METAPLECTIC GROUPS AND QUADRATIC HAMILTONIANS

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1. The metaplectic group

Definition 1.1. Let n be a positive integer. The metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$ is the Lie group defined up to isomorphism as the double cover of the symplectic group.

By definition, $M_{P_{2n}}(\mathbb{R})$ fits into the following short exact sequence of Lie groups:

$$\mathbf{1} \to \boldsymbol{\mu}_2 \to \mathrm{Mp}_{2n}(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R}) \to \mathbf{1} \ , \tag{1}$$

where:

- $\mathbf{1}$ is the trivial group (the group with a single element);
- $\boldsymbol{\mu}_2 := \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}$ is the two-element central subgroup of the metaplectic group consisting of $+1$ and -1, where 1 is the unit of $M_{P_{2n}}(\mathbb{R})$;
- $\mathrm{Sp}_{2n}(\mathbb{R})$ is the symplectic group in dimension $2n$, defined as the multiplicative group of $2n \times 2n$ real matrices whose action as linear operators on $\mathbb{R}^{2n}$ preserves the canonical symplectic form of the latter.

Remarks.

- Metaplectic groups play an important role in symplectic geometry and harmonic analysis.
- They are also important in algebraic geometry and number theory, since modular forms of half-integer weight arise from automorphic representations of multiplectic groups.

Let us recall the definitions of some notions which will be used in the following.

- **Symplectic Matrices**. These are defined as those square matrices of size $2n$ whose action as linear operators on $\mathbb{R}^{2n}$ preserve the canonical symplectic form of the latter. The canonical symplectic form is the alternate (i.e. bilinear and antisymmetric) and non-degenerate pairing ω_n on $\mathbb{R}^{2n}$ defined through:

$$\omega_n(x, y) = x^T J_n y \ \forall x, y \in \mathbb{R}^{2n} \ , \tag{2}$$

where J_n is the square matrix:

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \ . \tag{3}$$

It is easy to check that the real square matrix M of size $2n$ is symplectic if it satisfies the condition $M^T J_n M = J_n$. The Darboux theorem implies that any symplectic pairing on $\mathbb{R}^{2n}$ is equivalent to ω_n under up to a general linear transformation of $\mathbb{R}^{2n}$. Symplectic matrices have the following *key properties*:

(1) Any symplectic matrix M satisfies $\det M = 1$;
(2) The set of all symplectic matrices of size $2n$ forms a group called the symplectic group in 2n-dimensions, which is denoted $\mathrm{Sp}_{2n}(\mathbb{R})$. By the first property above, this is a subgroup of the special linear group

$SL_2(\mathbb{R})$, so the action of a symplectic matrix as an operator on $\mathbb{R}^{2n}$ preserves the canonical orientation of the latter.

(3) In two dimensions (i.e. when $n = 1$), the canonical symplectic form coincides with the area form, which is defined as the bilinear form which gives the oriented area of the parallelogram constructed on an ordered pair of vectors in $\mathbb{R}^2$. This implies that the groups $Sp_2(\mathbb{R})$ and $SL_2(\mathbb{R})$ coincide.

Remark. A direct way to see that $Sp_2(\mathbb{R}) = SL_2(\mathbb{R})$ is as follows. For $n = 1$, we have:

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

Direct computation shows that the condition $M^T J_1 M = J_1$ that a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be symplectic reduces to $ad - bc = 1$, i.e. $\det M = 1$.

- The **Siegel upper half plane** $\mathbb{H}$ is defined as the set of all complex numbers whose imaginary part is positive. This is the particular case for $n = 1$ of the Siegel space $\mathbb{H}_n$ of all complex symmetric matrices whose imaginary part is positive-definite. The latter plays an important role in higher-dimensional hyperbolic geometry and in the theory of Abelian varieties due to Riemann's bilinear relations.
- **Diffeomorphism**. Two manifolds M and N are diffeomorphic if there exists a smooth bijective map $f : M \to N$ whose inverse f^{-1} is also smooth. Intuitively, this means that M can be smoothly deformed into N without tearing or gluing.

Example. The $2-$ sphere S^2 is diffeomorphic to an ellipsoid.

2. Connection to modular forms of half-integer weight and number theory

Naively, modular forms of fractional weight $k \in \mathbb{Q}$ are univariate holomorphic functions f defined on the Siegel upper half-plane $\mathbb{H}$ that satisfy the following transformation law (known as the naive *modularity condition*) under the action of the classical modular group $SL_2(\mathbb{Z}) = Sp_2(\mathbb{Z})$ by fractional transformations of $\mathbb{H}$:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \ . \tag{5}$$

Here:

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ . \tag{6}$$

The rational number k is called the *weight* of the modular form f.

Shimura studied modular forms of half-integer weight $k \in \frac{1}{2} + \mathbb{Z}$. In that case, the transformation property above must be clarified because $(c\tau + d)^k$ does not make sense globally when k is half-integer. More precisely, we need a choice of a square root of $c\tau + d$ which is a holomorphic function of $\tau \in \mathbb{H}$. Such a choice naturally "lives" on the complex plane, which provides a double cover of $\mathbb{H}$ branched along the real axis. This leads one to introduce a multiplier system χ (a suitable multiplicative character of the group $SL_2(\mathbb{Z})$ also known as the "metaplectic correction factor") so the naive modularity condition above is replaced by:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(\gamma)\,(c\tau + d)^k f(\tau) \ . \tag{7}$$

With this precise definition, one can develop a consistent theory of modular forms of half-integer weight which generalizes the classical theory of integer weight modular forms which goes back to Riemann, Weierstrass and Gauss.

How modular transformations work in practice. Let $k = \frac{1}{2} + n$; $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)(c\tau+d)^k f(\tau), k \in \frac{1}{2} + \mathbb{Z}$. Consider the classic example of the Jacobi θ function:

$$\theta(\tau) = \sum\nolimits_{-\infty}^{\infty} q^{n^2} \text{ , where } q = e^{2\pi i\tau} \text{ \& } \tau \in \mathbb{H} \text{ ,} \tag{8}$$

which obeys the modular transformation law:

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)(c\tau+d)^{\frac{1}{2}}\theta(\tau) \text{ .} \tag{9}$$

Here $\varepsilon(a,\ldots,d)$ is the fundamental 8th root of unity. The explicit transformations under the action of the canonical generators of the modular group $\mathrm{SL}_2(\mathbb{Z})$ by fractional transformations on the Siegel upper half plane are:

- Under $\tau \to \tau + 1$

$$\theta(\tau+1) = \sum_n e^{2\pi i n^2(\tau+1)} = \sum_n e^{2\pi i n^2\tau} e^{2\pi i n^2} = \theta(\tau) \text{ .} \tag{10}$$

- Under $\tau \to -\frac{1}{\tau}$:

$$\tau\left(-\frac{1}{\tau}\right) = \sqrt{-it}\theta(\tau) \text{ .} \tag{11}$$

Connection to automorphic forms and number theory. Let $f : \mathbb{H} \to \mathbb{C}$ be a modular form of weight k for the arithmetic group $\Gamma \supset \mathrm{SL}_2(\mathbb{Z})$:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}_2(\mathbb{R}) \text{ .} \tag{12}$$

Using the action of $\mathrm{SL}_2(\mathbb{R})$ given by:

$$(\pi(g) f)(\tau) = (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) \ \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \tag{13}$$

one finds that the automorphic representation corresponding to f is the irreducible piece of the representation π of $\mathrm{SL}_2(\mathbb{R})$ which contains f. Via the θ-correspondence, this leads to deep connections between metaplectic groups and number theory (and in particular the Langlands program).

Adelic version. Over the adeles $\mathbb{A}$, a classical modular form f for the arithmetic group Γ can be lifted to an automorphic form:

$$F : \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C} \tag{14}$$

such that

$$F(\gamma g) = F(g) \ \forall \gamma \in \mathrm{SL}_2(\mathbb{Q}) \ \forall g \in \Gamma \tag{15}$$

where $\mathrm{SL}_2(\mathbb{A})$ acts by right translation:

$$(\pi(h) F)(g) = F(gh) \ \ \forall h \in \mathrm{SL}_2(\mathbb{A}) \ \forall g \in \Gamma \text{ .} \tag{16}$$

The space generated by all such F forms an automorphic representation of $\mathrm{SL}_2(\mathbb{A})$.

3. Topological description of metaplectic groups

Recall special linear group in two dimensions is the Lie group:

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{R}) \;\middle|\; ad - bc = 1 \right\} . \tag{17}$$

The group operation is given by matrix multiplication and the differentiable structure which makes it into a Lie group comes from the natural embedding as three-dimensional submanifold of $\mathbb{R}^4$. The subgroup of two-dimensions rotations is diffeomorphic to the circle:

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; 0 \leq \theta \leq 2\pi \right\} . \tag{18}$$

Consider the real vector space of symmetric matrices with zero trace:

$$\mathfrak{p} := \left\{ \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix} \;\middle|\; \beta, \gamma \in \mathbb{R} \right\} . \tag{19}$$

Multiplication and the exponential give a diffeomorphism:

$$\phi : \mathrm{SO}_2(\mathbb{R}) \times \mathfrak{p} \to \mathrm{SL}_2(\mathbb{R}) \;,\; \phi(r, X) := r \cdot e^X \;. \tag{20}$$

This implies that the fundamental groups of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SO}_2(\mathbb{R})$ (both taken with base point 1) are isomorphic. Since $\mathrm{SO}_2(\mathbb{R})$ is diffeomorphic with the circle S^1, we obtain:

$$\pi_1(\mathrm{SL}_2(\mathbb{R})) \equiv \pi_1(\mathrm{SO}_2(\mathbb{R})) = \pi_1\left(\mathrm{S}^1\right) = \mathbb{Z} \;. \tag{21}$$

From the fact that the fundamental group of $\mathrm{SL}_2(\mathbb{R})$ is $\pi_1(\mathrm{SL}_2(\mathbb{R})) := \mathbb{Z}$, we can take its quotient by $2\mathbb{Z}$, which gives the cyclic group $\mathbb{Z}/2\mathbb{Z}$. This quotient corresponds to passing to the two-fold cover space of $\mathrm{SL}_2(\mathbb{R})$. By general Lie group theory, the 2-fold covering space carries a natural induced Lie group structure, thereby becoming a Lie group denoted $\mathrm{Mp}_2(\mathbb{R})$. The underlying manifold structure of this Lie group makes the covering projection $\pi : \mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ is a surjective submersion. This new group $\mathrm{Mp}_2(\mathbb{R})$ is the metaplectic group and, by construction, it fits into the short exact sequence:

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{Mp}_2(\mathbb{R}) \xrightarrow{\pi} \mathrm{SL}_2(\mathbb{R}) \to 1 \;. \tag{22}$$

The term *metaplectic* was coined by Andre Weil in a broader framework, where metaplectic groups serve as covering groups of the symplectic groups. In fact, *Cartan's diffeomorphism* (20) also applies to the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{R})$, leading to the identification:

$$\pi_1(\mathrm{Sp}_{2n}(\mathbb{R})) = \pi_1(U(n)) = \mathbb{Z} \tag{23}$$

for $n \geq 1$.

General proof of Cartan's diffeomorphism. We start by defining the Lie algebra $\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, and the Cartan involution $\theta(X) = -X^T$, where:

$$\mathfrak{k} = \left\{ \mathrm{X} \in \mathfrak{sl}_2(\mathbb{R}) \middle| \theta(\mathrm{X}) = \mathrm{X} \right\} = \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\} \tag{24}$$

and

$$\mathfrak{p} = \{ X \in \mathfrak{sl}_2(\mathbb{R}) : \theta(\mathrm{X}) = -\mathrm{X} \} = \left\{ \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix} \;\middle|\; \beta, \gamma \in \mathbb{R} \right\} \tag{25}$$

$$K = e^{\mathfrak{k}} = \mathrm{SO}(2) \tag{26}$$

and:

$$e^{\mathfrak{p}} = \left\{ \begin{pmatrix} \cosh(t) + u\sinh(t) & v\sinh(t) \\ v\sinh(t) & \cosh(t) - u\sinh(t) \end{pmatrix} \;\middle|\; u^2 + v^2 = 1, t \in \mathbb{R} \right\} , \tag{27}$$

Since θ is a Cartan involution, we write

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p} \ , \tag{28}$$

and $\mathrm{SL}_2(\mathbb{R}) = Ke^{\mathfrak{p}}$. The map $\phi(r, X) = re^X$ is a global diffeomorphism. For $g \in \mathrm{SL}_2(\mathbb{R})$, we have $g^T g \in e^{2\mathfrak{p}}$, $X = \frac{1}{2}\log\left(g^T g\right) \in \mathfrak{p}$, $r = ge^{-X} \in \mathrm{SO}(2)$ and hence $g = re^X$. In this general context, the diffeomorphism reads:

$$\phi : K \times \mathfrak{p} \xrightarrow{\text{diffeo.}} G \ , \tag{29}$$

where K is the maximal compact group of G. □

The same topological reasoning that gives a two-fold covering of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ also allows us to construct a d-fold cover of $\mathrm{Sp}_{2n}(\mathbb{R})$ for any positive integer d. This corresponds to a projection of its fundamental group onto the cyclic group of order d:

$$\pi_1(\mathrm{Sp}_{2n}(\mathbb{R})) = \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \ . \tag{30}$$

Such covering groups of $Sp_{2n}(\mathbb{R})$ are sometimes called *higher metaplectic groups*, though this terminology is not always used consistently.

The situation is quite different for the special linear group $\mathrm{SL}_n(\mathbb{R})$. While $\mathrm{SO}_2(\mathbb{R})$ (the rotation group in two dimensions) is topologically a circle with fundamental group $\mathbb{Z}$, the rotation group $\mathrm{SO}_3(\mathbb{R})$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. This leads to the short exact sequence:

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R}) \to 1 \ . \tag{31}$$

In fact, for all $n \geq 3$, we have:

$$\mathrm{SO}_2(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \ . \tag{32}$$

Since $\mathrm{SO}_n(\mathbb{R})$ is a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$, it follows that $\mathrm{SL}_n(\mathbb{R})$ only admits twofold coverings when $n \geq 3$. Note that some authors refer to these coverings as metaplectic covers of $\mathrm{SL}_2(\mathbb{R})$ even though the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is no longer present.

Remark. For $n = 1$, we recover $\mathrm{Mp}_2(\mathbb{R})$, because $\mathrm{SL}_2(\mathbb{R}) = \mathrm{Sp}_2(\mathbb{R})$. More generally, the 2-fold cover $\mathrm{Mp}_{2n}(\mathbb{R})$ of the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{R})$ are referred to as metaplectic groups.
Summary of various symbols, terms and their meaning.

- $\pi_1(G)$: The fundamental group of a topological space (or Lie group) G, which measures how many distinct ways one can loop around inside G.
- $\mathbb{Z}/d\mathbb{Z}$: The cyclic group of order d, obtained by taking integers modulo d (here d is a positive integer).
- $d-$fold covering: A topological covering map where each point in the base space has exactly d distinct points above it in the covering space.
- $\mathrm{SO}_2(\mathbb{R}) \cong \mathrm{S}^1$: The 2-dimensional rotation group, topologically a circle, with fundamental group $\mathbb{Z}$.
- $\mathrm{SO}_3(\mathbb{R})$: The 3-dimensional rotation group, whose loops wrap twice before closing giving $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.
- $\mathrm{SU}(2)$: The double cover of $\mathrm{SO}_3(\mathbb{R})$; it is topologically a 3-sphere S^3.
- Maximal compact group: The largest compact subgroup of a Lie group, which is determined up to isomorphism. This is homotopy equivalent with the original Lie group and hence captures its homotopy invariants and in particular its homotopy groups. For $\mathrm{SL}_n(\mathbb{R})$, the maximal compact subgroup is $\mathrm{SO}_n(\mathbb{R})$.

4. Weil's Groups of Operators

In the introductory section, we described the metaplectic group topologically but did not provide an explicit construction.

Its construction can be compared to that of the spin groups. For example, the fundamental group of $\mathrm{SO}_n(\mathbb{R})$ has order 2 for $n > 2$. Consequently, we can write the corresponding central extension:

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R}) \to 1$$

where $\mathrm{Spin}_n(\mathbb{R})$ is the compact group known as the spin group.

A concrete construction of $\mathrm{Spin}_n(\mathbb{R})$ can be achieved through the *Clifford algebra* associated with $\mathbb{R}^2$, which has dimension 2^n.

This approach allows us to embed $\mathrm{Spin}_n(\mathbb{R})$ into the group of invertible $2^m \times 2^m$ matrices, where $m = \lfloor (n-1)/2 \rfloor$. Although this embedding involves high-dimensional representations, it provides an explicit realization of the spin groups as matrix groups.

One might be tempted to draw an analogy between $\mathrm{Spin}_n(\mathbb{R})$ and the metaplectic groups, yet the two are conceptually distinct. The classical metaplectic groups $\mathrm{Mp}_{2n}(\mathbb{R})$ do not admit any finite-dimensional faithful matrix representation — a fact established by a fundamental theorem.

Theorem 4.1. *Let N be a positive integer and let $\rho : \mathrm{Mp}_{2n}(\mathbb{R}) \to \mathrm{GL}(\mathbb{C})$ be a continuous morphism. Then* $\ker(\rho)$ *contains $\boldsymbol{\mu}_2$, i.e., ρ factors through* $\mathrm{Sp}_{2n}(\mathbb{R})$. *In particular, ρ is not injective.*

A complete proof of this result is hard to find in the literature; Bourbaki mentions it only as an exercise in "Lie groups and Lie algebras", Ch. III, Exercises for Sec. 6.

It is sufficient to consider the case $n = 1$, since $\mathrm{Mp}_2(\mathbb{R}) \subset \mathrm{Mp}_{2n}(\mathbb{R})$ for all $n \geq 1$. The finite-dimensional representations of $\mathrm{Mp}_2(\mathbb{R})$ are determined by the corresponding representations of its Lie algebra:

$$\mathfrak{mp}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R}) \ .$$

All such representations can be explicitly described, and each corresponding representation of $\mathrm{Mp}_2(\mathbb{R})$ factors through the quotient:

$$\mathrm{Mp}_2(\mathbb{R})/\boldsymbol{\mu}_2 = \mathrm{SL}_2(\mathbb{R})$$

Since there is no way to express all the elements of the metaplectic group using finite-dimensional matrices, one must instead consider groups of unitary operators acting on a Hilbert space. The first one to construct the metaplectic group explicitly through unitary operators was Andre Weil, in his paper "Sur certains groupes d'operateurs unitaires".

4.1. The standard unitary representation of the metaplectic group.

In what follows, we describe the metaplectic group in terms of unitary operators acting on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ of square-integrable functions. These operators arise naturally within the framework of harmonic analysis.

The example we start with is the Fourier transform of a function f:

$$[\mathbf{F}f](x) = \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i x y}\, \mathrm{d}y \ . \tag{33}$$

For convenience, we scale the Fourier transform by an eighth root of unity and define:

$$\mathbf{W}f = \zeta\, \mathbf{F}f \ , \tag{34}$$

where $\zeta = e^{i\pi/4}$ is a phase rotation. Applying the Fourier inversion formula, we have:

$$[\mathbf{F}\mathbf{F}f](x) = f(-x)\ ,\ [\mathbf{W}\mathbf{W}f](x) = if(-x)\ . \tag{35}$$

Note that $\mathbf{F}^4 = \mathrm{id}$ but $\mathbf{W}^4 = -\mathrm{id}$ in the group $U(\mathcal{H})$ of unitary operators of $\mathcal{H}$.

Another operator comes from multiplying by a complex function of absolute value 1. For $b \in \mathbb{R}$ define the quadratic exponential

$$\sigma_b(x) = e^{\pi i b x^2} \tag{36}$$

(a quadratic sinusoid) and the corresponding unitary operators $\mathbf{E}_+(b)$, $\mathbf{E}_-(b)$ as $[\mathbf{E}_+(b)f](x) = \sigma_b(x) \cdot f(x)$ and $\mathbf{E}_-(b) = \mathbf{W}\mathbf{E}_+(b)W^{-1}$.

Remark 4.2. The operator $\mathbf{E}_+(b)$ "twists" the function in the position space whereas $\mathbf{E}_-(b)$ "twists" the function in the momentum/Fourier space. Together, they generate all quadratic phase transformations that appear in the metaplectic group.

Using a standard computation in Fourier analysis, we find that $\sigma_b = |b|^{-1/2}\xi^{\mathrm{sign}(b)}\sigma_{-1/b}$. This allows us to express $\mathbf{E}_-(b)$ as:

$$[\mathbf{E}_-(b)f](x) = |b|^{-1/2}\xi^{\mathrm{sign}(b)} \cdot [\sigma_{-1/b} * f](x)\ , \tag{37}$$

where $*$ denotes the convolution.

For any nonzero real number u, we define the time-scaling operator $\mathbf{H}(u)$ by:

$$[\mathbf{H}(u)f](x) = \begin{cases} |u|^{-1/2}f(ux) & u > 0 \\ -i|u|^{-1/2}f(ux) & u < 0 \end{cases} \tag{38}$$

This operator stretches or compresses the function while preserving unitarity, with the factor 1 or $-i$ chosen so that the following identity holds:

$$\mathbf{E}_-(u^{-1}) = \mathbf{E}_+(-u)\mathbf{H}(u)W \forall u \in \mathbb{R}^\times \tag{39}$$

This implies

$$\mathbf{H}(u) = \mathbf{E}_+(u)\mathbf{E}_-(u^{-1})\mathbf{E}_+(u)\mathbf{W}^{-1}\ . \tag{40}$$

Proposition 4.3. *The operators defined above obey the following relations for all $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^\times$:*

(1) $\mathbf{W}^2 = \mathbf{H}(-1)$
(2) $\mathbf{E}_+(a + b) = \mathbf{E}_+(a)\mathbf{E}_+(b)$
(3) $\mathbf{H}(u)\mathbf{H}(v) = (u, v)_2\mathbf{H}(uv)$, *where the expression $(u, v)_2 \in \{-1, 1\}$ is the* Hilbert symbol, *which is defined as:*

$$(u, v)_2 = \begin{cases} 1 & \text{if } u > 0 \text{ or } v > 0 \\ -1 & \text{if } u < 0 \text{ and } v < 0 \end{cases} \tag{41}$$

(4) $\mathbf{H}(u)\mathbf{E}_+(a)\mathbf{H}(u^{-1}) = \mathbf{E}_+(au^2)$.

The operators and relations defined above are relevant to our representation, as they are analogous to the matrices and relations in $\mathrm{SL}_2(\mathbb{R})$. For an arbitrary field F, we define the following standard elements of SL_2:

$$w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_+(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \forall b \in F\ , \tag{42}$$

where the matrix w represents a fundamental involution exchanging coordinates—an algebraic analogue of the Fourier transform—while $e_+(b)$ corresponds to a shear transformation, analogous to the action of the operator $\mathbf{E}_+(b)$ defined earlier.

For all $b \in F$ and $u \in F^\times$ we define:

$$e_-(b) := we_+(b)\,w^{-1} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{43}$$

$$h\left(u\right):=e_{+}\left(u\right)e_{-}\left(u^{-1}\right)e_{+}\left(u\right)w^{-1}=\left(\begin{array}{cc}u & 0\\ 0 & u^{-1}\end{array}\right). \tag{44}$$

Proposition 4.4. *Let F be any field. The group $\mathrm{SL}_2\left(F\right)$ can be generated by the set $\{w, e_+\left(b\right) : b \in F\}$ subject to the following relations:*

(1) $w^2 = h(-1)$
(2) $e_+(a+b) = e_+(a)e_+(b)$ *for all* $a, b \in F$
(3) $h\left(u\right)h\left(v\right) = h\left(uv\right)$ *for all* $u, v \in F^{\times}$
(4) $h\left(u\right)e_+\left(a\right)h\left(u^{-1}\right) = e\left(au^2\right)$ *for all* $a \in F$ *and* $u \in F^{\times}$.

Upon closer examination, the only distinction between the last two propositions is related to the sign of the third relation. Combined, they lead to the following:

Theorem 4.5. *Let M be a subgroup of $U(\mathcal{H})$ generated by $\mathbf{W}$ and $\{\mathbf{E}_+(b)|b \in \mathbb{R}\}$. Then there is a unique isomorphism $\iota : \mathrm{SL}_2(\mathbb{R}) \to M/\{\pm 1\}$ which satisfies:*

$$\iota\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)=\mathbf{W} \tag{45}$$

and:

$$\iota\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)=\mathbf{E}_{+}(b) \tag{46}$$

Indeed, the defining relations $I - IV$ of $\mathrm{SL}_2\left(\mathbb{R}\right)$ are satisfied by the elements $\mathbf{W}$ and $\mathbf{E}_+(b)$ once we identify elements differing by a sign, i.e., after taking the quotient by $\{\pm 1\}$. This yields a unique group morphism $\iota : \mathrm{SL}_2(\mathbb{R}) \to M/\{\pm 1\}$.

Surjectivity follows immediately, since M is, by definition, generated by the elements $\{\mathbf{W}, \mathbf{E}_+\left(\mathrm{b}\right) : b \in \mathbb{R}\}$. To prove injectivity, we use the fact that the only nontrivial proper normal subgroup of $\mathrm{SL}_2(\mathbb{R})$ is $\{\pm 1\}$. Moreover, we have:

$$\iota\left(\begin{array}{cc}-1 & 0\\ 0 & -1\end{array}\right)=\iota\left(h(-1)\right)=\left[\mathbf{H}(-1)\right]\neq\left[\pm 1\right], \tag{47}$$

which shows that the kernel of ι is trivial.

Thus, the metaplectic group can be defined in a constructive manner as the subgroup of the unitary group $U\left(\mathcal{H}\right)$ generated by the operators $\mathbf{W}$ and $\mathbf{E}_+(\mathrm{b})$ for all $b \in \mathbb{R}$. In this formulation, the earlier topological definition of the group is replaced by an analytic one, based on explicit operators acting on a Hilbert space.

The same construction extends naturally from $\mathrm{Mp}_2(\mathbb{R})$ to the higher-dimensional case $\mathrm{Mp}_{2n}(\mathbb{R})$, when the Hilbert space $\mathcal{L}^2(\mathbb{R})$ is replaced by $\mathcal{L}^2(\mathbb{R}^n)$. More remarkably, this construction can be extended beyond the real numbers to any field that admits a form of harmonic analysis—for example, the fields $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ and their finite extensions (excluding fields of characteristic two). In this broader setting, the Hilbert space $\mathcal{L}^2(\mathbb{R})$ is replaced by $\mathcal{L}^2(F)$, where F denotes such a local field. The usual exponential function $x \mapsto e^{2\pi i x}$ is replaced by an appropriate additive character $x \mapsto \mathbf{e}(x) \in U(1)$, which allows one to define the Fourier transform $\mathbf{F}$ and the operator $\mathbf{E}_+(b)$ in this general context. This framework yields the metaplectic groups $\mathrm{Mp}_{2n}(F)$ for every local field of characteristic different from two, following the construction proposed by Andre Weil.

Remark 4.6. Fields of characteristic two are excluded becuse the quadratic and symplectic structures required for the metaplectic construction fail to satisfy the necessary algebraic properties (e.g., division by 2 and antisymmetry of the symplectic form).

Weil's operator-theoretic construction of the metaplectic group did more than formalize its definition—it paved the way for decades of breakthroughs in number theory. Extending the construction to all local fields and assembling them defines the adelic metaplectic group $\mathrm{Mp}_{2n}(\mathbb{A})$ mentioned above. This group forms a double cover of the adelic symplectic group $\mathrm{Sp}_{2n}(\mathbb{A})$.

In the theory of automorphic forms, one often studies the subgroup $\mathrm{Sp}_{2n}(\mathbb{Q})$ inside $\mathrm{Sp}_{2n}(\mathbb{A})$. The metaplectic cover $\mathrm{Mp}_{2n}(\mathbb{A})$ is constructed so that it splits over $\mathrm{Sp}_{2n}(\mathbb{Q})$; in other words, the natural inclusion of $\mathrm{Sp}_{2n}(\mathbb{Q})$ into $\mathrm{Sp}_{2n}(\mathbb{Q})$ can be lifted to a map that embeds $\mathrm{Sp}_{2n}(\mathbb{Q})$ as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$. These statements can be represented more clearly through the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Mp}_{2n}(\mathbb{R}) & \longrightarrow & \mathrm{Sp}_{2n}(\mathbb{A}) & \longrightarrow & 1 \\ & & & & {\scriptstyle \text{Split}}\uparrow & \nearrow{\scriptstyle \text{Inclusion}} & & & \\ & & & & \mathrm{Sp}_{2n}(\mathbb{Q}) & & & & \end{array}$$

This natural splitting encapsulates the principle of quadratic reciprocity and thus reveals a fundamental connection between harmonic analysis on local fields and adeles and the reciprocity laws of number theory.

5. Research topics

The splitting that concluded the previous section lays the groundwork for studying automorphic forms and representations of metaplectic groups—for example, through the spectral decomposition of $\mathcal{L}^2\left(\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{Mp}_2(\mathbb{A})\right)$. This approach eventually led to Waldspurger's development of the theta correspondence, a framework that links representations of metaplectic and orthogonal groups through the Weil representation. By analyzing how automorphic forms transform under this representation, Waldspurger established precise connections between theta lifts and the central values of L - functions, revealing deep arithmetic structure within the theory of automorphic representations.

Another approach that deserves attention avoids any use of topology or analysis. Instead, it constructs metaplectic groups entirely algebraically, using generators and relations. The starting point is a simply-connected Chevalley group $G(F)$ over a field F, which can be of classical types (A_l, B_l, C_l, D_l) or exceptional types (E, F, G). Such a group is presented via generators $e_\alpha(x)$ for each root α in the root system Φ and each $x \in F$, together with standard commutation and conjugation relations reflecting the structure of the root system. By omitting the relation that defines multiplication of certain Cartan elements $h_\alpha(u)$, one obtains a new group $G'(F)$ that is a universal central extension of $G(F)$. The kernel of this extension, (G, F), lies in the center of $G'(F)$ and is isomorphic to the Milnor K-theory group $K_2(F)$ in most cases. To construct a metaplectic or generalized covering group, one then uses the Hilbert symbol, a number-theoretic function $(\cdot,\cdot) : F^\times \times F^\times \to \boldsymbol{\mu}_d$, which factors through $K_2(F)$. Quotienting $G'(F)$ by the kernel of the Hilbert symbol produces a central extension:

$$1 \to \boldsymbol{\mu}_d \to \tilde{G}(F) \to G(F) \to 1 \tag{48}$$

which for $d = 2$ and $G = \mathrm{Sp}_{2n}$ recovers the classical metaplectic group $\mathrm{Mp}_{2n}(F)$.

For other Chevalley groups, these generalized metaplectic covers play an important role in number theory, representation theory, and the theory of automorphic forms. For instance, they appear in the study of half-integral weight modular forms on $\mathrm{SU}(1,2)$, the 3 - fold cover of $\mathrm{SL}_3(\mathbb{R})$, and minima representations of exceptional groups like G_2 or F_4. This purely algebraic approach thus provides a powerful framework for understanding metaplectic groups and their generalizations without ever invoking operators, topology, or analytic methods.

Explanation of some terms.

- **Chevalley group**: A Chevalley group is a group built from a semisimple Lie algebra which is defined over any field. It unifies and generalizes classical matrix groups like SL_n , SO_n , and Sp_n and its finite versions form a major family of simple groups.
- **Milnor K-theory group**: The Milnor K-theory group $K_n^M(F)$ of a field F is a group built from the multiplicative group $F^\times$ that reflects the multiplicative relations between field elements.

6. Quantum Quadratic hamiltonians

Quadratic hamiltonians play a crucial role in the theory of partial differential equations (providing nontrivial examples of wave propagation phenomena) and in quantum mechanics, where the evolution of coherent states under general classes of hamiltonians—including $-\hbar^2\nabla+V$ can be approximated by the evolution generated by time-dependent quadratic hamiltonians.

In the second part of this report, we focus on deriving exact formulas for time-dependent Schrodinger equations governed by quadratic hamiltonians in phase space.

In particular, we study these systems in relation to the metaplectic group, exploring the mathematical framework of the Weyl symbol of the propagator and its connection to metaplectic representations and quadratic quantum hamiltonians. We present formulas that link harmonic oscillators, Gaussian functions, and the symplectic group.

Our emphasis is on time-dependent quadratic Schrodinger equations and Gaussian coherent states, analyzed through the lens of the metaplectic group. This approach is central to the theory and has been fundamental in many studies of quantum quadratic hamiltonians.

6.1. **Weyl quantization.** Fix the Planck constant $\hbar > 0$ (in the homogeous quadratic case, it suffices to take $\hbar = 1$). For any topological vector spaces E and F, let $\mathcal{L}_w(E, F)$ denote the space of continuous linear maps from E to F, endowed with the weak topology. For any positive integer m, let $\mathcal{S}(\mathbb{R}^m)$ denote the Schwartz space of complex-valued rapidly-decreasing functions defined on $\mathbb{R}^m$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions on $\mathbb{R}^m$, which is defined as the topological dual of $\mathcal{S}(\mathbb{R}^m)$ endowed with the weak dual topology. The *Weyl quantization map* is a continuous linear map (denoted by Op^w or by a hat) defined on $\mathcal{S}'(\mathbb{R}^{2n})$ and valued in $\mathcal{L}_w(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$. We also consider the separable Hilbert space $\mathcal{H}_n := \mathcal{L}^2(\mathbb{R}^n)$ of square integrable complex-valued functions defined on $\mathbb{R}^n$.

We equip $\mathbb{R}^{2n}$ with the canonical symplectic form ω_n discussed in the first section and write a vector $X \in \mathbb{R}^{2n}$ as $X = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Weyl quantization is uniquely characterized by the following two conditions:

- (W_1) $(\hat{q}\psi)(q) = q\psi(q)$, $(\hat{p}\psi)(q) = (D_q\psi)(q)$ (where $D_q := \frac{\hbar}{i}\nabla_q$) for every $\psi \in \mathcal{S}(\mathbb{R}^{2n})$;
- (W_2) $e^{i\widehat{(\alpha\cdot q+\beta\cdot p)}} = e^{[i\cdot\hat{q}+\beta\cdot\hat{p}]}$ for all $\alpha, \beta \in \mathbb{R}^n$.

Remark 6.1. For any $\alpha, \beta \in \mathbb{R}$, the operator $\alpha\hat{q} + \beta\hat{p}$ admits a densely-defined self-adjoint extension in the Hilbert space $\mathcal{H}_n$, which implies that $e^{i(\alpha\hat{q}+\beta\hat{p})}$ extends to a unitary operator defined on $\mathcal{H}_n$. Setting $z = \alpha + i\beta \equiv (\alpha, \beta) = X$, this gives the following z-dependent unitary operator on $\mathcal{H}_n$, which is called the *Weyl operator* at parameter $z \in \mathbb{C}^n = \mathbb{R}^n \oplus \mathbf{i}\mathbb{R}^n$:

$$\hat{T}(z) := e^{i(\alpha\hat{q}-\beta\hat{p})} := T(X) ,$$

This operator represents the quantized phase-space translation defined by the vector $X = (\alpha, \beta)$:

$$(p, q) \to (p, q) + X = (p + \alpha, q + \beta) \ .$$

Recall that the Fourier transform of $A \in \mathcal{S}(\mathbb{R}^{2n})$ is the rapidly-decreasing function $\tilde{A} \in \mathcal{S}(\mathbb{R}^{2n})$ defined through:

$$\tilde{A}(Y) = \int_{\mathbb{R}^{2n}} e^{-iX\cdot Y} A(X)\, dX \ ,$$

where $\cdot$ denotes the canonical Euclidean scalar product of $\mathbb{R}^{2n}$.

Proposition-Definition 6.2. The *Weyl quantization* of $A \in \mathcal{S}(\mathbb{R}^{2n})$ is given by:

$$\hat{A} := (2\pi)^{-2n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{A}(\alpha, \beta)\, e^{i(\alpha\hat{q} + \beta\hat{p})}\, \mathrm{d}\alpha\, \mathrm{d}\beta = (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} \tilde{A} T\, \omega_n \ .$$

Definition 6.3. A is the *contravariant Weyl symbol* of the pseudodifferential operator $\hat{A}$ if they are related through formula (6.2).

By applying the explicit action of Weyl operators on $\mathcal{S}(\mathbb{R}^n)$ and using Fourier analysis, the operator $\hat{A}$ can be expressed as

$$(\hat{A}\varphi)(x) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\cdot\xi} A\Big(\frac{x+y}{2}, \xi\Big)\, \varphi(y)\, dy\, d\xi. \tag{49}$$

If we denote by K_A the Schwartz kernel of $\hat{A}$, then the corresponding Weyl symbol $A(q, p)$ is given by:

$$A(q, p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}u\cdot p}\, K_A\Big(q + \frac{u}{2}, q - \frac{u}{2}\Big)\, du \ . \tag{50}$$

In general, these formulas hold in the weak sense and they are valid pointwise when $\hat{A}$ is sufficiently regular (for example, if $A \in \mathcal{S}(\mathbb{R}^{2n})$). We note that the mapping $A \mapsto \hat{A}$ is well-defined and establishes a bijection between $\mathcal{S}'(\mathbb{R}^{2n})$ and $L_w(\mathcal{S}(\mathbb{R}^n)),$ $\mathcal{S}'(\mathbb{R}^n)$). In particular, the following inversion formula holds:

Proposition 6.4. *For every $\hat{A} \in L_w(\mathcal{S}(\mathbb{R}^n),\ \mathcal{S}'(\mathbb{R}^n))$, there exists a unique contravariant Weyl symbol $A \in \mathcal{S}'(\mathbb{R}^{2n})$, given by:*

$$A(X) = 2^n\, \mathrm{Tr}\big[\hat{A}\, \mathrm{Sym}(X)\big] \tag{51}$$

where $\mathrm{Sym}(X)$ *is the unitary operator on $L^2(\mathbb{R}^n)$ defined by:*

$$\big(\mathrm{Sym}(X)\, \phi\big)(q) = \pi^{-n}\, e^{-2i\,\beta\cdot(\alpha - q)}\, \phi(2\alpha - q),$$

for $X = (\alpha, \beta)$.

Proposition-Definition 6.5. For every $\hat{A} \in L_w(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$, there exists a unique tempered distribution $A^{\#}$ on $\mathbb{R}^{2n}$, called the *covariant Weyl symbol* of $\hat{A}$, such that:

$$\hat{A} = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\, \hat{T}(X)\, dX, \tag{52}$$

and the inverse formula is:

$$A^{\#}(X) = \mathrm{Tr}\big[\hat{A}\, \hat{T}(-X)\big]. \tag{53}$$

Moreover, the covariant and contravariant Weyl symbols satisfy the following relation:

$$A^{\#}(X) = (2\pi\hbar)^{-n}\, \tilde{A}(JX) \ , \tag{54}$$

The composition $\tilde{A} \circ J$ is called the *symplectic Fourier transform* of A.

These properties are easy to prove using the Gaussian coherent states φ_z defined as:

$$\varphi_z = \hat{T}(z)\varphi_0 \ \forall z = \alpha + i\beta \in \mathbb{C}^n \ , \tag{55}$$

where:

$$\varphi_0 = (\pi\hbar)^{-n/4}\, e^{-\hbar\alpha^2/2} \tag{56}$$

Corollary 6.6. *With the above notations, for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, we have:*

$$\langle \varphi \,|\, \hat{A}\psi \rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\, \langle \varphi \,|\, \hat{T}(X)\psi \rangle \, dX \ . \tag{57}$$

Moroever, we have:

$$\langle \varphi_z \,|\, \hat{A}\varphi_0 \rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^{\#}(X)\, e^{\left(-\frac{|X-z|^2}{4\hbar} - \frac{i}{2\hbar}\,\omega(X,z) \right)} \, dX. \tag{58}$$

which is a consequence of the following equalities:

$$\hat{T}(z)\,\hat{T}(z') = e^{\frac{i\omega(z,z')}{2\hbar}}\, \hat{T}(z+z'), \tag{59}$$

$$\langle \varphi_z \,|\, \varphi_0 \rangle = e^{-\frac{|z|^2}{4\hbar}}. \tag{60}$$

Definition 6.7. Let two operators $\hat{A}, \hat{B} \in L_w(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ be such that the operator composition $\hat{A}\hat{B}$ is well-defined. The *Moyal product* of A and B is the unique element $A\#B \in \mathcal{S}'(\mathbb{R}^{2n})$ satisfying:

$$\hat{A}\hat{B} = \widehat{A\#B}. \tag{61}$$

The Moyal product can be used to translate between the Weyl and deformation quantizations of the symplectic vector space $(\mathbb{R}^{2n}, \omega_n)$.

6.2. **Time evolution of quadratic hamiltonians.** Consider a quadratic time-dependent hamiltonian, $H_t(z) = \sum_{1\le j,k\le 2n} c_{j,k}(t)\, z_j z_k$, where the coefficients $c_{j,k}(t)$ are real and continuos and defined on $\mathbb{R}$. Writing $z = q+ip \equiv (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{C}^n$ define:

$$H_t(q,p) = \frac{1}{2}\left(Gq\cdot q + 2L_t q\cdot p + K_t p\cdot p \right) \ .$$

In phase space, the corresponding classical flow is determined by the linear ODE:

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} G_t & L_t^T \\ L_t & K_t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \ ,$$

where $L^T$is the transposed matrix of L. The classical flow consists of linear symplectic transformations F_t, with the property $F_0 = \mathrm{id}$. In the quantum setting, $\widehat{H}_t$ denotes a family of self-adjoint operators acting on the separable Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$. The time evolution of a quantum state is governed by the Schrodinger equation, beginning from an initial state $\varphi \in \mathcal{H}$:

$$i\frac{\partial \psi_t}{\partial t} = \widehat{H}_t \psi_t, \quad \psi_{t_0} = \varphi \ . \tag{62}$$

Assume that the existence and uniqueness of a solution to this equation holds. We then express the quantum state as $\psi_t = \widehat{U}\varphi$. In this setting, the correspondence between the classical and quantum evolutions is exact.

For every observable $A \in \mathcal{S}(\mathbb{R}^{2n})$, we identify:

$$\hat{U}_t\, \widehat{A}\, \hat{U}_t^{-1} = \widehat{A \circ F_t}. \tag{63}$$

6.3. **The metaplectic group and computation of the Weyl symbol** $R(F, X)$**.** A metaplectic transformation corresponding to a linear symplectic transformation $F \in \mathrm{Sp}(2n, \mathbb{R})$ is a unitary operator $\widehat{R}(F)$ on $L^2(\mathbb{R}^n)$ which satisfies any of the following equivalent properties:

(a) $\widehat{R}(F)^* \, \widehat{A} \, \widehat{R}(F) = \widehat{A \circ F}, \quad \forall A \in \mathcal{S}(\mathbb{R}^{2n})$
(b) $\widehat{R}(F)^* \, \widehat{T}(X) \, \widehat{R}(F) = \widehat{T}\left[F^{-1}(X)\right], \quad \forall X \in \mathbb{R}^{2n}$,
(c) $\widehat{R}(F)^* \, \widehat{A} \, \widehat{R}(F) = \widehat{A \circ F}$ for $A(q,p) = q_j$ $(1 \le j \le n)$ & $A(q,p) = p_k$ $(1 \le k \le n)$.

For every $F \in \mathrm{Sp}(2n)$, there exists a corresponding metaplectic transformation $\widehat{R}(F)$. If $\widehat{R}_1(F)$ and $\widehat{R}_2(F)$ are two such operators associated with the same symplectic map F, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\widehat{R}_1(F) = \lambda \widehat{R}_2(F)$. Thus, $F \mapsto \widehat{R}(F)$ defines a projective representation of $\mathrm{Sp}(2n, \mathbb{R})$.

Let $\mathrm{Mp}(n)$ denote the metaplectic group and $\pi_p : \mathrm{Mp}(n) \to \mathrm{Sp}(2n)$ its natural projection. The *metaplectic representation* is the group morphism:

$$F \mapsto \widehat{R}(F), \qquad \pi_p[\widehat{R}(F)] = F, \quad \forall F \in \mathrm{Sp}(2n, \mathbb{R}) \ ,$$

whose image is $\mathrm{Mp}(n)/\{1, -1\}$.

For each $F \in \mathrm{Sp}(2n)$, one can find a smooth path $F_t \in \mathrm{Sp}(2n)$, $t \in [0,1]$ satisfying $F_0 = 1$ and $F_1 = F$. Using the polar decomposition $F = V|F|$, where V is orthogonal symplectic and $|F| = (F^T F)^{1/2}$ is positive symplectic, both admit logarithms and we can write $F = e^K e^L$, with K, L hamiltonian matrices. Hence we can define $F_t := e^{tK} e^{tL}$, which corresponds to the linear flow generated by the quadratic hamiltonian $H_t(z) = \frac{1}{2} S_t z \cdot z, \qquad S_t = -J \dot{F}_t F_t^{-1}$.

Theorem 6.8. *For every $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^{2n}$, the following formulas hold:*

$$\widehat{U}_t \varphi_\Gamma(x) = \varphi_{\Gamma_t}(x) \tag{64}$$

and:

$$\widehat{U}_t \, \varphi_z^\Gamma(x) = \widehat{T}(F_t z) \varphi^{\Gamma_t}(x) \ , \tag{65}$$

where $\Gamma_t = (C_t + iD_t\Gamma)(A_t + iB_t\Gamma)^{-1}$ and $a_{\Gamma_t} = a_\Gamma \, \det(A_t + iB_t\Gamma)^{-1/2}$.

This result allows us to set $\widehat{R}(F) = \widehat{U}_1$, from which the standard properties of the metaplectic representation follow.

Proposition 6.9. *Let F_t and F'_t be two symplectic paths that join identity 1 at $t = 0$ to F at $t = 1$. Then we have $\widehat{U}_1 = \pm \widehat{U}'_1$. Moreover, for any $F_1, F_2 \in Sp(2n, \mathbb{R})$, we have:*

$$\widehat{R}(F_1) \, \widehat{R}(F_2) = \pm \widehat{R}(F_1 F_2).$$

We now compute the Weyl symbol of $\widehat{R}(F)$, considering the case $det(1 + F) > 0$. Consider an arbitrary path $t \mapsto F_t$ of class $\mathbb{C}^1$ that connects 1 at $t = 0$ to F at $t = 1$. It is known that $\mathrm{Sp}_+(2n) := \{F \in \mathrm{Sp}(2n) | \det(1 + F) > 0\}$ is an open and connected subset of $\mathrm{Sp}(2n)$. Therefore, we can choose a piecewise class $\mathcal{C}^1$ path $t \mapsto F'_t$ in $\mathrm{Sp}_+(2n)$ connecting 1 to F, and take $\mathcal{M}_0 = i\varepsilon J$.

We will use the following:

Lemma 6.10. *Let $t \mapsto F_t$ be a path in $Sp(2n)$. Then for every $\mathcal{M}_0 \in \mathfrak{sp}_+(2n, \mathbb{C})$, we have, for the real part:*

$$\mathfrak{R}\left[\int_0^t \frac{\dot{\delta}(F_s, \mathcal{M}_0)}{\delta(F_s, \mathcal{M}_0)} \, \mathrm{d}s\right] = \log\left(\frac{|\delta(F_t, \mathcal{M}_0)|}{|\delta(F_0, \mathcal{M}_0)|}\right).$$

Applying this lemma, we obtain the following for the imaginary part:

$$\mathfrak{F}\left(\int_0^1 \frac{\dot{\delta}(F_t, i\varepsilon J)}{\delta(F_t, i\varepsilon J)}\,\mathrm{d}t\right) = 2\pi\nu + \mathfrak{F}\left(\int_0^1 \frac{\dot{\delta}(F_t', i\varepsilon J)}{\delta(F_t', i\varepsilon J)}\,\mathrm{d}t\right) \ . \tag{66}$$

Since $\det(1+F_t')$ is real and never vanishes in $[0,1]$, the last term vanishes as $\varepsilon \to 0$, giving:

$$\lim_{\varepsilon\to 0} \mathfrak{F}\left(\int_0^1 \frac{\dot{\delta}(F_t, i\varepsilon J)}{\delta(F_t, i\varepsilon J)}\,\mathrm{d}t\right) = 2\pi\nu \ .$$

This shows that the Weyl symbol $R(F,X)$ of $\widehat{R}(F)$ is given by the Mehlig-Wilkinson formula:

$$R(F,X) = e^{i\pi\nu}|\det(1+F)|^{-1/2} e^{-iJ(1-F)(1+F)^{-1}X\cdot X} \ .$$