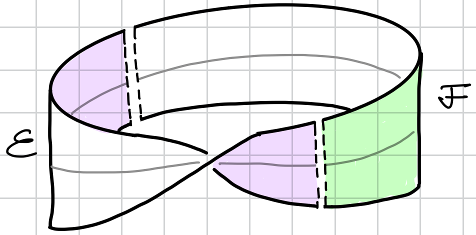
$$\mathcal{B} = \mathbb{R}^2/\mathbb{Z} \qquad k\cdot(x,y) = \left(x + k\cdot 2\pi,\ (-1)^k y\right)$$

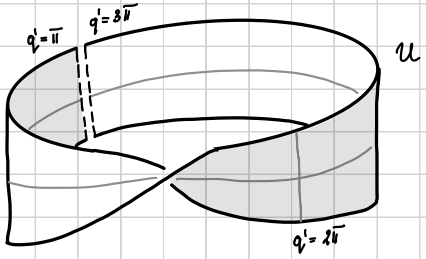
A LINE BUNDLE OVER S^1

THE ACTION IS LINEAR ON THE SECOND ARGUMENT

$$\boxed{\mathcal{M} = J^1\mathcal{B},\quad \mathcal{C} = \left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle}$$







$(q,p,z) \quad q \in]0,2\pi[$

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle$$

$\mathcal{O} \cap \mathcal{U} = \mathcal{E} \cup \mathcal{F}$

$(q',p',z') \quad q' \in]\pi, 3\pi[$

$$\mathcal{C} = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, \frac{\partial}{\partial p'} \right\rangle$$

IN $\mathcal{E}$ THE COORDINATE TRANSFORMATION IS IDENTITY

in $\mathcal{E}$ $\quad q=q',\ p=p',\ z=z'$

$$\frac{\partial}{\partial q} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial p} = \frac{\partial}{\partial p'}$$

in $\mathcal{F}$ $\quad q' = q + 2\pi \quad p' = -p,\ z' = z$

$$\frac{\partial}{\partial q} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial q'} + (-p')\left(-\frac{\partial}{\partial z'}\right) \qquad \frac{\partial}{\partial p} = -\frac{\partial}{\partial p'}$$

$$\left\langle \frac{\partial}{\partial q} + p\frac{\partial}{\partial z}, \frac{\partial}{\partial p} \right\rangle = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, -\frac{\partial}{\partial p'} \right\rangle = \left\langle \frac{\partial}{\partial q'} + p'\frac{\partial}{\partial z'}, \frac{\partial}{\partial p'} \right\rangle$$

IN $\mathcal{F}$ SOME COORDINATES CHANGE SIGN, BUT STILL $\mathcal{C}$ IS WELL DEFNED

$\eta_{\mathcal{O}} = dz - p\,dq \qquad \mathcal{C} = \ker \eta$

$\eta_{\mathcal{U}} = dz' - p'dq' \qquad \mathcal{C} = \ker \eta_{\mathcal{U}}$

$$\eta_{\mathcal{U}} = \eta_{\mathcal{O}} \longleftrightarrow \eta_{\mathcal{U}} = -\eta_{\mathcal{O}}$$

THE DISTRIBUTION $\mathcal{C}$ IS WELL DEFINED, BUT THE FORM η IS NOT. IF THERE WOULD EXIST A GLOBAL FORM η SUCH THAT $\ker \eta = \mathcal{C}$ THEN

ON $\mathcal{O}$: $\eta_{\mathcal{O}} = f\eta$, ON $\mathcal{U}$ $\eta = g\,\eta_{\mathcal{U}}$, ON $\mathcal{O}\cap\mathcal{U}$ $\quad \eta_{\mathcal{O}} = gf\,\eta_{\mathcal{U}}$

$gf \to +1$ ON $\mathcal{E}$

$gf \to -1$ ON $\mathcal{F}$

EXAMPLE 3. PROJECTIVIZED COTANGENT BUNDLE $\mathbb{P}T^*Q$

IN $(T^*Q)^\times = T^*Q \setminus \{0_Q\}$ WE DEFINE A RELATION : $p \sim p'' \iff \pi_M(p) = \pi_M(p'),\ \exists \lambda \neq 0\ \lambda p = p'$

$$M = T^*Q^\times /_{\sim} = \mathbb{P}T^*Q$$

$\mathcal{C}$ IS DEFINED AS A KERNEL OF LOCAL FORMS.

FOR SIMPLICITY : $Q = \mathbb{R}^n$, $T^*Q \ni (q^i, p_j)$

$$\mathcal{U}_k = \{ [(q^i, p_j)]_\sim : p_k \neq 0 \}$$

COORDINATES : $(q^i; \pi_1, \dots, \pi_{k-1}, \pi_{k+1}, \dots, \pi_n)$

$$\bigcup_{k=1}^{n} \mathcal{U}_k = M$$

$$q^i([q,p]) = q^i$$

$$\overset{(k)}{\pi}_j([q,p]) = p_j/p_k$$

$$\overset{(k)}{\eta} = dq^k + \sum_{j \neq k} \overset{(k)}{\pi}_j \, dq^j$$

ON $\mathcal{U}_k \cap \mathcal{U}_\ell$ WE HAVE $p_\ell \neq 0$, $p_k \neq 0$

COORDINATE CHANGE :

$$j \neq k,\ell \qquad \overset{(k)}{\pi}_j = \frac{p_j}{p_k}, \quad \overset{(\ell)}{\pi}_j = \frac{p_j}{p_\ell}, \quad \overset{(\ell)}{\pi}_j = \frac{\overset{(k)}{\pi}_j}{\overset{(k)}{\pi}_\ell}$$

$$\overset{(\ell)}{\pi}_k = \frac{1}{\overset{(k)}{\pi}_\ell}$$

$$\overset{(\ell)}{\eta} = dq^\ell + \sum_{j \neq \ell} \overset{(\ell)}{\pi}_j \, dq^j = dq^\ell + \sum_{\ell \neq j,k} \frac{\overset{(k)}{\pi}_j}{\overset{(k)}{\pi}_\ell} dq^j + \frac{1}{\overset{(k)}{\pi}_\ell} dq^k = \frac{1}{\overset{(k)}{\pi}_\ell} \left(\overset{(k)}{\pi}_\ell \, dq^\ell + \sum_{j \neq k,\ell} \overset{(k)}{\pi}_j \, dq^j + dq^k \right) =$$

$$= \frac{1}{\overset{(k)}{\pi}_\ell} \left\{ dq^k + \sum_{j \neq k} \overset{(k)}{\pi}_j \, dq^j \right\} = \frac{1}{\overset{(k)}{\pi}_\ell} \overset{(k)}{\eta}$$

$$\left.\begin{array}{l} \text{ON } \mathcal{U}_k \quad \overset{(k)}{\eta} \text{ IS NONVANISHING}, \quad \dim \ker \overset{(k)}{\eta} = 2n-2 \\ \text{ON } \mathcal{U}_k \cap \mathcal{U}_\ell \quad \overset{(\ell)}{\eta} = \dfrac{1}{\overset{(k)}{\pi}_\ell} \overset{(k)}{\eta} \quad \text{I.E.} \quad \ker \overset{(k)}{\eta} = \ker \overset{(\ell)}{\eta} \end{array}\right\}$$

$\mathcal{C} = \ker \overset{(k)}{\eta}$

$k \in \{1, \dots, n\}$

WE HAVE DEFINED A DISTRIBUTION $\mathcal{C}$ OF DIMENSION $2n-2$ ON A MANIFOLD $M = \mathbb{P}T^*\mathbb{R}^n$ OF DIMENSION $2n-1$. THE SAME CAN BE DONE ON $\mathbb{P}T^*Q$ FOR ANY Q

EXAMPLE 3a : $Q = S^2$ $\quad \mathbb{P}T^*S^2 \simeq S^3$ WE GET A CONTACT STRUCTURE ON THE TOTAL SPACE OF HOPF FIBRATION

OF CONSTANT DIMENSION LOCALLY SPANNED BY SMOOTH VECTOR FIELDS

DEFINITION: A **CONTACT MANIFOLD** IS A MANIFOLD M TOGETHER WITH A REGULAR DISTRIBUTION $\mathcal{C}$ OF CODIMENSION 1 WHICH IS MAXIMALLY NONINTEGRABLE

$\dim \mathcal{C}_x = \dim M - 1$

$$\varrho : TM \longrightarrow TM/\mathcal{C}$$

$$V, W \in Sec(\mathcal{C})$$

$$\nu(V, W) = \varrho([V, W])$$

NOTE THAT ν DEPENDS ONLY ON $V(x)$, $W(x)$, I.E IT IS A TWO-FORM ON $\mathcal{C}$

$$\nu(V, fW) = \varrho([V, fW]) = \varrho(f[V,W] + V(f)W) =$$

$$= f\varrho([V,W]) = f\nu(V,W)$$

MAXIMAL NONINTEGRABILITY MEANS ν IS NONDEGENERATE

$\downarrow$ $\mathcal{C}$ IS OF EVEN DIMENSION, M IS OF ODD DIMENSIO

DEFINITION: LET $(M,\mathcal{C})$ BE A CONTACT MANIFOLD. LOCALLY DEFINED 1-FORM η SUCH THAT $\mathcal{C}=\ker\eta$ IS CALLED **A CONTACT FORM**

PROPOSITION: A CONTACT FORM η SATISFIES $\eta\wedge(d\eta)^{\wedge n}\neq 0$ IF $\dim M=2n+1$

PROOF:

$\mathcal{O}\subset M$ $\quad$ $TM/\mathcal{C}$ AND $\mathcal{C}^\circ\subset T^*M$ ARE A PAIR OF DUAL BUNDLES

$\eta\in\Omega^1(\mathcal{O})$ $\quad$ $\eta\in Sec(\mathcal{C}^\circ_{|\mathcal{O}})$

$\mathcal{C}_{|\mathcal{O}}=\ker\eta$

USING η WE DEFINE A TWO-FORM ON $\mathcal{C}_{|\mathcal{O}}$:

THE SAME UP TO A SIGN

$$\mathcal{C}\times_M\mathcal{C}\ni(v,w)\longmapsto\langle\eta,\nu(v,w)\rangle\in\mathbb{R}$$

LET US TAKE $V,W\in Sec(\mathcal{C})$ SUCH THAT $V(x)=v$, $W(x)=w$ AND CALCULATE

$$d\eta(V,W)=V\langle\eta,W\rangle-W\langle\eta,V\rangle-\eta([V,W])=-\langle\eta,\nu(v,w)\rangle$$

$d\eta_{|\mathcal{C}}$ COINCIDES WITH ν IF WE TRIVIALIZE $TM/\mathcal{C}\simeq M\times\mathbb{R}$ AND $\mathcal{C}^\circ\simeq M\times\mathbb{R}$ USING η

IT FOLLOWS THAT $d\eta$ IS NOT DEGENERATE ON $\mathcal{C}$, $\ker d\eta$ IS THEN 1-DIMENSIONAL AND HAS TRIVIAL INTERSECTION WITH $\mathcal{C}$

$(d\eta)^{\wedge m}$ DOES NOT VANISH ON $\mathcal{C}$, $\eta\wedge(d\eta)^{\wedge n}$ DOES NOT VANISH ON TM

NOTE THAT IF η IS GLOBAL, THEN $\eta\wedge(d\eta)^{\wedge m}$ IS A VOLUME FORM THEREFORE M IS ORIENTABLE

THERE ARE SEVERAL USEFUL NOTIONS IN CONTACT GEOMETRY DEFINED TRADITIONALY IN A LANGUAGE OF CONTACT FORMS. LETS THINK ON THEM FOR A WHILE!

THEOREM: (DARBOUX) LET η BE A CONTACT FORM. FOR EVERY $x \in M$ THERE EXISTS AN OPEN SUBSET $M \supset \mathcal{O} \ni x$ AND COORDINATES $(q^i; p_j, z)$ IN $\mathcal{O}$ SUCH THAT $\eta = dz - p_i dq^i$

AS A CONSEQUENCE $\mathcal{C}|_{\mathcal{O}} = \langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_j} \rangle$

- **REEB VECTOR FIELD**

$R_\eta \in \mathfrak{X}(\mathcal{O})$: $R_\eta \lrcorner d\eta = 0$, $\langle \eta, R_\eta \rangle = 1$ IN DARBOUX COORDINATES $R_\eta = \frac{\partial}{\partial z}$

REEB VECTOR FIELD SEVERELY DEPENDS ON η. CHANGING η TO $\eta' = f\eta$ WE RADICALLY CHANGE R_η – IT IS NOT TRUE THAT $R_{\eta'}$ IS PROPORTIONAL TO R_η, THE DIRECTION CHANGES IF ONLY f IS NOT A CONSTANT FUNCTION

⟶ WEINSTEIN CONJECTURE

- **CONTACT HAMILTONIAN VECTOR FIELDS**

$H \in \mathcal{C}^\infty(\mathbb{R})$ THE FOLLOWING CONDITIONS FOR $X^c_H \in \mathfrak{X}(M)$

$X^c_H \lrcorner d\eta = dH - R_\eta(H)\eta$ $\langle \eta, X^c_H \rangle = -H$ DEFINE X^c_H UNIQUELY.

X^c_H IS CALLED **A CONTACT HAMILTONIAN VECTOR FIELD**

- **PROPERTIES OF X^c_H**

$\mathcal{L}_{X^c_H}\eta = ?$ $\mathcal{L}_{X^c_H}\eta = d\langle \eta, X^c_H \rangle + i_{X^c_H} d\eta =$

$= -dH + dH - R_\eta(H)\eta = R_\eta(H)\eta$

$Y \in \operatorname{Sec}\mathcal{C}$ $[X^c_H, Y]$?

$$d\eta(X^c_H, Y) = X^c_H \overset{0}{\cancel{\langle \eta, Y \rangle}} \underbrace{- Y\langle \eta, X^c_H \rangle}_{+Y(H)} - \langle \eta, [X^c_H, Y] \rangle$$

$$\shortparallel$$

$$Y(H) - R_\eta(H)\overset{0}{\cancel{\langle \eta, Y \rangle}} \Rightarrow \langle \eta, [X^c_H, Y] \rangle = 0 \Rightarrow [X^c_H, Y] \in \mathcal{C}$$

X^c_H DOES NOT PRESERVE η, BUT IT PRESERVES $\mathcal{C}$