Phase transitions and the Renormalization Group

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Abstract

This internship explored the calculation of critical exponents and the theoretical frameworks used to describe critical phenomena. Starting from mean-field and Landau theories applied to the Ising model, we examined their qualitative predictions and limitations. To achieve quantitative agreement with experiments, we then introduced the Renormalization Group (RG) approach, which systematically explains scaling behavior and universality near phase transitions. Comparing these frameworks highlights how the RG extends mean-field theory to provide a deeper and more accurate understanding of critical behavior.

Contents

1 Introduction

In this internship report, we review topics in statistical physics, focusing on techniques and challenges that arise in the study of critical phenomena.

We begin with an introduction to phase transitions, with particular attention to the critical point. This single point contains a rich amount of physics and has motivated the development of a field of study devoted to it.

A central issue in testing theory against experiment in this context is the determination of the correct critical exponents, which will be defined later. Historically, obtaining these exponents was a difficult task. We will therefore examine different approximation methods and calculation schemes, such as mean-field theory and related approaches, culminating in the development of the Renormalization Group (RG). This summer internship project is devoted to studying these methods as a way of testing the validity of theoretical approaches. Our analysis will be carried out using the Ising model, a classic model which exhibits critical behavior. By reviewing and contrasting traditional approaches, we will emphasize how the Renormalization Group provides a general and widely applicable framework, capable of yielding the correct critical exponents.

1.1 Motivation and relevance

The primary focus of this report is the Renormalization Group, which constitutes a central tool in modern theoretical physics for analyzing systems across scales and has significantly influenced our understanding of physical theories. The RG formalism offers a systematic way to study how physical laws evolve with changes in scale, providing insights into universality, critical behavior, and the effective description of nature across different regimes.

As emphasized by Nigel Goldenfeld in his widely regarded text—which serves as the principal reference for this work—renormalizable models are particularly powerful because they produce a consistent and predictive phenomenology that is insensitive to microscopic details. This insensitivity manifests structurally: changes in the microscopic physics only affect the numerical values of a finite set of phenomenological parameters.[1] An interesting

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[1] Following Goldenfeld, a phenomenological parameter is not predicted by the theory and must be determined experimentally, capturing the effects of unknown microscopic physics.

corollary is that all our current fundamental physical theories contain phenomenological parameters.[1] This perspective not only clarifies the structure of existing theories but also guides the formulation of new physical descriptions, particularly in areas where microscopic details are unknown or intractable.

In this project, the RG will be used both as the main theoretical framework and as the standard of comparison for testing how well classical methods can determine critical exponents in the study of the Ising model. Although our focus here is on statistical physics and the Ising model, the RG has also played a major role in other domains of physics. These broader applications will be outlined in the historical overview that follows.

1.2 Short History

The history of the Renormalization Group (RG) unfolds along two distinct but ultimately converging lines of research: **statistical physics** and **quantum field theory (QFT)**. Both fields faced challenges that could not be resolved with existing methods, and their convergence in the 1970s, largely through Kenneth Wilson's work, produced the modern formulation of the Wilsonian RG, for which he ultimately received the Nobel prize in physics in 1982.

His Nobel lecture[2] and Kadanoff's review[3] were used in preparing this historical overview.

1. Statistical Physics and Critical Phenomena

The scientific study of thermodynamic phases of matter and their transformations began in earnest in the late 19th century. A milestone came with van der Waals' mean-field model of fluids (1873), later refined by Maxwell, which successfully described liquid–gas transitions and the existence of a critical point.

Although mean-field theory provided qualitative insights, it consistently failed to predict **critical exponents** accurately. Landau later unified these mean-field approaches in the 1930s–40s, showing that models such as van der Waals' fluid and Weiss' ferromagnetism could be treated under a common framework.[4] Yet the discrepancy in exponents remained unresolved.

A breakthrough came in 1944 with **Lars Onsager's exact solution of the two-dimensional Ising model**[5], which produced results incompatible with mean-field predictions. This, together with experimental evidence, made it clear that a theory beyond mean-field approximations was needed to describe critical phenomena.

2. Quantum Field Theory and Renormalization

Meanwhile, in the 1940s, particle physics faced its own theoretical difficulties. Quantum electrodynamics (QED) suffered from infinities in its calculations. The development of **renormalization theory** by Schwinger, Tomonaga, Feynman, Dyson, and Bethe provided a way to systematically remove these divergences by redefining (*renormalizing*) physical parameters.

Soon after, Stueckelberg and Petermann (1953) introduced the term **"renormalization group"**, noting that the transformations between different parametrizations of a theory had a group-like structure.[6] Later, in the 1950s, Gell-Mann and Low studied scaling behavior in QED, formulating what became the **Gell-Mann–Low equation**.[7] These advances suggested that renormalization could provide deep insights into how physical theories behave across scales, but a comprehensive framework was still lacking.

3. Wilson's Synthesis

The decisive step occurred when Wilson, initially working in particle physics, encountered the challenges of critical phenomena. He was strongly influenced by the work of **Leo P. Kadanoff**, who in 1966 proposed the idea of "block spins" in the Ising model. By grouping spins together and rescaling the lattice, Kadanoff showed how systems near criticality might exhibit scale invariance and universal behavior. [8]

Wilson recognized that Kadanoff's heuristic picture could be put on firm theoretical footing using tools from field theory. He developed a **systematic renormalization group procedure**, where effective theories are obtained by integrating out short-distance degrees of freedom. This approach explained **Widom's scaling laws**[9] for critical phenomena, gave a clear framework for calculating critical exponents, and resolved long-standing discrepancies between mean-field predictions and exact/experimental results. [10] [11]

At the same time, Wilson's RG formalism provided a unifying language for particle physics and statistical mechanics. It clarified the meaning of renormalization in QFT, distinguished between renormalizable and non-renormalizable theories, and laid the foundation for modern quantum field theory and the theory of critical phenomena.

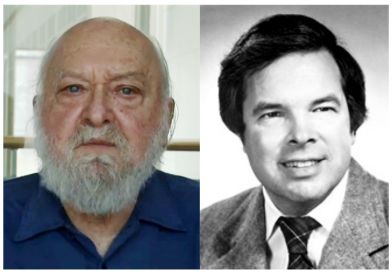


Figure 1.1: Leo Kadanoff (left)[12] and K.G.Wilson (right)[13]

2 Phase transitions and critical phenomena

It is a well-established fact that matter can exist in a variety of forms, or phases. The study of these phases and the transitions between them—known as phase transitions—is most effectively carried out within the framework of statistical mechanics.

One might argue that, since phase transitions are observed at the macroscopic level, the natural language to describe them should be that of thermodynamics. While thermodynamics indeed provides a powerful macroscopic description, it offers only a limited perspective: it characterizes the outcome of phase transitions but does not explain their microscopic origin or underlying mechanisms.

For this reason, and in order to present the theory of phase transitions—and later that of critical phenomena—in a comprehensive way, we will employ the tools of statistical mechanics, which build upon the microscopic physics. From this microscopic foundation, we will then recover the familiar macroscopic behavior in the appropriate limits.

2.1 Statistical mechanics and thermodynamics

As Kadanoff emphasizes in his review, two properties stand out in the study of matter: irreversibility and the existence of sharply distinct thermodynamic phases.[3]

Although the microscopic laws governing individual particles are time-reversible, a system with many degrees of freedom exhibits a clear loss of time-reversal invariance at the macroscopic scale. Such systems evolve irreversibly toward an unchanging state known as statistical equilibrium. This property is well-defined only in limiting cases—either when the number of degrees of freedom becomes infinite or when the observation time tends to infinity—making it a crucial caveat for any consistent theory.[3]

Statistical mechanics restricts itself to systems that have already reached equilibrium, since its formulation is built upon time-independent ensembles (microcanonical, canonical, grand canonical). These ensembles are statistical constructions that assign probabilities to all possible microscopic states of a system under given physical constraints.[3]

Once equilibrium is established through irreversible dynamics, the framework of statistical mechanics becomes applicable. In the thermodynamic limit, where the system size tends to infinity, equilibrium matter can then display the second key property: the emergence of distinct thermodynamic phases.[3]

2.1.1 Theoretical framework

Access to the microscopic description of matter is obtained by specifying the **Hamiltonian**, a function that encodes the total energy of the system in terms of the positions and momenta of its constituent particles.

To connect this microscopic level with the **macroscopic description** provided by thermodynamic functions, we introduce the **partition function**. Evaluating the partition function is the central task of statistical mechanics, as it serves as the bridge between microscopic laws of motion and macroscopic observables.

Following Goldenfeld's notation, we present the most general form of a system, which will serve as the basis for our subsequent calculations. Our system of interest will be denoted as some sample region Ω, where we define the Hamiltonian H_Ω.[1]

The statistics appropriate for the systems we consider are **Maxwell–Boltzmann statistics**, which describe distinguishable classical systems. This allows us to assign a probability for the system to be in a particular **microstate**, i.e., a complete specification of the configuration of the system.

We write the Hamiltonian as

$$\frac{-H_\Omega}{k_B T} = \sum_n K_n \Theta_n, \tag{2.1}$$

where K_n are the coupling constants and Θ_n are combinations of the dynamical degrees of freedom, which are summed over in the partition function.

The probability for the system to be in a given microstate is given by the **Boltzmann distribution**:

$$P_\Omega(\{K_n\}) = \frac{e^{-\beta H_\Omega(\{\Theta_n\})}}{Z_\Omega}, \tag{2.2}$$

where $\beta = (k_B T)^{-1}$, the $\{\cdots\}$ denotes the full set of coupling constants, and Z_Ω is the **partition function**, defined as

$$Z_\Omega[\{K_n\}] \equiv \operatorname{Tr} e^{-\beta H_\Omega}, \tag{2.3}$$

with the trace taken over all degrees of freedom.

Physically, this corresponds to the **canonical ensemble**, in which the system is in thermal contact with a heat bath at fixed temperature. We can also characterize the system by a linear dimension L, so that $V_\Omega \propto L^d$ and the surface area $S_\Omega \propto L^{d-1}$, where

d is the system's dimensionality.[1]

From the partition function, we recover the macroscopic thermodynamics. For example, the **Helmholtz free energy** is defined as

$$F_\Omega[\{K_n\}] = -k_B T \ln Z_\Omega, \tag{2.4}$$

from which all other macroscopic thermodynamic properties can be obtained via differentiation. Importantly, the equilibrium configuration of the system corresponds to the minimum of the free energy.

So far, we have not specified the explicit dimension of Ω. However, for a system of finite size, there is no theoretically grounded basis to discuss phase transitions rigorously. The next section will explain the importance of taking the limit of an infinite number of degrees of freedom and the next subsection one way we can achieve it.[1]

2.1.2 The thermodynamical limit

Before defining phase transitions explicitly, we first introduce the *thermodynamic limit*, which provides a way of letting the system acquire infinitely many degrees of freedom. As we have emphasized, this limit is a prerequisite for a well-founded discussion of phases and, by implication, of phase transitions. In this limit, macroscopic thermodynamics is valid.

The free energy defines the thermodynamics of the system. Empirically and theoretically, it scales with the system size as

$$F_\Omega \propto V(\Omega),$$

so that for a finite system one may write

$$F_\Omega = V(\Omega) f_b + S(\Omega) f_s + O(L^{d-2}),$$

where f_b is the bulk free energy per unit volume, f_s is the surface free energy per unit area, and L is the linear extent of the system in d dimensions.

When the limits exist and are independent of the choice of region Ω, we define the thermodynamic limit of the bulk free energy as

$$f_b[K] \equiv \lim_{V(\Omega)\to\infty} \frac{F_\Omega[K]}{V(\Omega)},$$

and, for lattice systems, equivalently as

$$f_b[K] \equiv \lim_{N(\Omega)\to\infty} \frac{F_\Omega[K]}{N(\Omega)}.$$

The way this limit is taken depends on the physical system under study. Its importance has a notable historical background: at the 1937 Amsterdam conference commemorating van der Waals, Kramers famously posed the question of whether the partition function could describe sharp phase transitions or only a single phase. The audience was evenly split. Subsequent progress — particularly Kramers' own work and Onsager's exact solution of the two-dimensional Ising model — demonstrated that the thermodynamic limit is crucial: in this limit, non-analytic behavior arise, thereby allowing the definition of phase transitions.[3]

2.2 Phase transitions

Definition. A phase transition occurs when there is a singularity in the free energy(or related thermodynamic potentials) or one of it's derivatives.[14]

To quantitatively analyze the possible behaviors of a system, we employ the *phase diagram*. A phase diagram is a graphical representation that shows the regions of stability for different phases of matter, along with the boundaries where phase transitions occur.

Formally, when the bulk free energy density $f_b[K]$ exists, we can define phase boundaries in a precise way. The phase diagram is constructed by taking as axes the coupling constants of the system, $K_1, K_2, \ldots, K_D$, where D is the dimensionality of the parameter space (the number of independent couplings). [1]

We further define the *co-dimension* C as

$$C = D - D_s \, ,$$

where D_s is the dimensionality of the singular locus, i.e. the set of points in parameter space where $f_b[K]$ becomes non-analytic. The co-dimension has the important property of being invariant under an enlargement of the parameter space: if additional coupling constants are introduced, C remains unchanged.[1]

Within this framework, the phase diagram is partitioned as follows: regions where $f_b[K]$ is analytic correspond to distinct thermodynamic phases. The phase boundaries correspond to singular loci of co-dimension one ($C = 1$). By contrast, loci of higher co-dimension ($C > 1$) cannot separate phases, since they do not have the correct dimensionality to divide parameter space.[1]

To grasp the concepts and to have a concrete example, we can look at the phase diagram of a typical fluid. Such a diagram usually displays the regions corresponding to the solid, liquid, and gaseous phases in the pressure–temperature plane, together with the lines of first-order phase transitions separating them.

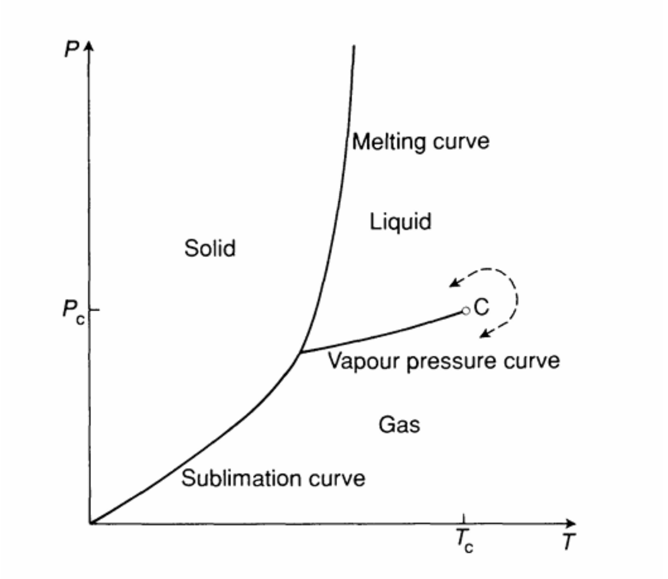


Figure 2.1: Phase diagram of a typical fluid.[14]

In this representation, the first-order transition lines correspond to singular loci of co-dimension one ($C = 1$), while special points such as the critical point or the triple point have co-dimension two ($C = 2$). The critical point marks the end of the liquid–gas coexistence line, beyond which the distinction between liquid and gas no longer exists: the two phases become analytically connected through a smooth crossover, conventionally represented by a dashed line in the phase diagram.

We used the term first-order phase transition implying that there are multiple forms in which a transition can take place. Indeed this is the case and a classification can be made.

2.2.1 Types of phase transitions

Even though model dependent, as we shall see explicitly, the bulk free energy density $f_b[K]$ remains everywhere continuous - this turns out to be true in the general case. This implies that the only way a phase transition can occur is through the appearance of non-analytic behavior at phase boundaries. Such singularities can be classified into two general types:

1. **First-order phase transitions:** In this case, one or more of the first derivatives of the free energy, $\partial f_b/\partial K_i$, are discontinuous across the phase boundary. This discontinuity signals a sudden change in the thermodynamic state, such as the coexistence of liquid and gas. First-order transitions are typically associated with latent heat and phase coexistence.[1]

2. **Continuous phase transitions:** Here, all first derivatives $\partial f_b/\partial K_i$ remain continuous across the boundary, but higher-order derivatives become non-analytic. These transitions are characterized by a smooth change in the order parameter and the absence of latent heat, while exhibiting critical phenomena such as diverging susceptibilities and correlation lengths.[1]

Feature	**First-order**	**Continuous**
Non-analyticity	$\partial f_b/\partial K_i$ discontinuous	Higher derivatives singular
Latent heat	Finite	Zero
Order parameter	Discontinuous jump	Smooth, vanishes at T_c
Phase coexistence	Yes	No
Correlation length	Finite	Diverges at T_c

Table 2.1: Key differences between first-order and continuous phase transitions.

The *order parameter* reflects Landau's view that a phase transition occurs when a mathematical symmetry of the system is broken. This quantity, which is different from system to system, characterizes both the magnitude and the nature of the broken symmetry.[3] It will be a central concept in the analysis that follows.

Having now defined the necessary terms to formally discuss phase transitions, we will restrict our attention to a specific region of the phase diagram: the vicinity of a continuous (second-order) phase transition, commonly referred to as the *critical point.* The study of such points has led to the development of an entire field known as *critical phenomena*, which has introduced fundamental ideas such as *universality.*

In the following sections, we will introduce the notion of the *correlation length* and explain why it plays a central role in the study of critical points. Before doing so, however, we will describe in detail the model system that will serve as the basis for all subsequent calculations. (we shall simply denote from now on $f_b[K]$ as simply f for simplicity)

2.3 The Ising Model

Models play a central role in statistical mechanics, as they provide a simplified description of complex physical systems with many degrees of freedom. By reducing such systems to their essential features—retaining only what is necessary to capture their qualitative behavior—we uncover *universal* behavior. This universality implies that the large-scale properties of a system depend only on a few key ingredients: the symmetry of the order parameter, the dimensionality of the system, and the range of interactions.[14]

This principle of universality highlights why models are so valuable: by studying a simple representative model, we gain insight into the behavior of a whole class of more complex systems. In this sense, the simplest possible model is often the most useful, as it captures the essential physics common to many different systems.

As Goldenfeld makes the case, the Ising model is the drosophila melanogaster of statistical mechanics as it is the simplest interacting two-state system.[1]

The Ising model provides a simplified description of a ferromagnet (or, with appropriate modifications, an anti-ferromagnet). It is defined on a d-dimensional lattice, typically taken to be hypercubic, where each lattice site i is assigned a classical spin variable $S_i \in \{+1, -1\}$. These spins constitute the degrees of freedom of the system.

Physically, the spins represent the orientation of the microscopic magnetic moments, which can point either "up" ($S_i = +1$) or "down" ($S_i = -1$). They can be visualized as arrows restricted to two possible directions, making the Ising model a minimal model that captures the essence of magnetic ordering.

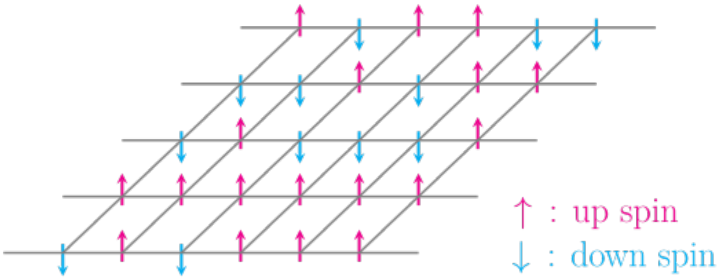


Figure 2.2: Schematic representation of a possible configuration of the Ising 2d model[15]

Assuming that f exists ans restricting the interactions of the spins between only the nearest neighbors (denoted in calculations as $< ij >$) we can write the Hamiltonian of the system as

$$-H_\Omega = H \sum_{i=1}^{N(\Omega)} S_i + J \sum_{\langle ij \rangle} S_i S_j \tag{2.5}$$

where here H is the applied magnetic field and J is the coupling that controls how strongly spins interact between them.

We can readily observe that, if we calculate the free energy F using the Hamiltonian defined in Eq. (2.4), the result is necessarily a smooth function: the logarithm of a smooth function is itself smooth. However, we previously defined a phase transition as a *singularity* in the free energy. At first glance, this appears to be a contradiction.[3]

The resolution lies in the *thermodynamic limit*. When the system size becomes infinite, the number of terms in the partition function also becomes infinite, and it is in this limit that true singularities in F can appear. In real physical systems, which are always finite, we do not see perfectly sharp singularities but rather very sharp crossovers. Nevertheless, because the number of particles is typically of order 10^{23}, the thermodynamic limit provides an excellent approximation to reality.

2.3.1 Thermodynamics of the Ising model

Having distinguished between first-order and continuous phase changes, it is important to study how thermodynamic functions behave in the vicinity of such changes. For the continuous cases that interest us, we must analyze the asymptotic behavior of these functions close to the transition. This approach highlights the role of critical phenomena and allows for a clear characterization by introducing critical exponents, which quantify how these functions diverge or vanish near criticality. In order to define these critical exponents we must first characterize the system from a thermodynamical point of view.

For a magnetic system, the first law of thermodynamics can be written as

$$dU = TdS - MdH,$$

so that the free energy naturally depends on the convenient variables T and H. In the presence of an uniform external field, the magnetization—defined as the magnetic moment per site—is

$$M \equiv \frac{1}{N(\Omega)} \sum_{i=1}^{N(\Omega)} \langle S_i \rangle, \tag{2.6}$$

where $\langle S_i \rangle$ is the statistical average of S_i. Since it can be obtained by differentiating the free energy, we can equivalently write

$$M = -\frac{1}{N(\Omega)} \frac{\partial F_\Omega}{\partial H}. \tag{2.7}$$

Similarly, all other thermodynamic quantities follow from appropriate derivatives of the free energy, and it is convenient to summarize them in a table for later use.

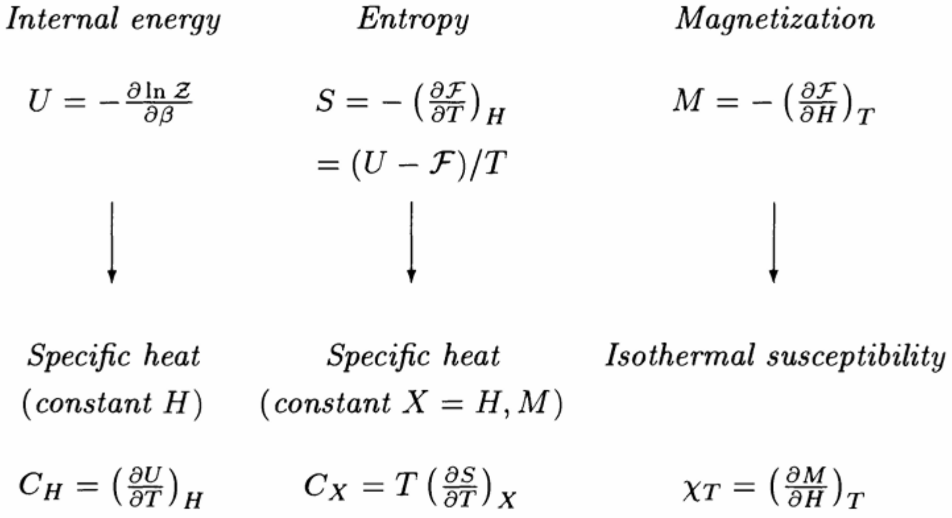


Figure 2.3: Relevant thermodynamic variables[14]

The free energy itself in the context of the Ising model is:

$$F_\Omega(T, \{H_i\}, \{J_{ij}\} \ldots) = -k_B T \log \operatorname{Tr} e^{-\beta H_\Omega} \tag{2.8}$$

in the most general form. Now that we have a notion of thermodynamics and an understanding of the importance of the free energy it is important to study its analytic properties, specifically in the case of the Ising model.

The main analytic properties of the free energy density f (we are interested in the properties with the thermodynamic limit) can be summarized as follows:

1. $f < 0$.
2. $f(H, J, T, \dots)$ is continuous.
3. The partial derivatives $\partial f/\partial T, \partial f/\partial H, \dots$ exist almost everywhere, with left and right derivatives equal almost everywhere.
4. The entropy per site is given by $S = -\partial f/\partial T \geq 0$ almost everywhere.
5. $\partial f/\partial T$ is monotonically non-increasing as a function of T, which implies

   $$\frac{\partial^2 f}{\partial T^2} \leq 0, \tag{2.9}$$

   and therefore the specific heat at constant magnetic field is non-negative:

   $$C_H \equiv T\left.\frac{\partial S}{\partial T}\right|_H = -T\left.\frac{\partial^2 F}{\partial T^2}\right|_H \geq 0. \tag{2.10}$$

6. $\partial f/\partial H$ is monotonically non-increasing as a function of H, which implies

   $$\frac{\partial^2 f}{\partial H^2} \leq 0, \tag{2.11}$$

   and thus the isothermal susceptibility is positive:

   $$\chi_T \equiv \left.\frac{\partial M}{\partial H}\right|_T = -\frac{\partial^2 f}{\partial H^2} \geq 0. \tag{2.12}$$

[1]

All properties can be derived from statistical mechanics; in particular, properties (2), (3), (5), and (6) follow directly from convexity.

Convexity is defined as follows: if we have a function f(x), then it is convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

for all x_1 and x_2. If the inequality is reversed then we say that the function is concave. More usefully, if the second derivative of $f(x)$ exists and is non-negative, f is convex; if it is non-positive, f is concave. We see from the properties that f then is a concave function of both T and H.

From the definition of convexity, it can be shown that if $f(x)$ is bounded and convex or concave, then:

1. $f(x)$ is continuous.
2. $f(x)$ is differentiable almost everywhere.
3. $\frac{df}{dx}$ is monotonically non-increasing (or non-decreasing).

[1]

This is helpful as it gives us a notion on how the functions will look and also tells us about what discontinuities may appear. It is easy to see graphically that any discontinuity violates convexity, but if $f(x)$ has a discontinuity in the slope, can still remain convex (or concave in our case).

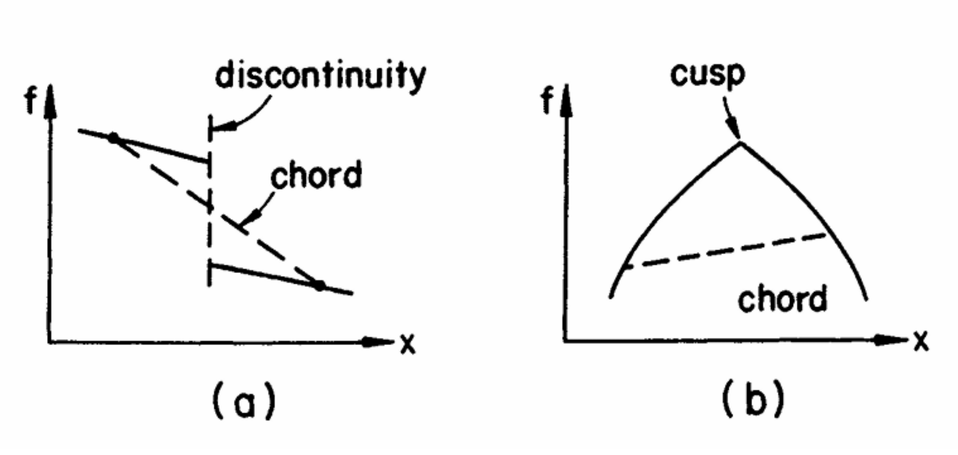


Figure 2.4: Discontinuity in f (on the left) and (on the right) discontinuity in the slope of f[1]

This observation supports our earlier statement that a first-order phase transition corresponds to a discontinuity in the first derivative of f, whereas continuous phase transitions involve higher-order non-analyticities.

2.3.2 Correlation functions

Up to this point, the discussion of the Ising model has mainly focused on thermodynamics. However, statistical mechanics provides the more fundamental description, offering the detailed framework needed to understand phase transitions and critical phenomena. To this end, we introduce correlation functions as a quantitative tool to describe microscopic physics. [14]

In the case of the Ising model, we are interested in the spin–spin correlation function, which measures the correlation between the spins at sites i and j. It is defined as

$$G(\vec{r}_i, \vec{r}_j) = \left\langle \left(S_i - \langle S_i \rangle \right) \left(S_j - \langle S_j \rangle \right) \right\rangle, \tag{2.13}$$

where $\vec{r}_i$ denotes the position vector of a lattice site, and the brackets $\langle \cdot \rangle$ denote a thermal average. This quantity is also called the two-point correlation function, and it measures the expectation value of the fluctuations around the mean spin. Moreover, if the system is translationally invariant, such that $\langle S_i \rangle = \langle S_j \rangle$, then G depends only on the separation $(\vec{r}_i - \vec{r}_j)$:

$$G(\vec{r}_i - \vec{r}_j) \equiv G_{ij} = \langle S_i S_j \rangle - \langle S \rangle^2. \tag{2.14}$$

Above the critical temperature T_c, which marks the continuous phase transition, the average spin vanishes, $\langle S \rangle = 0$, since the spins are no longer correlated. This is also true below T_c, but in that case $\langle S \rangle$ takes a nonzero constant value. In both regimes, the spins become uncorrelated as $r \to \infty$, and the correlation function tends to zero. Physically, this reflects the fact that below T_c the interaction energy dominates, favoring spin alignment, while above T_c the entropy contribution to the free energy dominates, favoring disorder. [14]

Away from criticality, the correlation function decays exponentially with the distance between spins as

$$G(\vec{r}) \sim \exp(-r/\xi), \tag{2.15}$$

where ξ is the correlation length, which can be interpreted as the typical size of regions where spins remain correlated.

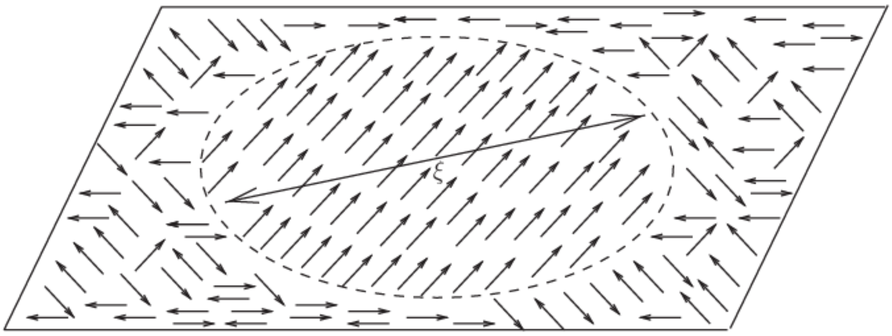


Figure 2.5: Illustration of magnetic domains, correlated over a length scale ξ.[16]

At the critical point, the behavior of the correlation function changes qualitatively: instead of exponential decay, it follows a power law,

$$G(\vec{r}) \sim \frac{1}{r^{d-2+\eta}}, \tag{2.16}$$

where d is the spatial dimension of the system. At criticality, long-range order develops, and the correlation length diverges, which is the defining feature of critical phenomena. The exponent η, known as the anomalous dimension of the order parameter, provides the first example of a critical exponent. This power-law form is supported by exactly solvable models and is a hallmark of scale invariance at criticality.

2.3.3 Phase transitions of the Ising model

The phase diagram of the Ising model is represented in the following figure.

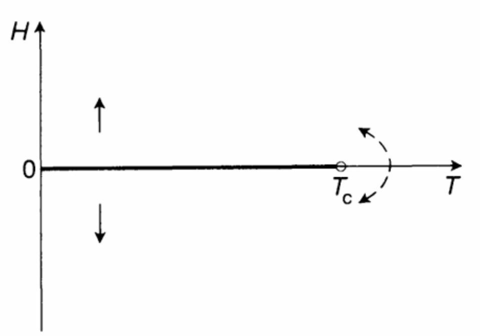


Figure 2.6: Phase diagram of the Ising model [14]

This diagram is representative for a infinite system in two or more dimensions and shows a first order phase transition at 0 applied magnetic field, ending in a critical point at T_c or also named the Curie temperature, where the magnetic properties are lost. There is no phase transition for a lower dimension of the system.

Furthermore we can look at the free energy, its first derivative - the magnetization and its second derivative - the susceptibility as a function of H, as we vary the temperature.

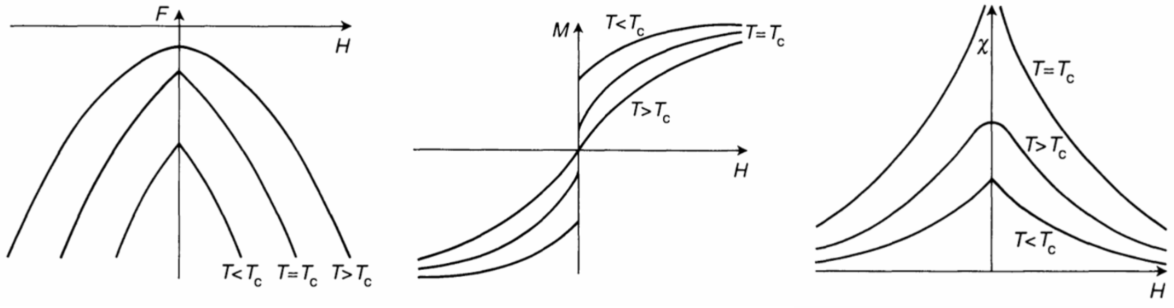


Figure 2.7: The free energy, the magnetization and the susceptibility (left to right)[14]

The thermodynamic functions have plots that match the analysis done in the previous discussion and we see the concavity of the free energy as well as the discontinuity in the first derivative, which is better seen in the magnetization. At the critical temperature, the magnetization is continuous at zero applied field but has infinite slope. We also note that at the critical point the susceptibility diverges and thus marks a characteristic behavior of a continuous phase transition. [14]

This is how the thermodynamic functions act specifically near a continuous phase transition, when such a transition exists. Just as the susceptibility, the specific heat defined before diverges as well at the critical point.

2.4 Critical exponents and universality

Critical exponents characterize how physical observables behave in the vicinity of a continuous phase transition, capturing the universal features of the system near criticality.

To begin to describe why they are so important we first define the reduced temperature, as a mean to see how far we are from the critical temperature:

$$t = \frac{T - T_c}{T_c} \tag{2.17}$$

Assuming that the limit exists, the critical exponent associated to a function F(t) is defined as

$$\lambda = \lim_{t \to 0} \frac{\ln |F(t)|}{\ln |t|} \tag{2.18}$$

or more usefully

$$F(t) \sim |t|^{\lambda}. \tag{2.19}$$

It is important to note that equation (2.19) captures only the asymptotic behavior of the function as it approaches the critical point.

Table 2.2: Most commonly used critical exponents for a magnetic system.[14]

Quantity	Definition
Zero-field specific heat	$C_H \sim \|t\|^{-\alpha}$
Zero-field magnetization	$M \sim (-t)^{\beta}$
Zero-field isothermal susceptibility	$\chi_T \sim \|t\|^{-\gamma}$
Critical isotherm ($t = 0$)	$H \sim \|M\|^{\delta} \operatorname{sgn}(M)$
Correlation length	$\xi \sim \|t\|^{-\nu}$
Pair correlation function at T_c	$G(\vec{r}) \sim \frac{1}{r^{d-2+\eta}}$

These relations originate from the scaling hypothesis and the assumed power-law behavior of thermodynamic quantities in the vicinity of a continuous phase transition. The scaling hypothesis will be discussed in a subsequent chapter.

The principle of universality states that different model systems can exhibit the exact same critical exponents, and therefore the same macroscopic behavior, when approaching criticality. This occurs because the microscopic details of the system are integrated out, leaving only three fundamental characteristics that determine the critical behavior:

1. The dimensionality of the system,
2. The symmetry of the order parameter,
3. The range of interactions between the system constituents.

Systems that share the same values of these characteristics belong to what is called a *universality class*. Universality classes are usually named after the simplest system belonging to them. A well-known example is the Ising universality class, defined by the Ising model, which is the simplest interacting two-state system. This universality class also includes, for instance, the liquid-gas critical point in three dimensions, which belongs to the same (3D Ising) universality class.[14]

The usefulness of this concept will become more evident once we introduce the theoretical framework of the renormalization group, which provides the justification for universality.

Table 2.3: Universality classes [14]

Universality class	Symmetry of order parameter	α	β	γ	δ	ν	η
2-d Ising	2-component scalar	0	1/8	7/4	15	1	1/4
3-d Ising	2-component scalar	0.10	0.33	1.24	4.8	0.63	0.04
3-d X-Y	2-dimensional vector	0.01	0.34	1.30	4.8	0.66	0.04
3-d Heisenberg	3-dimensional vector	-0.12	0.36	1.39	4.8	0.71	0.04
Mean field	n-component scalar	0	1/2	1	3	1/2	0
2-d Potts, $q = 3$	q-component scalar	1/3	1/9	13/9	14	5/6	4/15
2-d Potts, $q = 4$	q-component scalar	2/3	1/12	7/6	15	2/3	1/4

Next, we study the mean-field approach and observe an interesting phenomenon: for spatial dimensionality $d \geq 4$, the critical exponents of the Ising model, the XY model, and the Heisenberg model become identical and take the so-called mean-field values. This dimensionality-independent property will also be explained in the context of the RG. Critical exponents will also be calculated using the mean-field approach, as the simplest method, and will later be compared to the results obtained from the renormalization group.

3 Mean-field and Landau theory

From this point onward, the discussion will focus on the Ising model, as it serves as a representative example of its universality class.

3.1 Mean field theory

Up to this point, we have shown that only an infinite system can exhibit a true phase transition. Such a transition is characterized by a change in the order parameter. In the case of the Ising model, the relevant order parameter is the magnetization, which directly reflects the system's Z_2 symmetry under spin reversal. Because the Hamiltonian is invariant under flipping all spins simultaneously, phase transitions can occur only at zero external field ($H = 0$). Below the critical temperature, this symmetry is spontaneously broken, leading to a nonzero magnetization, whereas above the critical temperature, the symmetry is preserved and the magnetization vanishes.

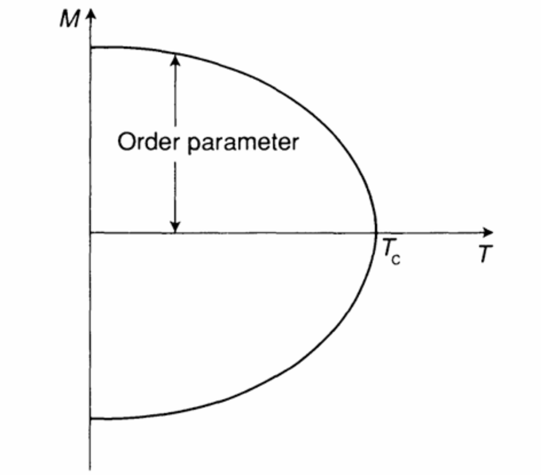


Figure 3.1: Magnetization as a function of temperature [14]

A first order phase transition can be seen as a discontinuous jump in the order parameter, while a continuous phase transition occurs when the discontinuity approaches zero. The order parameter is different from system to system.[3]

Mean-field theory provides an approximate framework for describing phase transitions, but it has a fundamental limitation: it neglects fluctuations, which play a crucial role in lower dimensions. In equilibrium statistical mechanics, a system can often be characterized by time-averaged quantities; however, fluctuations around these averages exhibit important spatial and temporal correlations. Since mean-field theory is based purely on averages, it fails to capture the impact of these fluctuations, leading to an incomplete description near criticality. This motivates the need for more refined approaches. On a quantitative level, mean-field theory predicts critical exponents that do not match those obtained from experiments or from exactly solved models, highlighting its shortcomings in describing universal properties of phase transitions. [3]

To this end, we now calculate the critical exponents in the mean-field framework, defining the Hamiltonian of the Ising model as before:

$$H_\Omega\{S\} = -J\sum_{(ij)} S_i S_j - H\sum_i S_i.$$

If we suppose that the interaction between spins is zero, i.e. they are independent, we could then describe the situation of a paramagnet, which is easier to solve mathematically:

$$Z_\Omega(0,H) = \prod_{i=1}^{N}\left(e^{\beta H} + e^{-\beta H}\right) = \left[2\cosh\left(\frac{H}{k_B T}\right)\right]^N .$$

From this we know that we can get the magnetization back by differentiating the partition function as

$$M = -\frac{1}{N}\frac{\partial F}{\partial H} = \tanh\left(\frac{H}{k_B T}\right). \tag{3.1}$$

Weiss considered the case where spins interact with one another and proposed that each spin experiences an effective field, H_{eff}, arising from the collective influence of all the other spins. This idea leads to a self-consistent description in which each spin is subject not only to the externally applied magnetic field but also to the effective field generated by the average magnetization M. Since no individual spin is special, every spin couples to this average moment, forming the basis of the mean-field approximation.[1]

Mean-field theory is not limited to magnetic systems but represents a general approach that can be adapted to a wide range of models. Its origins trace back to van der Waals, who introduced a mean-field description for fluids to account for intermolecular interactions. Later, Landau unified and generalized these ideas, providing a systematic framework for mean-field theories that applies broadly to different types of phase transitions.

Getting back on the calculation and to make the mean-field picture clearer, we can rewrite the Hamiltonian in a form that resembles that of independent paramagnetic spins placed in local, site-dependent effective fields. Specifically, we introduce an effective field

H_i acting on each spin S_i, which incorporates both the external magnetic field and the average interaction with the neighboring spins. With this definition, the Hamiltonian becomes

$$H_\Omega\{S\} = -\sum_i S_i H_i. \tag{3.2}$$

This expression shows that in the mean-field approximation, the interacting spin system can be mapped onto a collection of independent spins, each subject to its own effective field H_i.[1] The self-consistency of the theory arises because these effective fields are themselves determined by the average magnetization of the system.

Here H_i is:

$$H_i = H + \sum_j J_{ij}\langle S_j\rangle + \sum_j J_{ij}(S_j - \langle S_j\rangle). \tag{3.3}$$

where H is the external field, the second term is the mean field and the third is the fluctuations term which we will ignore. (Weiss model of ferromagnetism)

Defining the co-ordination number z as the number of total possible neighbors, on a hypercubic lattice of dimension d, eq. (3.3) will read:

$$H_i = H + 2dJM \tag{3.4}$$

and from eq. (3.1) we get

$$M = \tanh\left(\frac{H + 2dJM}{k_B T}\right) \tag{3.5}$$

and setting H = 0 we obtain the spontaneous magnetization:

$$M = \tanh\left(\frac{2dJM}{k_B T}\right). \tag{3.6}$$

This equation has a lot of insight graphically.

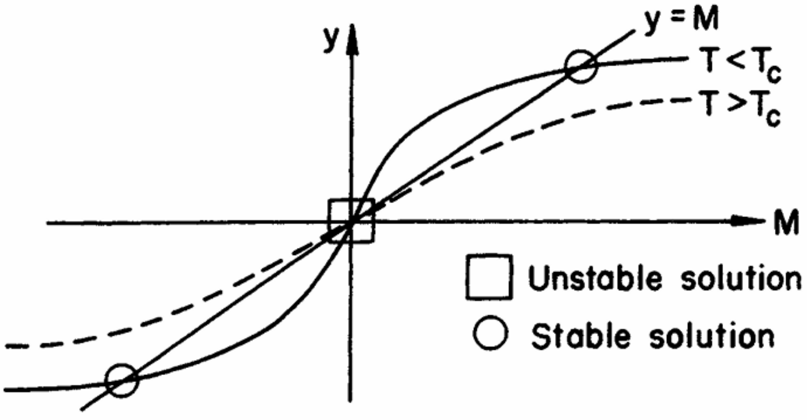


Figure 3.2: Spontaneous magnetization as the temperature is varied [1]

For temperatures above the critical point, $T > T_c \equiv 2dJ/k_B$, the hyperbolic tangent curve lies entirely below the line $y = M$, so the only possible solution is $M = 0$. When the temperature drops below the critical value, $T < T_c$, two additional nontrivial solutions appear, corresponding to $M = \pm M_s(T)$. The phase transition is signaled by the non-analytic change in these solutions as the temperature is varied. [1]

Thus, as stated, the critical temperature is

$$T_c = \frac{2dJ}{k_B} \tag{3.7}$$

We let $\tau = T_c/T$ and we study the critical behavior obtained by this description by expanding the equation of state near the critical temperature, after firstly inverting eq. (3.5) to obtain the equation of state:

$$M = \tanh(H/k_BT + M\tau) = \frac{\tanh H/k_BT + \tanh M\tau}{1 + \tanh H/k_BT \tanh M\tau} \tag{3.8}$$

thus

$$\tanh \frac{H}{k_BT} = \frac{M - \tanh M\tau}{1 - M \tanh M\tau}. \tag{3.9}$$

For small H and M, we can expand in powers of M to find the expanded equation of state :

$$\frac{H}{k_BT} \approx M(1-\tau) + M^3\left(\tau - \tau^2 + \frac{\tau^3}{3} + \dots\right) + \dots \tag{3.10}$$

Now we are ready to extract the critical exponents for the ferromagnetic transition, using the framework of mean field theory.

For $H = 0$ and $T \to T_c^-$ we get that eq. (3.10) simplifies to:

$$M^2 \approx 3\frac{(T_c - T)}{T_c} + \dots$$

From this we see that

$$M \sim \left(\frac{T_c - T}{T_c}\right)^{1/2}$$

and it matches the definition of the critical exponent being

$$M \sim (T_c - T)^{\beta}$$

so $\beta = 1/2$.

The curve in the H-M plane that corresponds to $T = T_c$ is called the critical isotherm and the shape of it is described by the the critical exponent δ.[1]

$$H \sim M^{\delta}$$

Setting $\tau = 1$ (exactly at the critical point), we find:

$$\frac{H}{k_B T} \sim M^3$$

showing hence that $\delta = 3$.

The isothermal magnetic susceptibility is a divergent quantity near the critical point.

$$\chi_T \equiv \left.\frac{\partial M}{\partial H}\right|_T \tag{3.11}$$

Doing the differentiation we obtain:

$$\frac{1}{k_B T} = \chi_T(1-\tau) + 3M^2\chi_T\left(\tau - \tau^2 + \frac{1}{3}\tau^3\right). \tag{3.12}$$

For $T > T_c$, $M = 0$ as the magnetic order is completely destroyed and

$$\chi_T = \frac{1}{k_B}\frac{1}{T - T_c} + \dots$$

and comparing with the definition for the critical exponent γ which is:

$$\chi_T \sim |T - T_c|^{-\gamma} \tag{3.13}$$

shows that the critical exponent $\gamma = 1$.

We can analyze the case of $T < T_c$ too, where

$$M = \sqrt{3}\left(\frac{T_c - T}{T}\right)^{1/2} + \dots$$

and substituting back into the susceptibility, eq. (3.12) we get

$$\chi_T = \frac{1}{2k_B}\frac{1}{T - T_c} + \dots$$

which shows that this is too, $\gamma' = \gamma = 1$. This means that the divergence of the susceptibility below the transition temperature is governed by the same critical exponent.[1]

There is one more critical exponent related to the thermodynamical functions, α, which governs the divergence of the heat capacity near the transition. This follows from differentiating the free energy twice which gives [14]

$$\begin{cases} C = \frac{3}{2}Nk_B + O(t), & T < T_c \\ C = 0, & T > T_c \end{cases}$$

The critical exponent is thus $\alpha = 0$, as there is a jump discontinuity in the specific heat.

Now we are left to calculate two more exponents: ν and η which are in the sector of microscopic physics as they describe fluctuations. This critical exponents describe namely the behavior of the correlation length and the correlation function near the critical point.[1]

We write the two-point correlation function in the form:

$$G(r_i - r_j) = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle. \tag{3.14}$$

We are gonna use the static susceptibility sum rule, relating the isothermal susceptibility to the two-point correlation function

$$\begin{aligned}
\chi_T = \frac{\partial M}{\partial H} &= \frac{1}{N\beta} \frac{\partial^2 \log Z_\Omega}{\partial H^2} \\
&= \frac{1}{N} k_B T \left[\frac{1}{Z_\Omega} \frac{\partial^2 Z_\Omega}{\partial H^2} - \frac{1}{Z_\Omega^2} \left(\frac{\partial Z_\Omega}{\partial H} \right)^2 \right] \\
&= \frac{1}{N} (k_B T)^{-1} \left[\sum_{ij} \langle S_i S_j \rangle - \left(\sum_i \langle S_i \rangle \right)^2 \right] \\
&= \frac{1}{N} (k_B T)^{-1} \sum_{ij} G(\mathbf{r}_i - \mathbf{r}_j) \\
&= (k_B T)^{-1} \sum_i G(\mathbf{x}_i) \\
&= (a^d k_B T)^{-1} \int_\Omega d^d\mathbf{r}\, G(\mathbf{r}).
\end{aligned}$$

This shows that G must reflect the divergence of χ_T.

It turns out, that for $|r| \ll \xi$, Ornstein-Zernike extension to Landau theory provides the following expression for the correlation function: [14]

$$G(\mathbf{r}) \sim \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|^{(d-1)/2} \xi^{(d-3)/2}}, \tag{3.15}$$

This provides a concrete result, letting us calculate the critical exponents in the mean-field approximation and using the result that $\gamma = 1$:

$$\begin{aligned}
\left(\frac{T - T_c}{T_c} \right)^{-1} &\sim \int \frac{r^{d-1} e^{-r/\xi}}{r^{(d-1)/2} \xi^{(d-3)/2}}\, dr \\
&\sim \left(\int z^{(d-1)/2} e^{-z}\, dz \right) \xi^2,
\end{aligned}$$

where $z = r/\xi$ and the integral turns out to be a constant so we see that the correlation length will diverge as:

$$\xi \sim \left(\frac{T - T_c}{T_c}\right)^{-\nu} \tag{3.16}$$

with $\nu = 1/2$.

The last exponent to be calculated, η, describes how the two-point correlation function behaves at large distances exactly at the critical point. It is predicted that the long distance behavior near the critical point follows a law that looks like:

$$G(r) \sim r^{-(d-2+\eta)} \tag{3.17}$$

with $\eta = 0$.

The anomalous dimension η is zero in mean-field theory because fluctuations are ignored, so correlations decay as in a simple Gaussian model. In lower dimensions ($d < 4$), fluctuations are strong enough to change how correlations fall off with distance, and that shift shows up as a nonzero η.[1]

In summary, mean field theory predicts the following critical exponents written in the table.

Table 3.1: Critical exponents for the Ising universality class[1]

Exponent	**Mean Field**	**Experiment**	**Ising (d = 2)**	**Ising (d = 3)**
α	0 (disc.)	0.110 – 0.116	0 (log)	0.110(5)
β	1/2	0.316 – 0.327	1/8	0.325±0.0015
γ	1	1.23 – 1.25	7/4	1.2405±0.0015
δ	3	4.6 – 4.9	15	4.82(4)
ν	1/2	0.625±0.010	1	0.630(2)
η	0	0.016 – 0.06	1/4	0.032±0.003

Although the theoretical model has clear limitations, since the predicted critical exponents do not depend on dimensionality and therefore disagree with experimental values(d<4), it still provides a useful approximation. At the qualitative level it captures the correct phase diagram, and it also offers a practical way to calculate the critical temperature.

We now briefly describe Landau's main idea, which was to generalize and unify mean-field theories within a single systematic framework.

3.2 Landau's generalization

A key property of phase transitions is that two phases of matter with different symmetries must be separated by a transition line. This follows from the fact that a given symmetry is either present or absent.[1] However, the way in which symmetry changes can differ: it may appear (or disappear) continuously or discontinuously as the control parameters are varied. This distinction underlies the classification of continuous and first-order phase transitions.

Lev Landau, noticing the similarities among mean-field theories, proposed a unified framework based on the idea that near a critical point a phase transition is a manifestation of symmetry breaking. To characterize this, he introduced the *order parameter*, which measures the magnitude and nature of the broken symmetry. [3]

Landau postulated that one can define a free energy functional — the *Landau free energy* — depending on the coupling constants $\{K_i\}$ and on the order parameter. The equilibrium state of the system corresponds to the minimum of this free energy with respect to the order parameter. Moreover, as in standard thermodynamics, differentiating the Landau free energy with respect to the relevant variables provides access to all thermodynamic functions.

To build the function L we must respect a set of assumptions:

- L must respect the symmetries of the system.
- Near T_c, $\mathcal{L} \equiv \frac{L}{V}$ (Landau free energy density) can be expanded as a power series in the order parameter η.
- In homogeneous systems, only powers of η appear; in inhomogeneous systems, $\mathcal{L}$ may also depend on $\eta(\mathbf{r})$ and a finite number of its derivatives.
- In the disordered phase ($T > T_c$), the minimum of $\mathcal{L}$ is at $\eta = 0$, while in the ordered phase ($T < T_c$) the minimum occurs at $\eta \neq 0$.
- The expansion is usually truncated at quartic order (η^4), which already captures the essential physics near T_c.

[1]

Here we denoted with η the order parameter.

Taking into account the Ising universality class and its underlying symmetries, and imposing the corresponding constraints on the construction of $\mathcal{L}$, we obtain:

$$\mathcal{L} = at\eta^2 + \frac{1}{2}b\eta^4 - H\eta \tag{3.18}$$

Here, a and b are phenomenological parameters that can be determined from an appropriate description of the underlying microscopic physics. This illustrates the principle of universality: the resulting equation for the order parameter, expressed as a function of the relevant environmental parameters, is universal in the sense that it depends only on the symmetry properties governing the phase transition. Other universality classes can be constructed in an analogous way, by considering different symmetries and constraints characteristic of the specific physical system under study.

To obtain the most general formulation, we must allow the order parameter η to vary in space. In practice, this is achieved through a coarse-graining procedure, where microscopic degrees of freedom are averaged over local regions to define a smoothly varying field that captures long-wavelength behavior. [1] Thus, to capture the correct physics, we consider that the magnetization varies slowly across the system on large length scales, allowing us to express the Landau free energy as an integral over space of an appropriate function of the order parameter. For the Ising universality class, this yields:

$$L = \int d^d r \left[\frac{\gamma}{2} (\nabla \eta)^2 + at\eta^2 + \frac{1}{2} b\eta^4 - H(\mathbf{r})\eta(\mathbf{r}) \right]. \tag{3.19}$$

where γ is a positive constant but its temperature dependence is so weak that it becomes negligible near the critical point. By applying the calculus of variations to minimize this free-energy functional, one obtains an equation for the order parameter. In the case of the Ising model, this reproduces the mean-field result for the magnetization near the critical point.[3]

Thus, while earlier theories already contained some of these results, Landau's approach provided a unified framework: a general theory tailored to the critical region, capable of encompassing all critical phenomena that share a given symmetry type. This formulation makes the principle of universality explicit. However, to sharpen these ideas and obtain the correct critical exponents, one must go beyond mean-field theory and employ the renormalization group.

In the following chapter, we briefly discuss different approaches to improving upon Landau theory, including alternative methods for determining critical exponents, the scaling hypothesis, and the reasons why the Landau framework breaks down in dimensions below four.

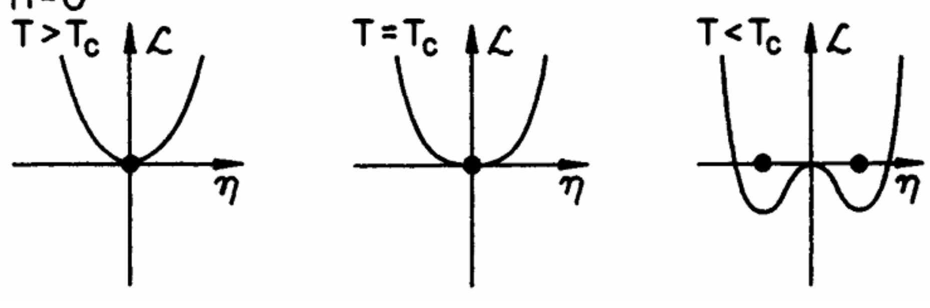


Figure 3.3: Example of the Landau free energy density depicting the continuous transition for the Ising universality class.[1]

4 Alternative Approaches and Scaling

Firstly, the theory breaks down for dimensions less than four because the average behavior of the order parameter is completely swept away by fluctuations in this quantity. Between the years 1959 and 1960 Vitaly Ginzburg and P. Levanyuk described a criterion - latter known as the Ginzburg criterion that indicated whether the behavior near a phase transition was dominated by average values or by fluctuations.[3] From this criterion we have three cases to consider:[1]

- **For $d > 4$:**
  - The Ginzburg criterion is satisfied.
  - Landau theory gives both the correct critical exponents and a good qualitative description.
- **For $d < 4$:**
  - The Ginzburg criterion is not satisfied.
  - Landau theory becomes inconsistent near the critical point.
  - A more advanced framework is required to describe the correct physics.
- **For $d = 4$ (marginal case):**
  - Landau theory is modified by logarithmic corrections due to fluctuations.
  - Example: the isothermal susceptibility behaves as

$$\chi_T \sim \frac{1}{t} |\log t|^{1/3}.$$

The dimension for which criterion breaks down is known as the upper critical dimension - in this case (of the Ising class) it is equal to four.

Along the way of the evolution of mean field theories, other calculation schemes developed, some analytical and some approximate, but nevertheless capable of getting critical exponents.

One method was the transfer matrix, which is an analytic technique that reformulates the partition function of lattice models, such as the Ising system, into a matrix eigenvalue problem; it has been successfully applied to exactly solvable 1d and 2d cases to extract thermodynamic quantities and critical exponents, though it becomes impractical in higher dimensions where closed-form solutions are not accessible. Onsager used this method to obtain the exact solution of the 2d Ising model.

Another important approach was the use of series expansions, which played a key role in discrediting mean-field theory, as the critical exponents obtained from these expansions differed from the mean-field predictions and agreed more with exactly solvable model solutions and experiment. The central idea is that by expanding thermodynamic quantities such as magnetization or heat capacity in power series of $K = J/T$ at high temperatures, or in e^{-K} at low temperatures, one can extract detailed information about the behavior of these models near criticality.[3]

We chose to discuss mean-field theory in more detail, as it is both more general and historically significant, and it also carries conceptual importance in shaping how critical phenomena were first understood. Furthermore, the Gaussian approximation can be formulated on top of mean-field theory, where fluctuations are included but assumed not to interact with one another. While this represents an improvement over the basic mean-field picture, it neglects the crucial role of interacting fluctuations and therefore does not generalize reliably to lower-dimensional systems or near the critical point. (in this approximation, just α is changed from the mean field values)

To conclude this short chapter, we note that critical exponents are not independent, but are constrained by a set of relations known as scaling laws, which were originally conjectured as inequalities and later understood as equalities. These relations follow from the static scaling hypothesis, which postulates that the singular part of the free energy is a generalized homogeneous function—a formalization and extension of Widom's original ansatz that the equation of state near criticality obeys homogeneous scaling. The validity of this hypothesis is confirmed by exact solutions, numerical simulations, and experiments, establishing it as a cornerstone of critical phenomena.[1]

Some of the most notable scaling laws are:

- **Rushbrooke equality:** $\alpha + 2\beta + \gamma = 2$
- **Widom equality:** $\gamma = \beta(\delta - 1)$
- **Josephson equality:** $d\nu = 2 - \alpha$ (for spatial dimension d)
- **Fisher equality:** $\gamma = \nu(2 - \eta)$
- **Hyperscaling relation:** $2 - \alpha = d\nu$

5 The Renormalization Group

After Widom proposed the scaling hypothesis, L. P. Kadanoff provided an argument showing how this hypothesis follows from the divergence of the correlation length. Kadanoff's key insight was that the coupling constants of an effective Hamiltonian depend on the length scale over which the order parameter is defined. Wilson later extended Kadanoff's idea, combining it with concepts from particle physics and field theory, to formulate a rigorous computational framework for critical phenomena — the theory of the renormalization group (RG).[3]

5.1 Kadanoff's Insight

We begin by defining our system Ω on a d-dimensional hypercubic lattice with spacing a, and Hamiltonian H_Ω given by:[1]

$$\beta H_\Omega = -K \sum_{<ij>} S_i S_j - h \sum_{i=1}^{N} S_i \tag{5.1}$$

where we define:

$$K \equiv \beta J, \qquad h \equiv \beta H. \tag{5.2}$$

We denote by $f_s(t, h)$ the singular part of the free energy per spin near the critical point.

Kadanoff's main idea is to understand critical phenomena through *coarse-graining* — grouping nearby spins into blocks and describing the system in terms of the new, averaged block spins.[1] This works because spins remain correlated over a length of order $\xi(T)$. Therefore, as long as

$$a \ll la \ll \xi(T),$$

the spins within a region of size la (with $l > 1$) act, in some sense, as a "single unit."

Each block spin contains l^d microscopic spins, so that the system consists of Nl^{-d} block spins in total. This procedure is known as the *block spin transformation.*[1]

We define the block spin S_I in block I as:

$$S_I = \frac{1}{|\bar{m_l}|}\frac{1}{l^d}\sum_{i\in I} S_i, \tag{5.3}$$

where $\bar{m_l}$, the average magnetization of block I, is defined by

$$\bar{m_l} = \frac{1}{l^d}\sum_{i\in I}\langle S_i\rangle. \tag{5.4}$$

Normalizing to preserve the physics, we require that the block spin have the same magnitude as the original spin:

$$\langle S_I\rangle = \pm 1. \tag{5.5}$$

This idea is built upon two key assumptions:

- Block spins, like the original spins, interact only with their nearest neighbors and experience an effective external field.[1]

This assumption implies that the block spins are governed by a new effective Hamiltonian with renormalized coupling constants K_l and h_l, subject to the boundary conditions $K_1 = K$ and $h_1 = h$ for $l = 1$.

The effective Hamiltonian for the block spins is given by:

$$-\beta H_l = K_l \sum_{<IJ>}^{Nl^{-d}} S_I S_J + h_l \sum_{I=1}^{Nl^{-d}} S_I. \tag{5.6}$$

This Hamiltonian has the same form as the original one but involves fewer spins separated by a larger distance la. Hence, the correlation length measured in units of the new lattice spacing, denoted by ξ_l, is related to the original correlation length ξ_1 by:

$$\xi_l = \frac{\xi_1}{l}. \tag{5.7}$$

Since $\xi_l < \xi_1$, the new Hamiltonian must be further from criticality, characterized by an effective reduced temperature t_l. The same reasoning applies to the magnetic field, leading to the relation:

$$h_l = h\bar{m_l}l^d. \tag{5.8}$$

The singular part of the free energy for the block spin system retains the same functional form as that of the original system, but the two are related by:

$$f_s(t_l, h_l) = l^d f_s(t, h). \tag{5.9}$$

- The second assumption is that:[1]

$$t_l = tl^{y_t}, \quad y_t > 0, \tag{5.10}$$

$$h_l = hl^{y_h}, \quad y_h > 0. \tag{5.11}$$

This assumption captures the power-law and scaling behavior near the critical point. In a rigorous RG treatment, one can compute the values of y_t and y_h.

Substituting this assumption into Eq. (5.10), we obtain:

$$f_s(t,h) = l^{-d} f_s(tl^{y_t}, hl^{y_h}). \tag{5.12}$$

We may choose l arbitrarily; setting

$$l = |t|^{-1/y_t}, \tag{5.13}$$

we find:

$$f_s(t,h) = |t|^{d/y_t} f(1, h|t|^{-y_h/y_t}). \tag{5.14}$$

Defining

$$\Delta \equiv \frac{y_h}{y_t}, \qquad 2-\alpha \equiv \frac{d}{y_t}, \tag{5.15}$$

we obtain

$$f_s(t,h) = |t|^{2-\alpha} f(1, h/|t|^{\Delta}), \tag{5.16}$$

where

$$F_f(x) \equiv f_s(1,x) \tag{5.17}$$

is a function depending only on x.

Equation (5.18) forms the starting point for the static scaling hypothesis. Thus, Kadanoff's argument provides a physical motivation for the scaling form of thermodynamic functions, based on the idea that coupling constants vary with the length scale. The way these couplings evolve under repeated elimination of short-length degrees of freedom was later formalized by Wilson. All these relations can also be derived within the RG framework. Correlation functions follow an analogous scaling form:[1]

$$G(r,t,h) = \frac{1}{r^{2(d-y_h)}} F_G(rt^{1/y_t}, ht^{-y_h/y_t}). \tag{5.18}$$

5.2 The Main Ideas of the RG Theory

The renormalization group provides a formal framework for systematically investigating how a system's properties change when analyzed at different scales. The process, in spirit similar to Kadanoff's coarse-graining, involves two main steps:[1]

- A concrete realization of the coarse-graining procedure (block spin transformation);
- Identification of the origin of singular behavior.

After each block spin transformation, the blocks are separated by a distance la. If we rescale lengths so that in the new units the block spins are again separated by the original distance a, the rescaled system looks similar to the original one in terms of its degrees of freedom, but it is governed by a different Hamiltonian.[1]

Iterating this procedure generates Hamiltonians progressively further from the critical point. The block spin transformations themselves are analytic, and non-analytic behavior arises only after an infinite number of iterations — corresponding to integrating out all degrees of freedom in the thermodynamic limit.[1]

We consider a general Hamiltonian of the form:

$$\mathcal{H} = -\beta H_\Omega = \sum_n K_n \Theta_n\{S\}, \tag{5.19}$$

where K_n are the coupling constants and $\Theta_n\{S\}$ are local operators, functionals of the spin configuration $\{S\}$.

To implement the RG procedure, we introduce a renormalization group transformation R_l, which coarse-grains the short-wavelength degrees of freedom, leaving an effective Hamiltonian for the long-wavelength modes. We then examine how $\mathcal{H}$ changes under such a transformation.

In essence, the renormalization group describes how the coupling constants are redefined when the system's scale changes and degrees of freedom are successively integrated out.

Under an RG transformation R_l (which is not unique for a given system), the set of coupling constants $K \equiv [K]$ transforms as:

$$[K'] \equiv R_l[K], \qquad l > 1. \tag{5.20}$$

This is known as the *recursion relation.* Since $l > 1$, there is no inverse transformation, and hence the RG possesses only a *semi-group* structure.[1]

This semi-group property follows from the fact that performing two successive transformations is equivalent to one transformation with a combined scale change:

$$[K'] = R_{l_1}[K],$$

$$[K''] = R_{l_2}[K'] = R_{l_2} \cdot R_{l_1}[K],$$

so that

$$R_{l_1 l_2}[K] = R_{l_2} \cdot R_{l_1}[K]. \tag{5.21}$$

This structure underlies the emergence of power laws in critical phenomena: physical quantities transform with scale according to these relations, and the power laws reflect the symmetries inherent in the scale transformations. [3]

Moreover, the partition function is invariant under RG transformations. The "free energy" per degree of freedom,

$$g[K] \equiv \frac{1}{N} \log Z_N[K],$$

transforms as

$$g[K] = l^{-d} g[K']. \tag{5.22}$$

It is important to note that new local operators can be generated under block-spin transformations during the RG flow.[1]

Finally, singular behavior arises only after infinitely many RG iterations, corresponding to the elimination of all degrees of freedom in the thermodynamic limit.

5.3 Renormalization Group Flows

A powerful way to visualize the renormalization group is through the lens of *dynamical systems theory*, describing how the coupling constants evolve under successive transformations. In this framework, we consider an abstract space whose axes correspond to the coupling constants $K_0, K_1, \ldots$.

After n RG iterations, the coarse-graining length scale is l^n, and the system is characterized by the couplings $K_0^{(n)}, K_1^{(n)}, \ldots$. As n varies, the system traces out a trajectory in this coupling-constant space.[1]

Each initial set of couplings defines one such trajectory, and the collection of all possible trajectories forms the *renormalization group flow*. This flow provides an insightful way to visualize the behavior of the system under successive coarse-graining and allows the phase diagram of the system to be determined.[1]

An very important feature of this representation are the fixed points. They are represented by points (or higher dimensional geometric structures characterized by their co-dimension) in the flow diagram with the property that the system remains invariant under further transformations. The flow usually tend to go towards this fixed points which come in two categories: attractive and repulsive, depending on how they make the flow behave. The set of initial conditions for which the flow goes toward a given fixed point is called the basin of attraction of that fixed point.[1]

In our formulation, the transformation is R_l and we say that we have a fixed point

$[K^*]$ in coupling constant space if

$$[K^*] = R_l[K^*]. \tag{5.23}$$

For a fixed point the correlation length follows:

$$\xi[K^*] = \frac{\xi[K^*]}{l} \tag{5.24}$$

which means that $\xi[K^*]$ is either 0 or ∞.

Thus we classify fixed points with $\xi = \infty$ as a critical fixed point and $\xi = 0$ as a trivial fixed point.

As a theorem, it can be proved that all points in the basin of attraction of a critical fixed point have infinite correlation length. The basin of attraction of a critical fixed point is called the critical manifold.[1]

From this flow diagram we can inherit the meaning of the physics behind it. The critical fixed points describe the the singular critical behavior and the trivial fixed points describe the bulk phases of the system. Moreover the location and the nature of the fixed points build up the phase diagram as mentioned before and RG flows near a critical fixed point determines the critical exponents.

As an illustrative example we can consider the following flow diagrams:

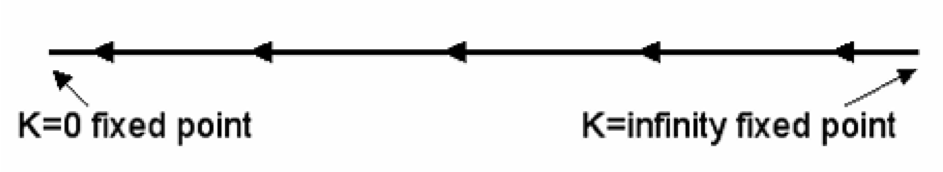


Figure 5.1: Flow diagram for one-dimensional Ising model.[3]

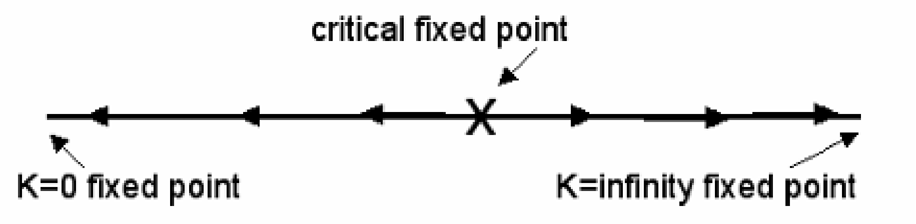


Figure 5.2: Flow diagram for two-dimensional Ising model.[3]

For the Ising 1d model, the renormalization transformation weakens the couplings, pushing it to the only fixed point at $K = 0$, while in the 2d case we have a critical fixed point in the middle, so three fixed points in total. If the initial couplings are below this value, then they will too be pushed toward a weak coupling fixed point ($K = 0$), but if they are above that critical fixed point values, renormalizations produce a flow towards a strong coupling fixed point, describing a ferromagnetic state.[3]

As our purpose is to calculate the critical exponents, we turn to analyze the local behavior of RG flows near a critical fixed point.[1] For this purpose, we let:

$$K_n = K_n^* + \delta K_n. \tag{5.25}$$

This makes the new Hamiltonian close to the one which describes a fixed point, $\mathcal{H} = \mathcal{H}[K^*] = \mathcal{H}^*$ and so the new Hamiltonian is $\mathcal{H} = \mathcal{H}^* + \delta\mathcal{H}$

After performing an RG transformation, the coupling constants will be

$$K'_n = K'_n[K] \equiv K_n^* + \delta K'_n \tag{5.26}$$

with $\delta K'_n$ given by Taylor's theorem:

$$K'_n\{K_1^* + \delta K_1, K_2^* + \delta K_2, \ldots\} = K_n^* + \sum_m \left.\frac{\partial K'_n}{\partial K_m}\right|_{K_m = K_m^*} \delta K_m + O((\delta K)^2) \tag{5.27}$$

so that we have

$$\delta K'_n = \sum_m M_{nm} \delta K_m \tag{5.28}$$

where

$$M_{nm} = \left.\frac{\partial K'_n}{\partial K_m}\right|_{K = K^*} \tag{5.29}$$

is the linearised RG transformation in the vicinity of the fixed point K^*. The matrix M is real, but not symmetric in general. If not mentioned otherwise we will take it to be symmetric for simplicity.[1]

With this formalism we study RG flows near the critical fixed point, using the linearised RG transformation M^l, where l represents the scale factor involved in the transformation. This turn to be a eigenvalue-eigenvector problem:

$$M_{nm}^{(\ell)}\, e_m^{(\sigma)} = \Lambda^{(\sigma)}\, e_n^{(\sigma)} \tag{5.30}$$

where $\Lambda^{(\sigma)}$ are the eigenvalues and $e_n^{(\sigma)}$ the eigenvectors. Here σ labels the eigenvalues and n labels the component of the vector e.

The semi-group property implies:

$$M^{(\ell)} M^{(\ell')} = M^{(\ell\ell')} \tag{5.31}$$

and so

$$\Lambda_{\ell}^{(\sigma)}\Lambda_{\ell'}^{(\sigma)} = \Lambda_{\ell\ell'}^{(\sigma)} \tag{5.32}$$

Solving this functional equation by turning it into a differential problem we get:

$$\Lambda_{(\ell)}^{(\sigma)} = \ell^{y_\sigma} \tag{5.33}$$

with the value of y_σ to be determined but independent of l.

The task at hand is to see how $[\delta K]$ transforms under M. We expand for this reason $[\delta K]$ in terms of the eigenvectors of M and we see how the components of $[\delta K]$ grow or shrink in the eigen-directions:[1]

$$\delta\mathbf{K} = \sum_{\sigma} a^{(\sigma)}\mathbf{e}^{(\sigma)} \tag{5.34}$$

where the coefficients $a^{(\sigma)}$ are calculated from the assumed orthogonality of the eigenvectors which is from the assumption of the symmetric matrix M.

Applying the linearised renormalization group transformation we get

$$\begin{aligned} \delta K' &= M\delta K && (5.35)\\ &= M\sum_{\sigma} a^{(\sigma)}e^{(\sigma)} && (5.36)\\ &= \sum_{\sigma} a^{(\sigma)}\Lambda^{(\sigma)}e^{(\sigma)} \equiv \sum_{\sigma} a^{(\sigma)\prime}e^{(\sigma)} && (5.37) \end{aligned}$$

where $a^{(\sigma)\prime}$ is the projection of $\delta K'$ in the direction of $e^{(\sigma)}$. This equation has the insight that some components of δK grow under M^l while others shrink.[1]

If we now order the eigenvalues by their absolute values,

$$|\Lambda_1| \geq |\Lambda_2| \geq |\Lambda_3| \geq \dots \tag{5.38}$$

we can distinguish three cases:[1]

1. $|\Lambda^{(\sigma)}| > 1$ i.e. $y^\sigma > 0$, which implies that $a^{(\sigma)\prime}$ grows as ℓ increases.
2. $|\Lambda^{(\sigma)}| < 1$ i.e. $y^\sigma < 0$, which implies that $a^{(\sigma)\prime}$ shrinks as ℓ increases.
3. $|\Lambda^{(\sigma)}| = 1$ i.e. $y^\sigma = 0$, which implies that $a^{(\sigma)\prime}$ does not change as ℓ increases.

After many iterations only the components of δK along directions $e^{(\sigma)}$ for which (1.) holds will be important.

The three cases have the following terminology:[1]

1. → **relevant** eigenvalues/directions/eigenvectors.
2. → **irrelevant** eigenvalues/directions/eigenvectors.
3. → **marginal** eigenvalues/directions/eigenvectors.

The distinctions are that the relevant eigenvalues are associated with directions out of the critical manifold in the vicinity of K^*, the irrelevant ones corresponds to directions of flow into the fixed point and the marginal ones correspond to logarithmic corrections to scaling. The terms used are always to be associated with respect to a particular fixed point in order to make sense as for example a Hamiltonian may be relevant at some fixed point but not at the other.[1]

To summarize, in the study of critical phenomena via the renormalization group (RG), the critical point corresponds to a fixed point of the RG transformation. At this fixed point, the system is scale-invariant. The RG flow near this fixed point is governed by its eigenvalues $\Lambda^{(\sigma)} = \ell^{y_\sigma}$ and eigenvectors.

The critical surface is the set of Hamiltonians that flow into the fixed point. It is defined by setting all *relevant* couplings (with positive exponents $y_\sigma > 0$) to zero. The unstable flow away from this surface, driven by relevant couplings, corresponds to deviating from criticality (e.g., by changing temperature).

Hamiltonians on the critical surface differ only in their *irrelevant* couplings (with negative exponents $y_\sigma < 0$). Since these couplings flow to zero under successive RG steps, all such Hamiltonians end up at the same fixed point. This universality in the flow underlies the **universality** of critical behavior: systems with the same symmetry and dimensionality, and thus the same set of relevant operators, belong to the same **universality class** and share identical critical exponents.

Thus, the RG acts as a focusing mechanism, showing how microscopic details (irrelevant operators) are washed out at long wavelengths, leaving only the universal macroscopic physics governed by the fixed point.

To conclude this section, we illustrate the renormalization-group (RG) flow in practice using the two-dimensional Ising model. A real-space renormalization is performed through a block-spin transformation, where groups of 3×3 spins are coarse-grained into single effective spins. This procedure reveals the RG flow diagram (Fig. 5.2): for temperatures below the critical point, the flow moves toward the ordered fixed point; above the critical point, toward the disordered fixed point; and at criticality, the system remains at the critical fixed point. This example clearly demonstrates the essence of real-space renormalization. This is shown diagrammatically in the following figure:

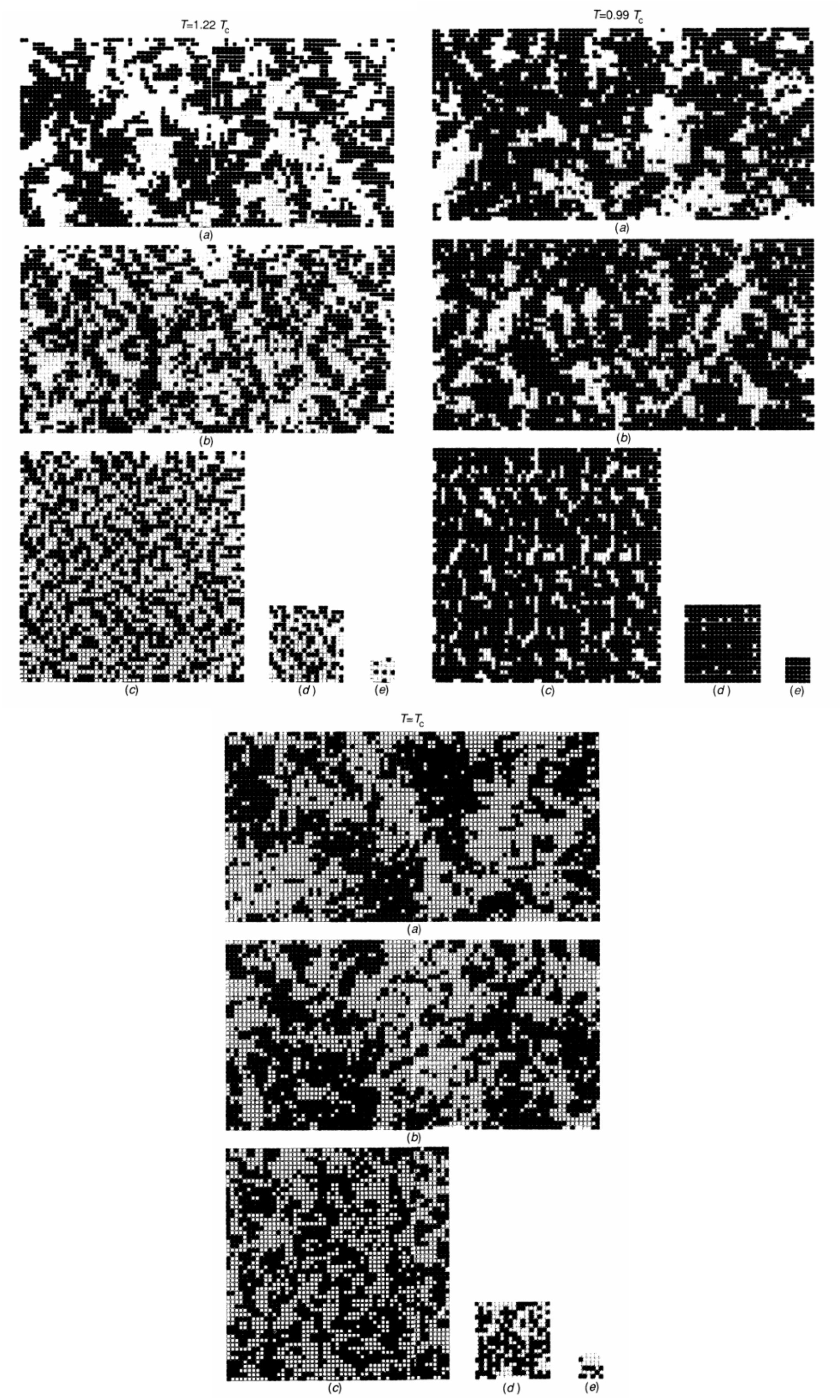


Figure 5.3: RG transformation for $T > T_c$, $T = T_c$ and $T < T_c$[14]

5.4 Real space RG

We now apply the theory described in the previous sections with an example of real space Renormalization Group in order to see the full computation procedure and calculate the critical exponents for the Ising 2-d model on a triangular lattice.[1] [17]

The Hamiltonian which governs our system is:

$$\mathcal{H} = K\sum_{\langle ij\rangle} S_i S_j + h\sum_i S_i. \tag{5.39}$$

We can inherit from the following picture the geometric way in which we group together the spins for the block spin transformation.

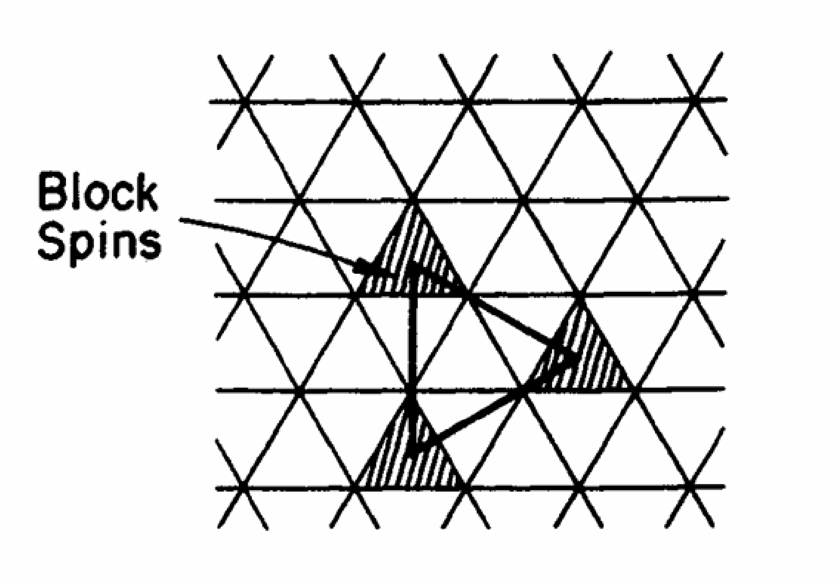


Figure 5.4: Block spin transformation for a triangular lattice - Ising 2-d model[1]

The block spin S_I in the block I is calculated with the majority rule:

$$S_I = \text{sign}\{S_1^I + S_2^I + S_3^I\} \tag{5.40}$$

where we take each spin from a block and we calculate if either there is a majority of "up" spins or "down" spins. The notation used is that S_j^I is represented by the j^{th} spin in the I-th block.

The S_I are normalized and the lattice spacings for the block spins have been enlarged by a factor of $l = \sqrt{3}$. With this description, our goal now is to build an approximate RG transformation.

Firstly we can look at Osanger's exact solution for the Ising 2-d model in order to calculate the exact values of the eigenvalues of the RG transformation and use it as benchmark to our approximation.[5] Thus the critical exponents and the eigenvalues will be compared to see how they differ.

The critical exponents for the exact solution are in Table (3.1) and the eigenvalues can be calculated to get:

$$y_t = 1,\ \Lambda^t = l = \sqrt{3} \tag{5.41}$$

$$y_h = 15/8,\ \Lambda^h = (\sqrt{3})^{15/8} \approx 2.80 \tag{5.42}$$

To actually calculate something we need a formally exact representation of the coarse-grained Hamiltonian $\mathcal{H}$.

Each value of S_I may be given by 4 possible configurations of the original spins:

$$S_I = +1 \quad \{\sigma_I\} = \begin{matrix} \downarrow\uparrow\uparrow \\ \uparrow\downarrow\uparrow \\ \uparrow\uparrow\downarrow \\ \uparrow\uparrow\uparrow; \end{matrix} \tag{5.43}$$

$$S_I = -1 \quad \{\sigma_I\} = \begin{matrix} \uparrow\downarrow\downarrow \\ \downarrow\uparrow\downarrow \\ \downarrow\downarrow\uparrow \\ \downarrow\downarrow\downarrow; \end{matrix} \tag{5.44}$$

where σ_I denotes the set of possible spins.

Following this procedure we preserve the total number of degrees of freedom, as each new value of the block spin has associated with it a number of possible configurations.

Thus the coarsed grained or effective Hamiltonian is given by:

$$e^{\mathcal{H}'\{S_I\}} = \sum_{\{\sigma_I\}} e^{\mathcal{H}\{S_I,\sigma_I\}} \tag{5.45}$$

The transformation is said to be approximate because we will estimate $\mathcal{H}$ using perturbation theory. We first consider the case with $h = 0$ (no external magnetic field). For convenience, we split the Hamiltonian in a part $\mathcal{H}_{\prime}$ to describe the interactions between spins within one block spin and V, for interactions between spins in different block spins.[1]

Thus,

$$\mathcal{H} = \mathcal{H}_{\prime} + V \tag{5.46}$$

where

$$\mathcal{H}_0 = K \sum_I \sum_{i,j\in I} S_i S_j \tag{5.47}$$

and

$$V = K \sum_{I \neq J} \sum_{\substack{i \in I \\ j \in J}} S_i S_j \tag{5.48}$$

$\mathcal{H}$, will serve as our unperturbed Hamiltonian, while we consider V a perturbation.

To see things in a more useful way, we can use the fact that the average of a quantity with respect to $\mathcal{H}$, is defined by:[1]

$$\langle A(S_i) \rangle_0 \equiv \frac{\sum_{\{\sigma_I\}} e^{\mathcal{H}_0\{S_I, \sigma_I\}} A(S_I, \sigma_I)}{\sum_{\{\sigma_I\}} e^{\mathcal{H}_0\{S_I, \sigma_I\}}}. \tag{5.49}$$

So that equation (5.45) becomes:

$$e^{\mathcal{H}'\{S_I\}} = \left\langle e^V \right\rangle_0 \sum_{\{\sigma_I\}} e^{\mathcal{H}_0(S_I, \sigma_I)}. \tag{5.50}$$

Then by writing the partition function for one block, subject to a given value of S_I,

$$Z_0(K) = \sum_{S_1 S_2 S_3} \exp\left\{K\left(S_1^l S_2^l + S_2^l S_3^l + S_3^l S_1^l\right)\right\}. \tag{5.51}$$

We can see that

$$\sum_{\{\sigma_I\}} e^{\mathcal{H}_0\{S_I, \sigma_I\}} = Z_0(K)^M \tag{5.52}$$

where M represents the total number of blocks in the system.

Using the possible spin configurations, we find that the partition function for one blocks spin is actually independent of S_I and has the value

$$Z_0(K) = 3e^{-K} + e^{3K} \tag{5.53}$$

So we reduced the problem of calculating $\mathcal{H}$ to that of evaluating

$$e^{\mathcal{H}'\{S_I\}} = \left\langle e^V \right\rangle_0 Z_0(K)^M \tag{5.54}$$

To be more specific, in order to solve the problem we need to calculate $\left\langle e^V \right\rangle_0$. We can use the cumulant expansion to get:

$$\begin{aligned} \langle e^V \rangle_0 &= \left\langle 1 + V + \frac{V^2}{2} + \cdots \right\rangle_0 \\ &= 1 + \langle V \rangle_0 + \frac{\langle V^2 \rangle_0}{2} + \cdots \end{aligned}$$

We consider V to be a perturbation so in some sense this quantity is small in order to be expanded in a power series, using

$$\log(1+x) = x - x^2/2 + O(x^3)$$

so that we have

$$\log\langle e^V\rangle_0 = \langle V\rangle_0 + \frac{1}{2}\langle V^2\rangle_0 - \frac{\langle V\rangle_0^2}{2} + O(V^3). \tag{5.55}$$

Then re-exponentiating we get:

$$\langle e^V\rangle_0 = \exp\left\{\langle V\rangle_0 + \frac{1}{2}[\langle V^2\rangle_0 - \langle V\rangle_0^2] + O(V^3)\right\}. \tag{5.56}$$

where the terms in the exponent are called cumulants.

Thus we have:

$$\mathcal{H}'\{S_I\} = M\log Z_0(K) + \langle V\rangle_0 + \frac{1}{2}[\langle V^2\rangle_0 - \langle V\rangle_0^2] + O(V^3). \tag{5.57}$$

The first term is regular and does not contribute to the singular behavior, while the term $\langle V\rangle_0$ couples nearest neighbor blocks.[1]

By writing

$$V = \sum_{I\neq J} V_{IJ} \tag{5.58}$$

and noting the structure of the following figure,

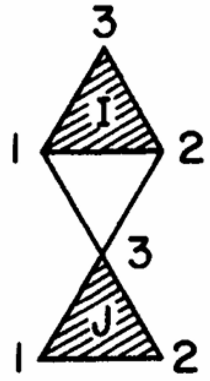


Figure 5.5: Interaction between nearest neighbor block spins to $O(V)$ [1]

we have

$$V_{IJ} = K(S_3^J)(S_1^I + S_2^I) \tag{5.59}$$

and thus

$$\langle V_J \rangle_0 = 2K\, \langle S_3^J\, S_1^I \rangle_0. \tag{5.60}$$

We can factorize this average as $\mathcal{H}$, does not couple to different blocks so

$$\langle V_{IJ} \rangle_0 = 2K \left\langle S_3^J \right\rangle_0 \left\langle S_1^I \right\rangle_0 \tag{5.61}$$

and

$$\langle S_3^J \rangle_0 = \frac{1}{Z_0} \sum_{\{\sigma_j\}} S_3^J e^{K[S_1^J S_2^J + S_2^J S_3^J + S_3^J S_1^J]}. \tag{5.62}$$

We need to evaluate this for each configuration of eq. (5.43) and eq. (5.44) - the possible configurations.

For $S_j = 1$:

$$\left\langle S_3^J \right\rangle_0 = \frac{e^{3K} - e^{-K}}{e^{3K} + 3e^{-K}} \tag{5.63}$$

and for $S_J = -1$:

$$\left\langle S_3^J \right\rangle_0 = -\frac{e^{3K} - e^{-K}}{e^{3K} + 3e^{-K}} \tag{5.64}$$

Hence

$$\left\langle S_3^J \right\rangle_0 = S_J \left[\frac{e^{3K} - e^{-K}}{e^{3K} + 3e^{-K}} \right] \tag{5.65}$$

and similarly for S_I,

$$\left\langle S_1^I \right\rangle_0 = S_I \left[\frac{e^{3K} - e^{-K}}{e^{3K} + 3e^{-K}} \right]. \tag{5.66}$$

Thus

$$\langle V \rangle_0 = 2K\Phi(K)^2 \sum_{\langle IJ \rangle} S_I S_J \tag{5.67}$$

with

$$\Phi(K) \equiv \frac{e^{3K} - e^{-K}}{e^{3K} + 3e^{-K}} \tag{5.68}$$

To summarize we found out that

$$\mathcal{H}'\{S_I\} = M \log Z_0(K) + K' \sum_{\langle IJ \rangle} S_I S_J + O(V^2) \tag{5.69}$$

with

$$K' = 2K\Phi(K)^2. \tag{5.70}$$

We have calculated thus an explicit RG transformation, by deriving a crude approximation to the recursion relation for the coupling constant.

Now that we obtained the transformation, we need to find the fixed points of the RG transformation. The fixed points need to satisfy[1]

$$K^* = 2K^*\Phi(K^*)^2 \tag{5.71}$$

which gives $K^* = 0$, $K^* = \infty$ or $\Phi(K^*) = \frac{1}{\sqrt{2}}$.

Inverting $\Phi(K^*)$ we find the non-trivial fixed point which we will denote by:

$$K_c = \frac{1}{4}\log(1 + 2\sqrt{2}) \approx 0.34. \tag{5.72}$$

This non-trivial fixed point compares in a good approximation with the exact result of Osanger for which he got $K_c = \frac{1}{4}\log 3 = 0.27$.

We can calculate the eigenvalue

$$\Lambda_t = \left.\frac{\partial K'}{\partial K}\right|_{K_c} = 1.62 \tag{5.73}$$

which is close to the exact value of about 1.73 as mentioned above.

A closer approximation is made by calculating terms of order $O(V^2)$ in the perturbation expansion series. With more terms, the more we converge towards the real values, although not uniformly in this case.[1]

In order to get Λ_h we need to consider the case $h \neq 0$ by applying a small deviation $\delta h = h - h^*$ in order to see how the calculation of $\mathcal{H}'$ is affected.[1] We let the change in $\mathcal{H}$ be a small external field δh to $\mathcal{H}'$. By definition we have:

$$e^{\mathcal{H}'\{S_I\}} = \sum_{\{\sigma_I\}} e^{\mathcal{H}\{S_I,\sigma_I\}} \tag{5.74}$$

$$e^{\mathcal{H}'+\delta\mathcal{H}'} = \sum_{\{\sigma_I\}} e^{\mathcal{H}\{S_I,\sigma_I\}+\delta\mathcal{H}\{S_I,\sigma_I\}}. \tag{5.75}$$

By subtracting and writing the power series for an exponential $e^z = 1 + z + \frac{z^2}{2} + O(z^3)$ we obtain:

$$\delta\mathcal{H}'\{S_I\} = \frac{\sum_{\{\sigma_I\}} e^{\mathcal{H}\{S_I,\sigma_I\}}\delta\mathcal{H}\{S_I,\sigma_I\}}{\sum_{\{\sigma_I\}} e^{\mathcal{H}\{S_I,\sigma_I\}}} \tag{5.76}$$

and now again by definition

$$\delta\mathcal{H}\{S_I,\sigma_I\} = \delta h \sum_i S_i = \delta h \sum_I \sum_{i\in I} S_i^I; \tag{5.77}$$

$$\delta\mathcal{H}' = \delta h' \sum_I S_I \tag{5.78}$$

so from eq. (5.76) it is implied that the zeroth order in V is:

$$\delta\mathcal{H}'\{S_I\} = \left\langle \delta h \sum_I \sum_{i\in I} S_i^I \right\rangle_0 \tag{5.79}$$

$$= \delta h \sum_I \left\langle S_1^I + S_2^I + S_3^I \right\rangle_0 \tag{5.80}$$

and from the previously calculated $\left\langle S_1^I \right\rangle_0$ we obtain

$$\langle S_i^I \rangle_0 = S_I \Phi(K) \quad \text{for } i = 1,2,3. \tag{5.81}$$

Thus we obtain the recursion relation

$$\delta h' = 3\Phi(K)\delta h \tag{5.82}$$

which at the critical fixed point yields the eigenvalue

$$\Lambda_h = 3\Phi(K_c) = \frac{3}{\sqrt{2}} \simeq 2.12 \tag{5.83}$$

which is in good agreement with the exact result $\Lambda_h \simeq 2.8$. Like in the previous case, calculating more orders in V would give closer values to the right answer.[1]

From the zeroth order approximation eigenvalues we can calculate y_h, y_t and from that we can calculate all other critical exponents with the help of the scaling laws mentioned before. The result which are easy calculations are summarized in the following table:

Critical Exponent	**Symbol**	**Exact**	**Approx.**
Thermal RG	y_t	1.000	0.879
Magnetic RG	y_h	1.875	1.367
Correlation-length	ν	1.000	1.138
Specific-heat	α	0 (log)	-0.277
Magnetization	β	0.125	0.721
Susceptibility	γ	1.75	0.835
Critical isotherm	δ	15.0	2.16
Anomalous dimension	η	0.25	1.27

The values are only approximate and are not that exact but this is expected as for the zeroth order approximation. Better results are obtained for further order terms in the perturbation series.

5.5 Momentum space RG

The two traditional ways of setting up a Hamiltonian which then will give a microscopic description of the system are to describe the system in coordinate space like in the previous section or to employ Fourier transforms in order to get the momentum space representation. With this description one can analyze analytically critical phenomena near four dimensions by the so-called ϵ expansion.[3]

We define ϵ as

$$\epsilon = 4 - d \tag{5.84}$$

The approach used is called momentum shell RG, which is related to the real space renormalization group that we saw in the previous section. The main idea is to build the RG recursion relations by performing a partial trace over the short wavelength degrees of freedom, on lengths less than the block size. In this approach, the partial trace is done on the Fourier components of the order parameter, so that the range of integration satisfies $\Lambda/l < |k| < \Lambda$. Here Λ serves as an ultra-violet(UV) cutoff to the integrals and represents the coarse graining length. $k = |k|$ is the wavenumber. This range of integration is useful, as there cannot be infra-red(IR) divergences, as k is restricted away from zero and the RG recursion relation can be calculated by means of perturbation theory. We can regard the spatial dimension d as a variable and for a $\epsilon \ll 1$ an expansion of the recursion relation can also be made in powers of ϵ. The second step is to find the fixed points of the recursion relations and as we shall see we will find a non-trivial fixed point called the Wilson-Fisher fixed point which governs the critical behavior near four dimensions and gives a good approximation for the critical exponents if we set $\epsilon = 1$.[1]

The starting point is the effective Hamiltonian of Landau theory for the Ising universality class in the form of:

$$-\mathcal{H}\{S\} = \int d^d\mathbf{r} \left[\frac{1}{2}(\nabla S)^2 + \frac{1}{2}r_0 S^2 + \frac{1}{4}u_0 S^4 - h_0 S\right] \tag{5.85}$$

where r_0 is a temperature-like coupling and u_0 controls the strength of the quartic self-interaction.

We want to compute the critical exponents in the complete Landau theory, near $d = 4$.

In order to achieve this, we define the long and short wavelength components of $S(r)$ respectively by

$$\hat{S}'_\ell(k) = \hat{S}_k \quad \text{for } 0 < |k| < \frac{\Lambda}{\ell}, \tag{5.86}$$

$$\hat{\sigma}_\ell(k) = \hat{S}_k \quad \text{for } \frac{\Lambda}{\ell} < |k| < \Lambda \tag{5.87}$$

Thus the Hamiltonian can be split apart into long and short wavelength parts. To get an analogy with the real space RG and to paint a clear picture, S'_l corresponds to the block spins while σ_l correspond to the microscopic degrees of freedom within a block. After integrating out the part with the microscopic degrees of freedom we want to read off the recursion relations for the coupling constant. In order to do this, we need to get the Hamiltonian back in its original form by rescaling the space so that the new Hamiltonian is defined in terms of the original degrees of freedom.[1]

To start our calculation, we mention that the integral is taken over $0 < |k| < \Lambda$ and that the external magnetic field h_0 is set to zero.[1]

Using Fourier transform to get to the momentum space representation, we write the effective Hamiltonian as:

$$-\mathcal{H}\{S\} = \frac{1}{2}\int_k (r_0 + k^2)|\hat{S}_k|^2 + \frac{1}{4}u_0 \int_{k_1,\ldots,k_4} \hat{S}_{k_1} \ldots \hat{S}_{k_4} \times (2\pi)^d \delta(k_1 + \cdots + k_4). \tag{5.88}$$

where

$$\int_{\Bbbk} \equiv \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \equiv \int_0^\Lambda \bar{d}k. \tag{5.89}$$

Now the task is to split short and long wavelength components. The difficulty for the problem arises in dealing with the interaction term u_0 because the quartic term is the only one that couples $\hat{S}'_\ell$ and $\hat{\sigma}_\ell$.[1]

We write the effective Hamiltonian as

$$-\mathcal{H}\{S\} = -\mathcal{H}_{S'}\{\hat{S}'_\ell\} - \mathcal{H}_\sigma\{\hat{\sigma}_\ell\} - V(\hat{S}'_\ell, \hat{\sigma}_\ell). \tag{5.90}$$

We begin by calculating the partition function

$$\begin{aligned} Z(r_0, u_0) &= \int \mathcal{D}S\, e^{\mathcal{H}} && (5.91)\\ &= \int \mathcal{D}S\, e^{\mathcal{H}_S + \mathcal{H}_\sigma + V} && (5.92)\\ &= \int \mathcal{D}S'_\ell\, e^{\mathcal{H}_{S'}} \int \mathcal{D}\sigma_\ell\, e^{\mathcal{H}_\sigma + V} && (5.93) \end{aligned}$$

where we integrate over all possible field configurations.

Thus after doing an integration over σ_l, a wave-function renormalization of S'_l and rescaling of k we get:

$$Z(r_0, u_0) = \int \mathcal{D}S_\ell\, e^{\mathcal{H}_\ell} \tag{5.94}$$

where $\mathcal{H}_l$ is the effective Hamiltonian for the coarse grained order parameter S_l.

Following what we have done in the previous section, using the definition of the average:

$$\langle A(S'_\ell)\rangle_0 \equiv \frac{\int \mathcal{D}\sigma_\ell \, e^{\mathcal{H}_\sigma} A(S'_\ell, \sigma_\ell)}{\int \mathcal{D}\sigma_\ell \, e^{\mathcal{H}_\sigma}} \tag{5.95}$$

which integrates out the short wavelength degrees of freedom we obtain the partition function

$$Z = Z_\sigma(r_0) \int DS'_\ell e^{\mathcal{H}_{S'}} \langle e^V \rangle_0 \tag{5.96}$$

and we ignore $Z_\sigma(r_0)$ as is a non-singular contribution to the partition function and doesn't enter in the calculation of the critical exponents.

As before we remain with the expression for the partition function which describes the long-wavelength degrees of freedom and it can be approximated by the cumulant expansion to obtain:[1]

$$\begin{aligned} Z &= \int \mathcal{D}S'_\ell \, e^{\mathcal{H}_{S'}} \langle e^V \rangle_0 &(5.97)\\ &= \int \mathcal{D}S'_\ell \, e^{\mathcal{H}_{S'}} e^{\langle V\rangle_0 + \frac{1}{2}\left[\langle V^2\rangle_0 - \langle V\rangle_0^2\right] + O(V^3)}. &(5.98)\end{aligned}$$

Also to get the renormalised Hamiltonian we need to make a wave-function renormalization of S'_l as

$$\hat{S}_\ell(\mathbf{k}) = z\hat{S}_\ell(\mathbf{k}_\ell); \quad \mathbf{k}_\ell = \ell\mathbf{k}, \tag{5.99}$$

where z is called a wave-function renormalization, introduced in order to have the same coefficient for the gradient term in the Hamiltonian.

The renormalised Hamiltonian is

$$\mathcal{H}_l\{\hat{S}_\ell(k_l)\} = \mathcal{H}_{S'}\{z\hat{S}_\ell\} + \langle V\rangle_0 + \frac{1}{2}\left[\langle V^2\rangle_0 - \langle V\rangle_0^2\right] + O(V^3) \tag{5.100}$$

and in order to read off the coupling constant recursion relations we write the renormalised Hamiltonian in the form:

$$-\mathcal{H}_\ell\{S_\ell\} = \frac{1}{2}\int_k u_{2,\ell}(k)|\hat{S}_\ell(k)|^2 + \frac{1}{4}\int_{k_1\ldots k_4} (2\pi)^d \delta\left(\sum_{i=1}^4 k_i\right) u_{4,\ell}(k_1\ldots k_4)\hat{S}_{k_1}\ldots\hat{S}_{k_4} + O(\hat{S}^6) \tag{5.101}$$

The quantities $u_{2,\ell}(\mathbf{k})$ and $u_{4,\ell}(\mathbf{k}_1, \ldots, \mathbf{k}_4)$ represent the renormalised counterparts of $r_0 + k^2$ and u_0. At the Gaussian fixed point—where interactions vanish ($u_0 = 0$) and the theory is purely quadratic—higher-order gradient terms, though irrelevant there, can be generated under renormalization. These introduce a momentum dependence in $u_{2,\ell}$ and $u_{4,\ell}$, and may produce additional couplings such as $u_{6,\ell}, u_{8,\ell}, \ldots$, which generally depend on momenta.[1]

However, to first order in ϵ these higher-order effects can be neglected. After deriving the recursion relations and locating the critical fixed point, the leading behavior of the couplings is

$$\begin{aligned} r^* &= O(\epsilon), \\ u^* &= O(\epsilon), \\ u_2^*(\mathbf{k}) &= r^* + k^2 + O(\epsilon^2), \\ u_4^*(\mathbf{k}) &= u^* + O(\epsilon^2), \\ u_6^*,\ u_8^*,\ \ldots &= O(\epsilon^2). \end{aligned}$$

Again in the like of the previous section we wish to compute $\langle V\rangle_0$. In order to do this we need to compute:

$$V = \frac{1}{4}u_0 \int_{k_1k_2k_3k_4} (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \hat{S}_{\mathbf{k}_1}\hat{S}_{\mathbf{k}_2}\hat{S}_{\mathbf{k}_3}\hat{S}_{\mathbf{k}_4} \tag{5.102}$$

where

$$\int_{\mathbf{k}} \hat{S}_{\mathbf{k}} = \int_0^{\Lambda/\ell} \frac{d^d\mathbf{k}}{(2\pi)^d}\, \hat{S}'_\ell(\mathbf{k}) + \int_{\Lambda/\ell}^{\Lambda} \frac{d^d\mathbf{k}}{(2\pi)^d}\, \hat{\sigma}_\ell(\mathbf{k}) \tag{5.103}$$

The strategy is to substitute eq. (5.102) in eq. (5.101) and then perform the average $\langle \ldots \rangle_0$. We wish to calculate the terms that follow from this procedure and then calculate the renormalised coupling constants $u_{2,\ell}$, $u_{4,\ell}$ and so on if needed.[1]

The idea to generate this terms is to think symbolically of eq. (5.102) and write it as:

$$\hat{S} = \hat{S}'_\ell + \hat{\sigma}_\ell \tag{5.104}$$

and substituting into the product of S_k in the Hamiltonian will give terms such as :

$$S'S'S'S' + 4S'\sigma S'S' + 6\sigma\sigma S'S' + \cdots .$$

Once we have expanded V in this way we shall take the average $\langle \ldots \rangle_0$ of the expression for V and since this average is taken only over $\hat{\sigma_l}$ we are able to factor out the S'.

We remain with terms like

$$\frac{u_0}{4} \int_0^{\Lambda/\ell} đk_1 \int_0^{\Lambda/\ell} đk_2 \int_{\Lambda/\ell}^{\Lambda} đk_3 \int_{\Lambda/\ell}^{\Lambda} đk_4\, (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\, \hat{S'}_\ell(\mathbf{k}_1)\hat{S'}_\ell(\mathbf{k}_2)\langle \hat{\sigma}_\ell(\mathbf{k}_3)\hat{\sigma}_\ell(\mathbf{k}_4)\rangle_0.$$

where the quantity $\langle \hat{\sigma}_\ell(\mathbf{k}_3)\hat{\sigma}_\ell(\mathbf{k}_4)\rangle_0$ is just some function of k_3 and k_4 and is calculated by applying Wick's theorem.[1]

If we were to do the integrals over k_3 and k_4 we would be left with a term like,

$$\frac{u_0}{4}\int_0^{\Lambda/l} đk_1 \int_0^{\Lambda/l} đk_2 \hat{S'}_\ell(\mathbf{k}_1)\hat{S'}_\ell(\mathbf{k}_2) \times (\text{some function of } k_1,\, k_2) \tag{5.105}$$

following from our considered example term.

Rescaling the momentum and performing a wave function renormalization this term will take the form

$$\int_k u_{2,\ell}(k)|\hat{S}_\ell(k)|^2 \tag{5.106}$$

So we see that terms from this expansion end up making a contribution to the renormalised coupling constant $u_{2,l}$ in the renormalised Hamiltonian. Doing this for all the terms in $\langle V\rangle_0$ and $[\langle V^2\rangle_0 - \langle V\rangle_0^2]$ we can calculate the renormalised coupling constants in a systematic way.[1]

$\langle V\rangle_0$ can be expanded as a sum of terms such as $\langle V\rangle_0^{SSSS}$, $\langle V\rangle_0^{SS\sigma\sigma}$, and so on, where the superscripts denote the number of $\hat{S}$ and $\hat{\sigma}$ fields involved. Terms containing an odd number of $\hat{\sigma}$ fields vanish by Wick's theorem which simplifies the problem. Specifically, $\langle V\rangle_0^{SSSS}$ contributes to $u_{4,\ell}$, $\langle V\rangle_0^{SS\sigma\sigma}$ to $u_{2,\ell}$, and $\langle V\rangle_0^{\sigma\sigma\sigma\sigma}$ to the regular part of the free energy. We are interested in the renormalised coupling constants and as an example we calculate the contributions to $O(V^1)$ to $u_{4,l}$, which we denote by $u_{4,l}^{(1)}$.[1] Starting with

$$\langle V\rangle_0^{SSSS} = \frac{1}{4}u_0\int_0^{\Lambda/l} đk_1 \ldots đk_4 (2\pi)^d \delta(k_1 + \cdots + k_4)\hat{S}'_l(k_1)\ldots\hat{S}'_l(k_4) \tag{5.107}$$

and renormalising as before and noting that $\delta(k_l/l) = l^d\delta(k_l)$ we find:

$$\frac{1}{4}u_{4,\ell}^{(1)} = \frac{1}{4}u_0 z^4 \ell^{-3d} \tag{5.108}$$

A similar algorithm is used for calculating $u_{2,l}$ but additionally using Wick's theorem and collecting all the terms which contribute to it and noting that $z = l^{1+d/2}$ we obtain the renormalised coupling constants:

$$u_{2,\ell}^{(1)} = r_0\ell^2 + k_\ell^2 + 3u_0\ell^2\int_{\Lambda/\ell}^{\Lambda}\frac{đk}{(r_0+k^2)}, \tag{5.109}$$

$$u_{4,\ell}^{(1)} = u_0\ell^{4-d} = u_0\ell^\epsilon \tag{5.110}$$

This is a great next step but in order to find a fixed point which is not trivial we must calculate at least to order $O(V^2)$. The calculation are long so we employ the framework of Feynman diagrams to ease the process of doing perturbation theory.

Feynman diagrams are a useful schematic way of writing how different fields are paired up. They have a one-to-one correspondence between them and a term in the perturbation series. One uses the Feynman rules in order to draw all the appropiate diagrams fit to compute.[1]

As a demonstration, our example term

$$\langle V \rangle_0^{SSSS} = \frac{1}{4} u_0 \int_0^{\Lambda/l} \text{đ}k_1 \dots \text{đ}k_4 (2\pi)^d \delta(k_1 + \dots + k_4) \hat{S}'_l(k_1) \dots \hat{S}'_l(k_4) \tag{5.111}$$

can be diagrammatically represented as

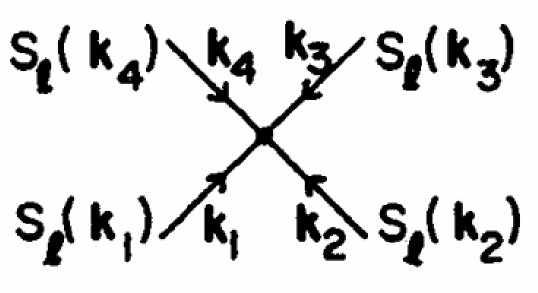


Figure 5.6: Feynman diagram of eq. (5.111)[1]

where each external leg is a S'_l field and k_i the momentum associated. The center vertex represents the factor:

$$\frac{u_0}{4} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \tag{5.112}$$

and in general each vertex is connected to four legs. Momentum is conserved and an integral is performed over each k_i in the external leg with $0 < |k_i| < \Lambda/l$. We should get back an algebraic expression after decoding the diagram.[1]

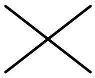
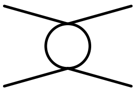
For example in this case we can write our result of the renormalised coupling constant for this term as:

$$u_{4,\ell}^{(1)} = z^4 \ell^{-3d} \left[\quad \times \quad \right] \tag{5.113}$$

In the order $O(V^1)$ a similar thing is done for $u_{2,l}$ and for $O(V^2)$ we have the same procedure but the value of the diagram is actually a product of two separate diagrams. The majority of the diagrams, because disconnected, do not contribute to $\langle e^V \rangle_0$ because of the linked cluster theorem.

With only the relevant Feynman diagrams remaining we can write the recursion relations as:

$u^{(2)}_{2,\ell} = z^2 \ell^{-d} \Big[(r_0 + k^2) + 12$  $\Big]$; (5.114)

$u^{(2)}_{4,\ell} = z^4 \ell^{-3d} \Big[$  $- 36$  $\Big]$. (5.115)

where the last diagram has the value of

$$\left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} đk_1 \cdots đk_4 (2\pi)^d \delta(k_1 + \cdots + k_4) \hat{S}'_\ell(k_1) \cdots \hat{S}'_\ell(k_4) \tag{5.116}$$

$$\times \int_{\Lambda/\ell}^{\Lambda} \frac{đk_5}{[r_0 + k_5^2][r_0 + (k_1 + k_2 - k_5)^2]}. \tag{5.117}$$

where the wave-vector dependence is not important for the $O(\epsilon)$ calculation.[1]

After momentum rescaling and wave-function renormalisation, the recursion relations become

$$r' = \ell^2 \left[r_0 + 3u_0 I_1 \right], \tag{5.118}$$

$$u' = u_0 \ell^{\epsilon} \left[1 - 9u_0 I_2 \right], \tag{5.119}$$

where

$$I_1 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{đk}{r_0 + k^2}, \tag{5.120}$$

$$I_2 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{đk}{(r_0 + k^2)^2}. \tag{5.121}$$

Now in order to analyze the recursion relations we evaluate I_1 and I_2 in $\epsilon = 4 - d$ dimensions, thus treating the dimensionality as a continuous variable so we can expand the integrals a a power series about four dimensions ($\epsilon = 0$). Such an expansion is divergent but Borel summable.[1] We expand thus

$$I_{1,2}(\epsilon) = I_{1,2}(0) + \epsilon I'_{1,2}(0) + O(\epsilon^2) \tag{5.122}$$

We only need the lowest order results in ϵ. We anticipate the fixed point value to be to $O(\epsilon)$ so we can expand the integrand in powers of r_0 also and after that and writing

$$\ell^{\epsilon} = e^{\epsilon \log \ell} = 1 + \epsilon \log \ell + \frac{1}{2!}\epsilon^2 (\log \ell)^2 + O(\epsilon^3), \tag{5.123}$$

we finally get the recursion relations for suitable use:

$$r' = l^2 \left[r_0 + \frac{3}{4\pi^2} \frac{u_0}{4} \Lambda^2 \left(1 - \frac{1}{l^2}\right) - \frac{3}{2\pi^2} \frac{u_0}{4} r_0 \log l \right]; \tag{5.124}$$

$$\frac{u'}{4} = \frac{u_0}{4} + \frac{u_0}{4} \left[\epsilon - \frac{9}{2\pi^2} \frac{u_0}{4} \right] \log l. \tag{5.125}$$

and as the fixed points we find the solution:

$$r^* = u^* = 0 \tag{5.126}$$

which corresponds to the Gaussian fixed point and the non-trivial fixed point:

$$\frac{u^*}{4} = \frac{2\pi^2}{9}\epsilon \tag{5.127}$$

and

$$r^* = -\frac{\epsilon \Lambda^2}{6}. \tag{5.128}$$

This point is called the Wilson-Fisher fixed point. Linearising the RG at this fixed points we get:[1]

$$\Lambda_t = \ell^{(2-\epsilon/3)} = \ell^{2(1-\epsilon/6)} \Rightarrow y_t = 2(1 - \epsilon/6).$$
$$\Lambda_2 = \ell^{-\epsilon} \Rightarrow y_2 = -\epsilon.$$

And after finding the right eigenvectors we can build the flow diagram:

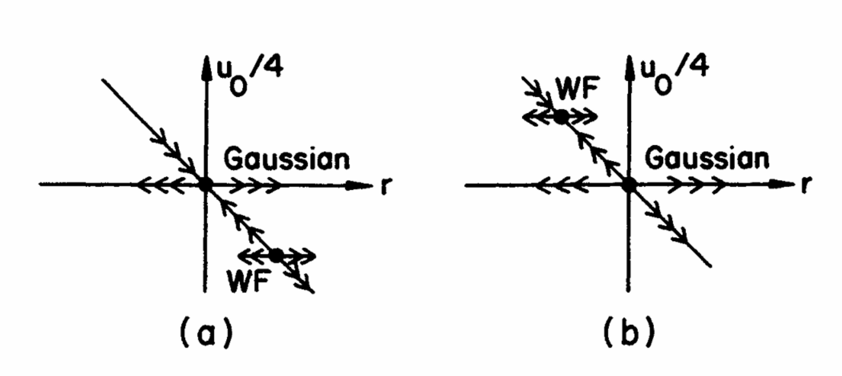


Figure 5.7: RG flows near 4d for $d > 4$ and $d < 4$ specifically[1]

For $d > 4$, the WF fixed point has a negative value which is unphysical but for $d < 4$ the point is in a physical region of the parameter space and controls the critical behavior.

The critical exponents are calculated using the scaling laws as before:[1]

$$\begin{aligned}
\gamma &= 1 + \epsilon/6 \\
\eta &= 0 \\
\alpha &= \epsilon/6 \\
\beta &= \frac{1}{2} - \frac{\epsilon}{6} \\
\delta &= 3 + \epsilon.
\end{aligned}$$

and setting $\epsilon = 1$ we get a good approximation for the critical exponents as calculated in the framework of the Renormalization Group and summarized in the following table.

Table 5.1: CRITICAL EXPONENTS FOR THE ISING UNIVERSALITY CLASS IN THREE DIMENSIONS[1]

Exponent	**ϵ-expansion to $O(\epsilon)$**	**Mean Field**	**Experiment**	**Ising ($d = 3$)**
α	0.167	0 (disc.)	0.110 – 0.116	0.110(5)
β	0.333	1/2	0.316 – 0.327	0.325±0.0015
γ	1.167	1	1.23 – 1.25	1.2405±0.0015
δ	4.0	3	4.6 – 4.9	4.82(4)
ν	0.583	1/2	0.625±0.010	0.630(2)
η	0	0	0.016 – 0.06	0.032±0.003

6 Conclusions and discussion

In conclusion, this internship project has explored the fundamental ideas, applications, and modern relevance of the Renormalization Group (RG) theory, one of the most profound conceptual advances in theoretical physics of the twentieth century. We began by introducing the historical context and motivation behind the development of RG theory, emphasizing its dual importance in both particle physics and statistical mechanics. Originally conceived to understand the behavior of systems across vastly different length scales, the RG formalism provides a unified framework for describing how physical laws and parameters evolve with scale transformations.

Within the scope of this report, particular attention was given to the application of the RG in statistical mechanics, especially in the theory of phase transitions and critical phenomena. These phenomena represent points where a physical system undergoes abrupt changes in macroscopic properties—such as magnetization or density—arising from microscopic interactions. The study of these transitions has historically introduced deep and conceptually rich ideas, among them the notion of universality, which states that systems with completely different microscopic details can exhibit identical critical behavior. This remarkable property underscores the power of the RG approach: it reveals that what matters near criticality is not the fine structure of microscopic interactions, but rather the symmetries and dimensionality of the system.

To understand under what conditions a phase transition can occur and to analyze the structure of the associated phase diagram, we focused on the Ising model, one of the most paradigmatic models in statistical physics. This model served as a concrete example to illustrate the connection between macroscopic observables and microscopic configurations. Through the analysis of the Ising model, we explored both the mean-field approximation and the Landau theory of phase transitions, computing the corresponding critical exponents. Although mean-field theory provides a useful starting point, it yields critical exponents that deviate from experimental results, revealing the necessity of a more refined theoretical framework.

This shortcoming motivated the introduction of the Renormalization Group itself. Building on Kadanoff's insight that physical systems can be understood by coarse-graining, grouping microscopic degrees of freedom into effective "blocks" that capture long-range behavior—Wilson formalized these ideas into a systematic computational scheme. Wilson's formulation of the RG not only provided a method to calculate critical exponents with remarkable accuracy, but also fundamentally changed our understanding of how physical theories behave under changes of scale. Through the RG lens, the structure of physical laws acquires a hierarchical organization, where the effective behavior at large scales emerges naturally from the microscopic dynamics.

Beyond its historical and conceptual achievements, the RG framework has continued to evolve and find applications in diverse fields of contemporary research. In recent years, it has found new resonance in machine learning, particularly in the context of deep learning architectures. The hierarchical feature extraction in neural networks bears a strong analogy to the coarse-graining process in RG: both involve identifying relevant information at different levels of resolution while discarding irrelevant details. This conceptual parallel has inspired attempts to formalize connections between RG transformations and the structure of deep neural networks, opening the door to a cross-disciplinary exchange of ideas between statistical physics and data science.

Furthermore, modern developments such as the Functional Renormalization Group (FRG) have extended the RG formalism into a powerful and flexible tool applicable beyond perturbation theory. The FRG provides a framework for studying systems with strong correlations, quantum field theories, and even complex systems outside traditional physics domains. Its versatility has led to active research in condensed matter, cosmology, and quantum gravity, where it offers a consistent way to interpolate between microscopic and macroscopic scales.

Overall, this internship has provided an opportunity to appreciate not only the technical structure of the Renormalization Group but also its broader conceptual significance. The RG represents more than a computational tool—it is a way of thinking about the organization of nature across scales. From explaining the universality of critical phenomena to inspiring new methodologies in modern data analysis, the RG remains a cornerstone of theoretical physics and a vibrant area of ongoing research. Its continued development, through both theoretical innovations like the Functional RG and interdisciplinary bridges with fields such as machine learning, ensures that it will remain a guiding framework for understanding complex systems.

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