Towards testing quantum gravity using cosmological observations

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Outline

- String theory and supergravity compactifications typically produce many moduli fields. These are scalar fields "living" on the uncompactified part of spacetime, which are either real or can be decomposed into their real components.
- Such fields will generally be dynamical in the very early universe, which therefore contains gravity and a finite number of scalar fields at energies lower than the Planck and string scale but higher than the Hubble scale.
- This leads one to consider multifield cosmological models.
- Neglecting higher order corrections, such a model has canonical coupling of the scalar fields to gravity. The target space of the scalar fields is a manifold $\mathcal{M}$ whose dimension equals the number of those fields (target manifold). This manifold need not be topologically trivial.
- The kinetic term of the scalars is described a Riemannian metric $\mathcal{G}$ defined on $\mathcal{M}$ (target space metric)
- In general, the scalars interact though a nontrivial potential, which is modeled by a function $V : \mathcal{M} \to \mathbb{R}$ (scalar potential).

Two-field cosmological models with oriented target space

Definition

A two-dimensional oriented scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected, **oriented** and borderless Riemann surface (called scalar manifold)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (this ensures conservation of energy)
2. $V > 0$ on $\mathcal{M}$ (this avoids technical problems but can be relaxed)

Each scalar triple defines a model of gravity coupled to scalar fields on $\mathbb{R}^4$:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|} \left[\frac{M^2}{2} R(g) - \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) - V\circ\varphi\right] \quad .$$

Define the *rescaled Planck mass* $M_0 \stackrel{\mathrm{def.}}{=} \sqrt{\frac{2}{3}} M$, where M is the reduced Planck mass. Take g to describe a spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \quad , \quad \vec{x} = (x^1, x^2, x^3) \quad , \quad a(t) > 0 \ \forall t)$$

and φ to depend only on the cosmological time: $\varphi = \varphi(t)$.

The cosmological equation and geometric dynamical system

Define the *Hubble parameter* $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$ and the *rescaled Hubble function*:

$$\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0} \quad , \quad \mathcal{H}(u) \stackrel{\text{def.}}{=} \sqrt{||u||^2 + 2V(\pi(u))} \;\; \forall u \in T\mathcal{M} \quad ,$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ is the bundle projection.

Proposition

When $H > 0$, the equations of motion are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\mathcal{H}(\dot{\varphi}(t))\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0 \quad ,$$

together with the Hubble condition:

$$H(t) = \frac{1}{3M_0}\mathcal{H}(\dot{\varphi}(t)) \quad .$$

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation are called cosmological curves. The cosmological equation defines an autonomous dissipative geometric dynamical system on $T\mathcal{M}$. Any cosmological curve φ defines a *cosmological orbit* $\mathcal{O}_\varphi : I \to T\mathcal{M}$ given by $\mathcal{O}_\varphi(t) \stackrel{\text{def.}}{=} (\varphi(t), \dot{\varphi}(t))$, which describes the *state evolution* of this dynamical system.

Let $j^k(\mathcal{M})$ be the k-th jet bundle of curves in $\mathcal{M}$; notice that $j^1(\mathcal{M}) = T\mathcal{M}$.

Definition

A *classical cosmological observable* of order k is a function $f : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U}$ is an open subset of $j^k(\mathcal{M})$. Observables of order 1 are called *basic*.

Any observable of order k can be reduced to a basic observable using the cosmological equation (on-shell reduction of observables). In particular, slow roll & turn parameters of various orders can be reduced on-shell to produce basic observables. Thus local conditions on a cosmological curve which constrain these parameters can be formulated as conditions on points in $T\mathcal{M}$ (conditions on the state of the dynamical system).

Let $f_1, \ldots, f_4 : \mathcal{U} \subset T\mathcal{M} \to \mathbb{R}$ be smooth basic observables which are functionally independent for generic model parameters $(\mathcal{G}, V)$. Since $\dim T\mathcal{M} = 4$, the simultaneous conditions

$$f_1(u) = f_2(u) = f_3(u) = f_4(u) = 0 \quad (u \in \mathcal{U})$$

select a discrete set of points u in $T\mathcal{M}$ for generic $(\mathcal{G}, V)$. Hence no cosmological orbit $\mathcal{O}$ can satisfy these conditions unless $\mathcal{G}$ and V satisfy a constraint (a differential equation) which renders the model non-generic. If we require $|f_j| \ll 1$ instead, the same argument shows that $\mathcal{G}$ and V must *approximately* satisfy a differential equation.

Let (T, N) be the positive Frenet frame of a cosmological curve $\varphi : I \to \mathcal{M}$:

$$T(t) \stackrel{\text{def.}}{=} \frac{\dot{\varphi}(t)}{||\dot{\varphi}(t)||} \quad , \quad N(t) = JT(t) \quad ,$$

where $J \in \mathrm{End}(T\mathcal{M})$ is the complex structure determined on $\mathcal{M}$ by the conformal class of $\mathcal{G}$:

$$\omega(u, v) = \mathcal{G}(Ju, v) \quad , \quad \text{where} \quad \omega \stackrel{\text{def.}}{=} \mathrm{vol}_{\mathcal{G}}$$

and let σ be an increasing proper length parameter for φ:

$$\mathrm{d}\sigma = ||\dot{\varphi}(t)||\mathrm{d}t \quad .$$

Projecting the cosmological equation along T and N gives respectively the *adiabatic* and *entropic* equations:

$$\ddot{\sigma} + \frac{1}{M_0}\mathcal{H}(\sigma, \dot{\sigma})\dot{\sigma} + V_T(\sigma) = 0 \quad , \quad \Omega(\sigma) = \frac{V_N(\sigma)}{\dot{\sigma}} \quad ,$$

where

$$\mathcal{H}(\sigma, \dot{\sigma}) = \sqrt{\dot{\sigma}^2 + 2V(\sigma)} \quad ,$$

$$V_T(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(T(\sigma)) \quad , \quad V_N(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(N(\sigma))$$

and we defined the signed turn rate of φ through:

$$\Omega(t) \stackrel{\text{def.}}{=} -\mathcal{G}(N, \nabla_t T) \quad .$$

Definition

Consider the following functions of t associated to the cosmological curve φ:

- The first, second and third Hubble slow roll parameters:

$$\varepsilon = -\frac{\dot{H}}{H^2} \ , \ \eta_{\parallel} = -\frac{\ddot{\sigma}}{H\dot{\sigma}} \ , \ \xi = \frac{\dddot{\sigma}}{H^2\dot{\sigma}} \ .$$

- The first and second turn parameters:

$$\eta_{\perp} \stackrel{\text{def.}}{=} \frac{\Omega}{H} \ , \ \nu \stackrel{\text{def.}}{=} \frac{\dot{\eta}_{\perp}}{H\eta_{\perp}} \ .$$

- The first IR parameter κ and the conservative parameter c:

$$\kappa \stackrel{\text{def.}}{=} \frac{\dot{\sigma}^2}{2V} \ , \ c \stackrel{\text{def.}}{=} \frac{H\dot{\sigma}}{||\mathrm{d}V||} \ .$$

Remark

The *opposite relative acceleration vector* $\eta \stackrel{\text{def.}}{=} -\frac{1}{H\dot{\sigma}}\nabla_t\dot{\varphi}$ decomposes as $\eta = \eta_{\parallel} T + \eta_{\perp} N$ and we have:

$$\varepsilon = \frac{3\kappa}{1+\kappa} \ .$$

Slow roll and rapid turn conditions

For simplicity, we take $M = 1$ i.e. $M_0 = \sqrt{\frac{2}{3}}$.

Definition

- The first, second and third slow roll conditions are the conditions $\epsilon \ll 1$, $|\eta_{||}| \ll 1$ and $|\xi| \ll 1$.
- The second order slow roll regime is defined by the joint conditions $\epsilon \ll 1$ and $|\eta_{||}| \ll 1$.
- The third order slow roll regime is defined by the joint conditions $\epsilon \ll 1$, $|\eta_{||}| \ll 1$ and $|\xi| \ll 1$.

Definition

- The rapid turn condition is the condition $|\eta_{\perp}| \gg 1$.
- The sustained rapid turn regime is defined by the joint conditions $|\eta_{\perp}| \gg 1$ and $|\nu| \ll 1$.

Proposition

Suppose that the second slow roll condition $|\eta_{||}| \ll 1$ is satisfied. Then the rapid turn condition $|\eta_{\perp}| \gg 1$ is equivalent with the conservative condition $c \ll 1$.

Let $\mathcal{M}_0 \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid (\mathrm{d}V)(m) \neq 0\}$ be the complement of the critical locus.

Definition

The adapted frame of $(\mathcal{M}, \mathcal{G}, V)$ is the oriented orthonormal frame (n, τ) of $\mathcal{M}_0$ defined by the vector fields:

$$n \stackrel{\text{def.}}{=} \frac{\mathrm{grad}\, V}{||\mathrm{grad}\, V||} \quad , \quad \tau = Jn \quad .$$

Definition

The characteristic angle $\theta \in (-\pi, \pi]$ of φ is the angle of rotation from the adapted frame (n, τ) to the Frenet frame (T, N):

$$T = n \cos\theta + \tau \sin\theta \quad , \quad N = -n \sin\theta + \tau \cos\theta \quad .$$

The quantity $s \stackrel{\text{def.}}{=} \mathrm{sign}(\sin\theta) \in \{-1, 0, 1\}$ is called the characteristic sign of φ.

Proposition

We have:

$$\eta_{||} = 3 + \frac{\cos\theta}{c} \quad , \quad \eta_{\perp} = -\frac{\sin\theta}{c} \quad .$$

For any vector fields X, Y, we use the notation $V_{XY} \stackrel{\mathrm{def.}}{=} \mathrm{Hess}(V)(X, Y)$, where $\mathrm{Hess}(V) \stackrel{\mathrm{def.}}{=} \nabla \mathrm{d} V$ is the Riemannian Hessian of V.

Proposition

$$\frac{V_{TT}}{3H^2} = \frac{\Omega^2}{3H^2} + \varepsilon + \eta_{\|} - \frac{\xi}{3}$$
$$\frac{V_{TN}}{H^2} = \frac{\Omega}{H}\left(3 - \varepsilon - 2\eta_{\|} + \nu\right) \quad .$$

Theorem

Suppose that the third order slow roll conditions $\varepsilon \ll 1$, $|\eta_{\|}| \ll 1$ and $|\xi| \ll 1$ as well as the small rate of turn condition $|\nu| \ll 1$ are satisfied. In this case, we have $\cos\theta \approx -3c, \sin\theta \approx s\sqrt{1-9c^2}$ and:

$$V_{TN}^2 \approx 3VV_{TT}$$
$$V_{TT} \approx 9c^2 V_{nn} - 6sc\sqrt{1-9c^2}\, V_{n\tau} + (1-9c^2) V_{\tau\tau}$$
$$V_{TN} \approx -3sc\sqrt{1-9c^2}(V_{\tau\tau} - V_{nn}) - (1-18c^2) V_{n\tau} \quad .$$

These equations admit a solution c with $c \ll 1$ iff:

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2$$

Corollary

The cosmological curve φ satisfies the sustained rapid turn conditions with third order slow roll at cosmological time t iff the following condition is satisfied at the point $m = \varphi(t)$ of $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2 \quad .$$

Definition

The SRRT equation is the following condition which constrains the target space metric $\mathcal{G}$ and scalar potential V on the noncritical submanifold $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} = 3VV_{nn}^2$$

A metric $\mathcal{G}$ on $\mathcal{M}_0$ which satisfies this equation for a fixed scalar potential V is called an SRRT metric relative to V.

The SRRT equation can be written as a nonlinear differential equation for the pair $(\mathcal{G}, V)$ on $\mathcal{M}_0$. When $\mathcal{G}$ is fixed, it can be viewed as a nonlinear second order PDE for V. When V is fixed, it can be viewed as a nonlinear first order PDE for $\mathcal{G}$.

Let $S \stackrel{\text{def.}}{=} \mathrm{Sym}^2(T^*\mathcal{M})$ and $S_+ \subset S$ be the fiber sub-bundle consisting of positive-definite tensors. When V is fixed, the SRRT equation has the form:

$$\mathcal{F}(j^1(\mathcal{G})) = 0 \ \ ,$$

where $\mathcal{F} : j^1(S_+) \to \mathbb{R}$ is a smooth function which depends on V.
Let $L = \det T^*\mathcal{M} = \wedge^2 T^*\mathcal{M}$ be the real determinant line bundle of $\mathcal{M}$ and L_+ be its sub-bundle of positive vectors. Fixing the complex structure J determined by $\mathcal{G}$, the map $\mathcal{G} \to \omega$ gives an isomorphism of fiber bundles $S_+ \stackrel{\sim}{\to} L_+$ which induces an isomorphism $j^1(S_+) \stackrel{\sim}{\to} j^1(L_+)$. Use this to transport $\mathcal{F}$ to a function $F := F_V^J : j^1(L_+) \to \mathbb{R}$. Then the SRRT equation becomes:

$$F(j^1(\omega)) = 0 \ \ .$$

This is a contact Hamilton-Jacobi equation for $\omega \in \Gamma(L_+)$ relative to the Cartan contact structure of $j^1(L_+)$. F restricts to a cubic polynomial function on the fibers of the natural projection $j^1(L_+) \to L_+$.

In local isothermal coordinates (U, x^1, x^2) on $\mathcal{M}$ relative to J, we have:

$$\mathrm{d}s^2_{\mathcal{G}} = e^{2\phi}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \ \ , \ \ \omega = e^{2\phi}\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and one can write the contact HJ equation as a nonlinear first order PDE for the conformal exponent ϕ, which is cubic in the partial derivatives $\partial_1\phi$ and $\partial_2\phi$. A change of local isothermal coordinates corresponds to a contact transformation.

The contact Hamiltonian in isothermal Liouville coordinates

Let $\mathcal{G}_0$ be the locally-defined flat metric with squared line element $\mathrm{d}s_0^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2$ and define the modified Euclidean gradient of V through:

$$\mathrm{grad}_0^J V \stackrel{\text{def.}}{=} J\mathrm{grad}_0 V \quad ,$$

where $\mathrm{grad}_0 V = \mathrm{grad}_{\mathcal{G}_0} V = \partial_1 V \partial_1 + \partial_2 V \partial_2$ is the ordinary Euclidean gradient. Let $\cdot$ denote the Euclidean scalar product defined by $\mathcal{G}_0$, thus $\partial_i \cdot \partial_j = \delta_{ij}$. Let:

$$H_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, \mathrm{grad}_0 V) = \partial_i \partial_j V \partial_i V \partial_j V \quad ,$$
$$\tilde{H}_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, J\mathrm{grad}_0 V) = -\partial_i \partial_j V \partial_i V \varepsilon_{jk} \partial_k V \quad .$$

Let $U \subset \mathcal{M}_0$ and $U_0 \subset \mathbb{R}^2$ be the image of U in the isothermal chart (U, x^1, x^2). The isothermal Liouville coordinates $(U, x^1, x^2, u, p_1, p_2)$ induce an isomorphism of fiber bundles $j^1(L_+)|_U \simeq U_0 \times \mathbb{R} \times \mathbb{R}^2$. Consider the smooth functions $A, B : U_0 \times \mathbb{R}^2 \to \mathbb{R}$ defined through:

$$A(x,p) \stackrel{\text{def.}}{=} (\partial_i V)(x) p_i \quad , \quad B(x,p) \stackrel{\text{def.}}{=} -\epsilon_{ij} (\partial_j V)(x) p_i \quad .$$

The linear transformation $\mathbb{R}^2 \ni (p_1, p_2) \to (A(x), B(x)) \in \mathbb{R}^2$ is nondegenerate for $x \in U_0$, with inverse:

$$p_1 = \frac{\partial_1 V A - \partial_2 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad , \quad p_2 = \frac{\partial_2 V A + \partial_1 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad .$$

Theorem

In isothermal Liouville coordinates (x^1, x^2, u, p_1, p_2) *on* $j^1(L_+)|_U$, *the contact Hamiltonian is given by the smooth function* $F : U_0 \times \mathbb{R}^3 \to \mathbb{R}$ *given by:*

$$F(x, u, p) \stackrel{\text{def.}}{=} -[B(x) - \tilde{H}_0(x)]^2[A(x, p) + (\Delta_0 V)(x) - H_0(x)] - 3e^{2u} V[A(x, p) - H_0(x)]^2$$

and the contact Hamilton-Jacobi equation takes the form:

$$F(x_1, x_2, \phi, \partial_1\phi, \partial_2\phi) = 0 \quad .$$

Remark

- The contact HJ equation can be solved *locally* through the method of characteristics.
- The contact Hamiltonian is proper in the sense of Crandall & Lyons, i.e. is nondecreasing in u. Hence the Dirichlet problem can be approached *globally* using the theory of viscosity solutions.

We have:

$$-F = AB^2 - 3Ve^{2u}A^2 + (\Delta_0 V - H_0)B^2 - 2\tilde{H}_0 AB + (6Ve^{2u}H_0 + \tilde{H}_0^2)A + 2\tilde{H}_0(H_0 - \Delta_0 V)B - F_0 \quad ,$$

where:

$$F_0 = -\tilde{H}_0^2[(\Delta_0 V) - H_0] + 3Ve^{2u}H_0^2 \quad .$$

Define:

$$P_1 \stackrel{\text{def.}}{=} A - H_0 \quad , \quad P_2 = B - \tilde{H}_0 \quad ,$$

The momentum curve

The momentum curve is the curve $C_{x,u}$ defined by the condition $F(x, u, p) = 0$ in the p-plane. This curve passes through the origin of P-plane, i.e. through the point with coordinates:

$$p_1 := p_{01} \stackrel{\text{def.}}{=} -\frac{\mathrm{grad} V \cdot (-H_0, \tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_1 V H_0 - \partial_2 V \tilde{H}_0}{(\partial_1 V)^2 + (\partial_2 V)^2}$$

$$p_2 := p_{02} \stackrel{\text{def.}}{=} \frac{\mathrm{grad}_J V \cdot (-H_0, \tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_2 V H_0 + \partial_1 V \tilde{H}_0}{(\partial_1 V)^2 + (\partial_2 V)^2}$$

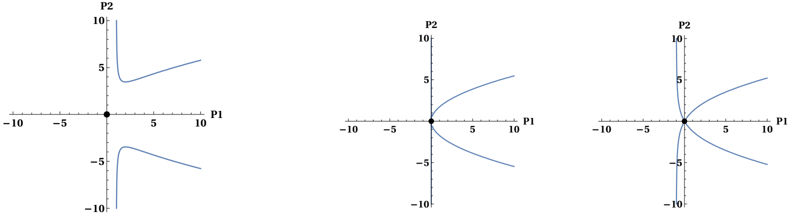
in the p-plane. The singular points of the momentum curve coincide with the characteristic points of the contact HJ equation.

Proposition

The origin of the P-plane is the only singular point of the momentum curve. When $(\Delta_0 V)(x) = 0$, the curve is reducible and F factorizes as:

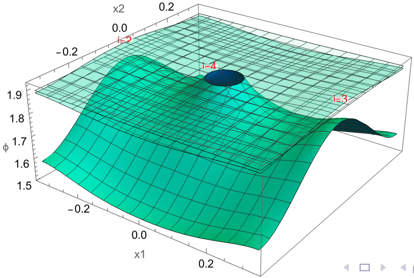
$$F = P_1(P_2^2 - 3Ve^{2u} P_1) \quad .$$

The curve is symmetric under reflection in the P_1-axis. When $(\Delta_0 V)(x) > 0$, it is connected and contained in the half-space $P_1 \geq -(\Delta V)(x)$, being the union of two embedded curves which intersect each other at the origin of the P-plane. When $(\Delta_0 V)(x) < 0$, it has three connected components, namely the origin of the (P_1, P_2)-plane (which is its only singular point) and two connected components which are nonsingular and contained in the half-space $P_1 > -(\Delta_0 V)(x)$.



() $(\Delta_0 V)(x) = -1.$ () $(\Delta_0 V)(x) = 0.$ () $(\Delta_0 V)(x) = 1.$

Figure: The momentum curve for $V(x)e^{2u(x)} = 1$ in the cases $(\Delta_0 V)(x) = -1, 0, 1$. The singular point of the curve is shown as a black dot.



Quasilinear approximation near an isolated critical point

Let $c \in U_0$ be an isolated critical point of V and λ_1, λ_2 be the principal values of $\mathrm{Hess}(V)(c)$. In principal isothermal coordinates centered at c, we have:

$$V(x) = V(c) + \frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2) + \mathcal{O}(||x||_0^3) \ .$$

Consider the following homogeneous polynomial functions of degree two in the variables x_1 and x_2, where $k \in \mathbb{Z}_{>0}$:

$$s_k(x) \stackrel{\text{def.}}{=} \lambda_1^k x_1^2 + \lambda_2^k x_2^2 \ .$$

Proposition

We have:

$$F(x,u,p) = -\frac{a_1(x,u)x^1 p_1 + a_2(x)x^2 p_2 - b(x,u)}{s_2(x)^3} + \mathcal{O}(||x||_0^2) \ ,$$

where a_i and b are homogeneous polynomial functions of degree six in x_1 and x_2 (whose coefficients depend on u) given by:

$$a_i(x,u) = \lambda_i s_2(x)\left[t_i(x) + 6V(c)e^{2u}s_2(x)s_3(x)\right]$$

with:

$$t_1(x) = \lambda_1\lambda_2^2(\lambda_1 - \lambda_2)x_2^2[s_2(x) - 3\lambda_2 s_1(x)]$$
$$t_2(x) = \lambda_2\lambda_1^2(\lambda_2 - \lambda_1)x_1^2[s_2(x) - 3\lambda_1 s_1(x)] \ .$$

and:

$$b(x,u) = -\lambda_1^3\lambda_2^3(\lambda_1 - \lambda_2)^2 x_1^2 x_2^2 s_1(x) + 3V(c)e^{2u}s_2(x)s_3(x)^2 \ .$$

Corollary

The contact HJ equation is approximated to first order in $||x||_0$ by the following quasilinear first order PDE:

$$a_1(x,\phi)x^1\partial_1\phi + a_2(x,\phi)x^2\partial_2\phi = b(x,\phi) \quad . \tag{1}$$

This quasilinear PDE can be studied by the Lagrange-Charpit method. Its scale-invariant solutions can be studied by reduction to a nonlinear ODE for a function defined on the unit circle.

Proposition

Suppose that ϕ satisfies the quasilinear equation (1) and that we have $\varphi(x) \gg 1$. Then ϕ is an approximate solution of the following linear first order PDE:

$$2s_2(x)\lambda_i x^i \partial_i\phi = s_3(x) \quad , \tag{2}$$

which it satisfies up to corrections of order $\mathcal{O}\left(\frac{e^{-2\phi}}{3V(c)}\right)$.

Consider the polar coordinate system (r, θ) defined though:

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad . \tag{3}$$

Proposition

Suppose that $\lambda_1 \neq \lambda_2$. Then the general smooth solution of the linear equation (2) *is:*

$$\phi(r, \theta) = \phi_0(\theta) + Q_0\left(\frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}\log r + \frac{1}{\lambda_1}\log|\cos\theta| - \frac{1}{\lambda_2}\log|\sin\theta|\right) \quad , \tag{4}$$

where:

$$\phi_0(\theta) = \frac{1}{4}\log(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta) - \frac{1}{2}\frac{\lambda_2\log|\cos\theta| - \lambda_1\log|\sin\theta|}{\lambda_2 - \lambda_1} \tag{5}$$

and Q_0 is an arbitrary smooth function of a single variable.

Proposition

Suppose that $\lambda_1 = \lambda_2 := \lambda$. Then the linear equation (2) *reduces to:*

$$x^i\partial_i\phi = \frac{1}{2} \quad , \tag{6}$$

whose general solution is:

$$\phi(r, \theta) = \frac{1}{2}\log r + Q_0(\theta) \quad , \tag{7}$$

where $Q_0 \in \mathcal{C}^\infty(\mathrm{S}^1)$ is an arbitrary smooth function.

Suppose that $\lambda_1 \neq \lambda_2$. The general solution (4) reads:

$$\phi(r,\theta) = \phi_0(\theta) + Q\left(\log r + \frac{\lambda_2 \log|\cos\theta| - \lambda_1 \log|\sin\theta|}{\lambda_2 - \lambda_1}\right)$$

and satisfies $\lim_{r\to 0}\phi(r,\theta) = +\infty$ iff $\lim_{w\to -\infty} Q(w) = +\infty$. In this case, we have:

$$\phi \approx Q(\log r) \quad \text{for} \quad r \ll 1 \quad ,$$

so ϕ is rotationally-invariant near c. The corresponding SRRT metric is asymptotically rotationally-invariant at c, with Gaussian curvature:

$$K \approx -e^{-2\phi}\Delta\phi \approx -e^{-2Q(\log r)}Q''(\log r) \quad \text{for} \quad r \ll 1 \quad .$$

Requiring $K = K_c$ for some constant K_c gives:

$$e^{-2Q(w)}Q''(w) = K_c \ .$$

Also require that $\mathcal{G}$ is geodesically complete at c. For $K_c = 0$, we can take $Q(w) = -w$, which gives $\phi(r,\theta) \approx_{r\ll 1} -\log r$ and:

$$\mathrm{d}s^2 \approx_{r\ll 1} \frac{1}{r^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) = \mathrm{d}\rho^2 + \mathrm{d}\theta^2 \quad , \quad \text{where} \quad \rho \stackrel{\text{def.}}{=} \log r \quad .$$

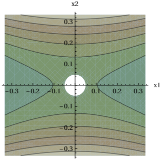
so $\mathcal{G}$ asymptotes at c to the metric on a flat cylinder. For $K_c = -1$, the SRRT metric $\mathcal{G}$ asymptotes to the hyperbolic cusp metric at c:

$$\mathrm{d}s^2 \approx \frac{1}{(r\log r)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) \quad \text{for} \quad r \ll 1 \quad . \tag{8}$$

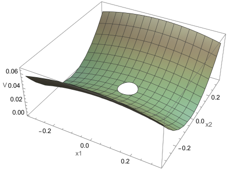
A natural Cauchy problem

Consider a circle $C_R \subset U_0$ of radius $R < 1$ centered at $0 \in U_0$ and the b.c.:

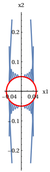
$$\phi|_{C_R} = -\log[R\log(1/R)] \ .$$



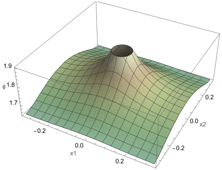
() Contour plot of the potential.



() 3D plot of the potential.

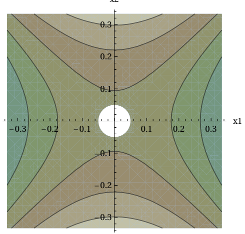


() Projected characteristic curves.

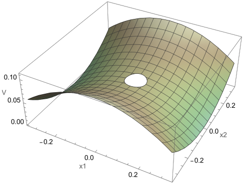


() Solutions of the Dirichlet problem for the viscosity perturbation with $\mathfrak{c} = e^{-7}$.

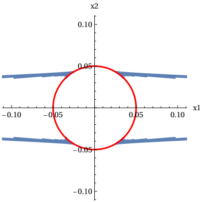
Figure: The potential, projected characteristics and a viscosity approximant of the solution of the Dirichlet problem for the contact Hamilton-Jacobi equation for $V_c = 1/90$ and $\lambda_1 = -1/5$, $\lambda_2 = 1$ with $R = 1/20$.



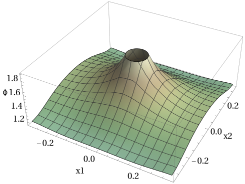
() Contour plot of the potential.



() 3D plot of the potential.



() Some characteristic curves projected on the (x_1, x_2)-plane.



() Solution of the Dirichlet problem for the viscosity perturbation with $\mathfrak{c} = e^{-8}$.

Figure: The potential, projected characteristics and a viscosity approximant of the solution of the Dirichlet problem for $V_c = 1/18$ and $\lambda_1 = -1$, $\lambda_2 = 1$ with $R = 1/20$.