FUNDAMENTALS OF HOMOTOPY THEORY

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Contents

1. The first homotopy group and covering space theory

1.1. Homotopy between two continuous maps.

Definition 1.1. Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. A **homotopy** from f to g is a continuous map

$$H : X \times [0,1] \to Y,\ (x,t) \mapsto H(x,t) = H_t(x)$$

such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for $x \in X$, i.e. $f = H_0$ and $g = H_1$.

In this case we write $H : f \simeq g$ and we say that f is **deformed** continuously into g. If we imagine the parameter t as representing time H represents a continuous "deforming" of f into g.

We call f and g **homotopic** if there exists a homotopy from f to g.

In what follows we shall use for convenience $I = [0,1]$.

If $f \simeq f'$ and f' is a constant map, we say that f is **nullhomotopic**.

The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$.

Proof. Reflexivity: Let $H(x,t) = f(x)$ be the constant homotopy. Then $f \simeq f$.

Symmetry: Let $H : f \simeq g$. Define the inverse of H as $H^- : (x,t) \mapsto H(x, 1-t)$. Since H is continuous, H^- is continuous. We have that $H^-(x,0) = H(x,1) = g(x)$ and $H^-(x,1) = H(x,0) = f(x)$, so H^- is a homotopy from g to f, which implies $g \simeq f$.

Transitivity: Let $K : f \simeq g,\ L : g \simeq h$ be given. The product homotopy $K * L$ is defined by

$$(K * L)(x,t) = \begin{cases} K(x,2t), & 0 \le t \le \frac{1}{2} \\ L(x, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

We have $(K * L)(x,0) = f(x)$ and $(K * L)(x,1) = h(x)$, so $K * L$ is a homotopy from f to $h \Rightarrow f \simeq h$.

□

The equivalence class of f under the homotopy relation is denoted by $[f]$ and is called the **homotopy class** of f. We denote by $[X, Y]$ the set of homotopy classes $[f]$ of maps $f : X \to Y$.

A homotopy is said to be **relative** to $A \subset X$ if the restriction $H_t|A$ does not depend on t (is constant on A).

Proposition 1.2. *If $f, g : X \to Y,\ h : X' \to X,\ k : Y \to Y'$ are continuous maps, then $f \simeq g \Rightarrow f \circ h \simeq g \circ h$ and $k \circ f \simeq k \circ g$.*

Proof. $f \simeq g \Rightarrow \exists H : X \times I \to Y$ continuous, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Define $H' : X' \times I \to Y$, by $H'(x',t) = H(h(x'),t)$. Then $H'(x',0) = H(h(x'),0) = f \circ h(x')$. Similarly $H'(x',1) = g \circ h(x')$. Then H' is a homotopy from $f \circ h$ to $g \circ h \Rightarrow f \circ h \simeq g \circ h$.

Next we define $H" : X \times I \to Y'$ by $H"(x,t) = k(H(x,t))$. We obtain $H"(x,0) = k \circ f(x)$ and $H"(x,1) = k \circ g(x)$, so $k \circ f \simeq k \circ g$, since $H"$ is a homotopy between these two maps. □

Definition 1.3. Let $f, g : X \to A,\ A \subset \mathbb{R}^n$. Suppose that the line segment from $f(x)$ to $g(x)$ is always contained in A. Then $H(x,t) = (1-t)f(x) + tg(x)$ is a **linear (or straight-line)** homotopy from f to g.

We now turn our attention to the case of paths.

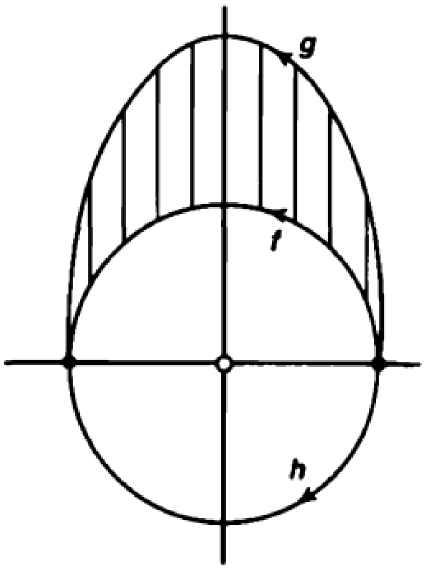


FIGURE 1. Example of path homotopy

Definition 1.4. Two paths f and f' mapping I into X are said to be **path homotopic** if they have the same initial point x_0 and final point x_1, and if there exists a continuous map $F : I \times I \to X$ such that

$$F(s,0) = f(s) \text{ and } F(s,1) = f'(s)$$

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1$$

for any $s, t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

Example 1.5. Let X denote the punctured plane $\mathbb{R}^2 - \{0\}$.The following paths in X are path homotopic:

$$f(s) = (cos\pi s, sin\pi s)$$

$$g(s) = (cos\pi s, 2sin\pi s)$$

Indeed, as shown in Figure 1 it is clear that the straight-line homotopy between them is an acceptable path homotopy. However, the straight-line homotopy between f and the path $h(s) = (cos\pi s, -sin\pi s)$ is not acceptable, for its image does not lie in the space $\mathbb{R}^2 - \{0\}$. Intuitively, one cannot deform f continuously into h without passing through 0, so f and h are not path homotopic.

Just like in the case of regular homotopy, one can prove that the relation $\simeq_p$ of homotopy between maps is an equivalence relation.

Definition 1.6. If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the **product** $f * g$ of f and g to be the path h given by

$$h(s) = \begin{cases} f(2s) \text{ for } s \in [0, \frac{1}{2}] \\ g(2s-1) \text{ for } s \in [\frac{1}{2}, 1]. \end{cases} \quad (1)$$

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by

$$[f] * [g] = [f * g]. \quad (2)$$

Proof. Let F be a path homotopy between f and f' and let G be a path homotopy between g and g'. Define

$$H(s,t) = \begin{cases} F(2s,t) \text{ for } s \in [0, \frac{1}{2}] \\ G(2s-1,t) \text{ for } s \in [\frac{1}{2}, 1] \end{cases}$$

$F(1,t) = x_1 + G(0,t)$ for any $t \Rightarrow H$ is well-defined. H is also continuous by the pasting lemma.

We leave it to the reader to verify that H is the required path homotopy between $f * g$ and $f' * g'$. □

The operation $*$ has the following properties, which we state without proof:
1.(Associativity) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$, if they are defined.
2.(Right and left identities) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \text{ and } [e_{x_0}] * [f] = [f].$$

3. (Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

We also have the following:

Theorem 1.7. *Let f be a path in X, and let $a_0, ..., a_n$ be numbers such that $0 = a_0 < a_1 < ... < a_n = 1$. Let $f_i : I \to X$ be a path that equals the positive linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then*

$$[f] = [f_1] * ... * [f_n].$$

1.2. **The fundamental group of a topological space.**

Definition 1.8. Let X be a topological space and x_0 a point in X. A path in X that begins and ends at x_0 is called a **loop** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the **fundamental group (or first homotopy group)** of X relative to the **base point** x_0. It is denoted by $\pi_1(X, x_0)$.

From the properties of the operation $*$ given in section 1.1, we see that when this operation is restricted in this way, it satisfies the axioms of the group. Before, it could not be an operation for a group since it was not defined for any two pairs of elements of X. Now, for any two loops f and g based at x_0, the product $f * g$ is always defined and is also a loop based at x_0.

Definition 1.9. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \text{ by}$$

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha].$$

We have

$$\bar{\alpha} * (f * \alpha) = \begin{cases} \bar{\alpha}(2s) \text{ for } s \in [0, \frac{1}{2}] \\ f(4s-2) \text{ for } s \in [\frac{1}{2}, \frac{3}{4}] \\ \alpha(4s-3) \text{ for } s \in [\frac{3}{4}, 1] \end{cases}$$

Since α is a path from x_0 to x_1 and $\bar{\alpha}$ is a path from x_1 to x_0 we obtain that $\bar{\alpha} * (f * \alpha)$ is a loop based at x_1. This means that $\hat{\alpha}$ maps $\pi_1(X, x_0)$ into $\pi_1(X, x_1)$.

Theorem 1.10. *The map $\hat{\alpha}$ is a group isomorphism.*

Proof. We have that

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] = [\bar{\alpha}] * [f * g] * [\alpha] = \hat{\alpha}([f] * [g]),$$

hence $\hat{\alpha}$ is a homomorphism. Next we show that $\hat{\alpha}$ is invertible. Let $\beta = \bar{\alpha}$; we show that $\hat{\beta}$ is an inverse for $\hat{\alpha}$: for any $[h] \in \pi_1(X, x_1)$,

$$\hat{\beta}([h]) = [\bar{\beta}] * [h] * [\beta] = [\alpha] * [h] * [\bar{\alpha}]$$

$$\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\bar{\alpha}] = [h].$$

Also, for any $[f] \in \pi_1(X, x_0)$

$$\hat{\beta}(\hat{\alpha}([f])) = [\bar{\beta}] * ([\bar{\alpha}] * [f] * [\alpha]) * [\beta] = [f].$$

□

Corollary 1.11. *If X is path-connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Proof. Since X is path connected, there exists a path α from x_0 to x_1. As before, we can define $\hat{\alpha}$, which is an isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$. □

Definition 1.12. A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one element) group for some $x_0 \in X$, and hence for every $x \in X$.

We express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma 1.13. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Proof. Let α and β be two paths in X from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, the loop is path homotopic to the constant loop at x_0. Then

$$[\alpha] = [\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta].$$

□

Proposition 1.14. *$\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, if X and Y are path-connected.*

Proof. A map $f : Z \to X \times Y$ is continuous iff the maps $g : Z \to X$ and $h : Z \to Y$ defined by $f(z) = (g(z), h(z))$ are both continuous. Hence a loop f in $X \times Y$ based at (x_0, y_0) can be thought of as a pair of loops g in X and h in Y based at x_0 and y_0 respectively. Similarly, a homotopy f_t of a loop in $X \times Y$ is equivalent to a pair of homotopies g_t and h_t of the corresponding loops in X and Y. Thus we obtain a bijection $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by $[f] \mapsto ([g], [h])$. We have $[f] * [f'] \mapsto ([g] * [g'], [h] * [h']) = ([g], [h]) * ([g'], [h'])$, so it is a homomorphism. □

If $h : X \to Y$ is a continuous map that carries $x_0 \in X$ to $y_0 \in Y$, we write $h : (X, x_0) \to (Y, y_0)$. If f is a loop in X based at x_0, then $h \circ f : I \to Y$ is a loop in Y based at y_0. The correspondence $f \to h \circ f$ gives rise to a map carrying $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$.

Definition 1.15. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. The **homomorphism induced by** h, relative to the base point x_0 is the map $(h_{x_0})_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $(h_{x_0})_*([f]) = [h \circ f]$.

In cases where no confusion may arise in regard to the base point we will denote $(h_{x_0})_*$ by h_*.

If F is a path homotopy between the paths f and f', then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$, so h_* is well-defined.

Since $(h \circ f) * (h \circ g) = h \circ (f * g)$, we have that

$$h_*([f] * [g]) = [h \circ (f * g)] = [(h \circ f) * (h \circ g)] = [h \circ f] * [h \circ g] = h_*([f]) * h_*([g]). \quad (3)$$

This proves that h_* is a homomorphism.

Theorem 1.16. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof.

$$\begin{aligned}(k \circ h)_*([f]) &= [(k \circ h) \circ f] = [k \circ (h \circ f)] \\ &= k_*([h \circ f]) = k_*(h_*([f])) = k_* \circ h_*([f]).\end{aligned} \quad (4)$$

$$i_*([f]) = [i \circ f] = [f]. \quad (5)$$

□

Corollary 1.17. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0). Also $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphims of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, then k_* is the inverse of h_*. □

1.3. **Covering spaces.**

Definition 1.18. Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriciton of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into **slices**.

Definition 1.19. Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a **covering map**, and E is said to be a **covering space** of B.

Here, by neighborhood of a point x_0 in a topological space X it is meant an open set of X that contains x_0.

Example 1.20. Let X be any topological space and $i : X \to X$ be the identity map. Then i is a covering map.

In order to see this, let x_0 be a point of X, and U a neighborhood of x_0 in X. Since $i^{-1}(U) = U$, we have that U is evenly covered by i, and since x_0 and U were arbitrary, we have that every point of X has a neighborhood (and, in fact, all of its neighborhoods) that is evenly covered by i, so i is a covering map for any topological space.

Let $p : E \to B$ be a covering map. Then p has the following properties:

(1) for each $b \in B$, the subspace $p^{-1}(b)$ of E has the discrete topology
(2) p is an open map
(3) p is a local homeomorphism. That is, each point e of E has a neighborhood that is mapped homeomorphically by p onto an open subset of B

Proof. (1) Let U be a neighborhood of b that is evenly covered by p. Then $p^{-1}(U)$ can be written as the union of disjoint sets V_α that are open in E. Since V_α is open in E we have that, by the subspace topology of $p^{-1}(b)$, $p^{-1}(b) \cap V_\alpha$ is open in $p^{-1}(b)$. But this intersection contains only one point, and any point of $p^{-1}(b)$ can be written as one such intersection. Therefore we have that all the one-element subsets of $p^{-1}(b)$ are open, so all their unions are open. This means that $p^{-1}(b)$ has the discrete topology.

(2) Let A be an open set of E and $x \in p(A)$. We choose a neighborhood U of x that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Let V_β be the slice containing the point y such that $p(y) = x$. The set $V_\beta \cap A$ is open in V_β as a subspace of E. Since p maps V_β homeomorphically onto U, and homeomorphisms are open maps, then $p(V_\beta \cap A)$ is open in U, and hence open in B, so it is a neighborhood of x contained in $p(A)$. Since x was an arbitrary point in $p(A)$, we have that $p(A)$ can be written as an union of open sets in B, so it is open. Then p is an open map.

(3) Let $e \in E$ and $x = p(e) \in B$. Let U be a neighborhood of x that is evenly covered by p and $\{V_\alpha\}$ a partition of $p^{-1}(U)$ into slices. Since $p(e) = x \in U$, we have that $e \in p^{-1}(U) = \cup_\alpha V_\alpha$. Then there exists α such that $e \in V_\alpha$. Since p is a covering map, it maps homeomorphically V_α onto U, so p is a local homeomorphism. □

Theorem 1.21. *The map $p : \mathbb{R} \to S^1$ given by*

$$p(x) = (cos2\pi x, sin2\pi x) \tag{6}$$

is a covering map.

Proof. Let U be the subset of S^1 of the points with positive first coordinate. Then

$$p^{-1}(U) = \cup_{n\in\mathbb{Z}} V_n \tag{7}$$

where $V_n = (n - \frac{1}{4}, n + \frac{1}{4})$. On the closed intervals $\bar{V}_n$, $sin2\pi x$ is strictly monotonic so p is injective. Also, p carries $\bar{V}_n$ onto $\bar{U}$ and V_n to U. Since $p|\bar{V}_n$ is bijective and $\bar{V}_n$ is compact we have that $p|\bar{V}_n$ is a homeomorphism of $\bar{V}_n$ with $\bar{U}$ and, in particular, $p|V_n$ is a homeomorphism of V_n with U. Proceeding similarly for the regions of S^1 with negative first coordinate and with positive and negative second coordinate we obtain open sets that cover S^1 and are evenly covered by p. This proves that p is a covering map. □

Theorem 1.22. *Let $p : E \to B$ be a covering map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.*

Proof. Let $b_0 \in B_0$ and U a neighborhood of b_0 in B that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. We have that $U \cap B_0$ is a neighborhood of b_0 in B_0 and $V_\alpha \cap E_0$ are disjoint open sets in E_0. Also, $\cup_\alpha(V_\alpha \cap E_0) = p^{-1}(U \cap B_0)$ so $U \cap B_0$ is evenly covered by p and any point b_0 of B_0 has a neighborhood in B_0 which is evenly covered by p. Since p maps V_α homeomorphically onto U, its restriction to $V_\alpha \cap E_0$ is a homeomorphism onto $U \cap B_0$. Then p is a covering map. □

Theorem 1.23. *Let $p : E \to B$ and $p' : E' \to B'$ be covering maps. Then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

Proof. Let $b \in B$, $b' \in B'$ and U and U' neighborhoods of b and b' that are evenly covered by p and p' respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $p'^{-1}(U')$ into slices. Then $(p \times p')^{-1}(U \times U') = \cup_{\alpha,\beta}(V_\alpha \times V'_\beta)$, which is a union of disjoint open sets of $E \times E'$. Since $V_\alpha \times V'_\beta$ is homeomorphically mapped into $U \times U'$ by $p \times p'$, $p \times p'$ is a covering map. □

1.4. **The fundamental group of the circle.**

Definition 1.24. Let $p : E \to B$ be a map. If f is a continuous mapping of some space X into B, a **lifting** of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

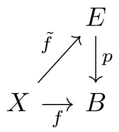


FIGURE 2. Correspondence between f, p, and $\tilde{f}$

Lemma 1.25. *Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. Then for any path $f : I \to B$ beginning at b_0, f has a unique lifting to a path $\tilde{f}$ in E beginning at e_0.*

Proof. We cover B by open sets U that are evenly covered by p. Since f is continuous and $[0,1]$ is compact, $f([0,1])$ is compact and we can find a division $s_0, ..., s_n$ of I such that $f([s_i, s_{i+1}])$ is in an open set U for any i and we consider one such case. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices.

Define $\tilde{f}(0) = e_0$. Since $\tilde{f}(s_i)$ lies in some V_α consider it lies in V_0.

Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by

$$\tilde{f}(s) = (p|V_0)^{-1}(f(s)). \tag{8}$$

Since the restriction of p to V_0 is a homeomorphism from V_0 onto U and f is continuous, we have that $\tilde{f}$ is continuous on $[s_i, s_{i+1}]$. Similarly we define $\tilde{f}$ on all of I. By the pasting lemma, $\tilde{f}$ is continuous. One can see from the definition that $p \circ \tilde{f} = f$.

In order to prove uniqueness, consider another lifting of f beginning at e_0, $\tilde{\tilde{f}}$. Then $\tilde{\tilde{f}}(0) = \tilde{f}(0) = e_0$. Suppose that $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for all s such that $0 \leq s \leq s_i$. As before, we define $\tilde{f}(s) = (p|V_0)^{-1}(f(s))$. Since $\tilde{\tilde{f}}$ is a lifting of f, it must carry $[s_i, s_{i+1}]$ into $p^{-1}(U) = \cup_\alpha V_\alpha$. $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is connected so it must lie entirely in one of V_α. But $\tilde{\tilde{f}}(s_i)$ is in V_0, so $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is also in V_0. So, for s in $[s_i, s_{i+1}]$, $\tilde{\tilde{f}}(s)$ must equal some point y of V_0 lying in $p^{-1}(f(s))$. But, since the restriction of p to V_0 is a homeomorphism, there is only one such point y, namely, $(p|V_0)^{-1}(f(s))$. Hence $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$. This can be done for all intervals $[s_i, s_{i+1}]$, so $\tilde{f}$ is unique. □

Lemma 1.26. *Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. Let the map $F : I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map*

$$\tilde{F} : I \times I \to E \tag{9}$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof. Define $\tilde{F}(0,0) = e_0$. By Lemma 1.25 we can extend $\tilde{F}$ uniquely to $0 \times I$ and to $I \times 0$.

Now, by the Lebesgue number lemma, we choose subdivisions

$$s_0 < s_1 < ... < s_m$$

$$t_0 < t_1 < ... < t_n$$

of I fine enough that each rectangle

$$I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \tag{10}$$

is mapped by F into an open set B that is evenly covered by p.

Given i_0 and j_0 we assume that $\tilde{F}$ is defined on the set A which is the union of $0 \times I$, $I \times 0$ and all the rectangles $I_i \times J_j$ such that either $j < j_0$ or $j = j_0$ and $i < i_0$. We also assume that $\tilde{F}$ is a continuous lifting of $F|A$. In order to define $\tilde{F}$ on $I_{i_0} \times J_{j_0}$ we choose an open set U of B that is evenly covered by p and contains the set $F(I_{i_0} \times J_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Since $\tilde{F}$ is defined on A, it is also defined on $C = A \cap (I_{i_0} \times J_{j_0})$. This set consists of the left and bottom edges of $I_{i_0} \times J_{j_0}$ so it is connected. This means that $\tilde{F}(C)$ is connected, so it is entirely contained in one of V_α, say V_0.

Now let $p_0 : V_0 \to U$ be the restriction of p to V_0. Since $\tilde{F}$ is a lifting of $F|A$, we have that for any $x \in C$

$$p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x) \tag{11}$$

so $\tilde{F}(x) = p_0^{-1}(F(x))$. Therefore, we may extend $\tilde{F}$ to $I_{i_0} \times J_{j_0}$ by defining $\tilde{F}(x) = p_0^{-1}(F(x))$ for x in this domain. Similarly, we can extend $\tilde{F}$ on $I \times I$. By the pasting lemma $\tilde{F}$ will be continuous.

For uniqueness, note that our construction allows the use of the previous lemma to prove that once the value of $\tilde{F}(0,0)$ is specified, $\tilde{F}$ is completely determined.

Now suppose that F is a path homotopy. Then $F(0,t) = F(0,0) = b_0$ for any $t \in I$, so F carries $0 \times I$ into b_0. Since $\tilde{F}$ is a lifting of F, we have $\tilde{F}(0 \times I) = p^{-1}(b_0)$. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, the set $\tilde{F}(0 \times I)$ is connected, so it must equal a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one-point set, so $\tilde{F}$ is a path homotopy. □

Theorem 1.27. *Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let f and g be two paths in B from b_0 to b_1 and let $\tilde{f}$ and $\tilde{g}$ be their respective liftings to paths in E beginning at e_0. If f and g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ are also path homotopic and a homotopy between them is given by the lifting of the homotopy between f and g.*

Proof. Let $F : I \times I \to B$ be a path homotopy between f and g. Then $F(0,0) = b_0$. Let $\tilde{F} : I \times I \to E$ be the lifting of F to E such that $\tilde{F}(0,0) = e_0$. By Lemma 1.26, $\tilde{F}$ is a path homotopy, so $\tilde{F}(0 \times I) = \{e_0\}$ and $\tilde{F}(1 \times I) = \{e_1\}$. The restriciton $\tilde{F}|I \times 0$ is a path on E beginning at e_0 that is a lifting of $F|I \times 0$. But $F(s,0) = f(s)$ and its lifting is unique, so $\tilde{F}(s,0) = \tilde{f}(s)$. Similarly, $\tilde{F}|I \times 1 = \tilde{g}$. This proves that $\tilde{F}$ is a homotopy between $\tilde{f}$ and $\tilde{g}$. □

Definition 1.28. Let $p : E \to B$ be a covering map, and let $b_0 \in B$. Choose e_0 such that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that begins at e_0. Let $\phi([f])$ denote the end point $\tilde{f}(1)$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0). \tag{12}$$

ϕ is called the **lifting correspondence** derived from the covering map p.

Theorem 1.29. *Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0). \tag{13}$$

is surjective. If E is simply connected, it is bijective.

Proof. Suppose E is path connected. Then for any $e_1 \in p^{-1}(b_0)$, there exists a path $\tilde{f}$ in E from e_0 to e_1. Since $p(e_1) = p(e_0) = b_0$, the path $f = p \circ \tilde{f}$ is a loop in B based at b_0, so $[f] \in \pi_1(B, b_0)$ and $\phi([f]) = e_1$. Since this holds for any $e_1 \in p^{-1}(b_0)$, ϕ is surjective.

Now suppose E is simply connected. Let $[f]$ and $[g]$ be elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be liftings of f and g to paths in E that begin at e_0. Then $\tilde{f}$ and $\tilde{g}$ have the same initial and final points. Since E is simply connected, there exists a homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g. So $[f] = [g]$, which means that ϕ is also injective. □

Theorem 1.30. *The fundamental group of S^1 is isomorphic to the additive group of integers.*

Proof. Let $p : \mathbb{R} \to S^1$ be the covering map (6), $e_0 = 0$, and $b_0 = p(e_0)$. Then $p^{-1}(b_0) = \mathbb{Z}$, the set of integers. Since $\mathbb{R}$ is simply connected, by Theorem 1.29, the lifting correspondence

$$\phi : \pi_1(S^1, b_0) \to \mathbb{Z} \tag{14}$$

is bijective. We must show that ϕ is a homomorphism.

Let $[f]$ and $[g]$ be elements of $\pi_1(S^1, b_0)$ and let $\tilde{f}$ and $\tilde{g}$ be liftings to paths on $\mathbb{R}$ beginning at 0. Since ϕ takes values in $\mathbb{Z}$, we may choose $\phi([f]) = \tilde{f}(1) = n$ and $\phi([g]) = \tilde{g}(1) = m$. Let $\tilde{\tilde{g}}$ be the path

$$\tilde{\tilde{g}}(s) = n + \tilde{g}(s) \tag{15}$$

on $\mathbb{R}$. Then we have

$$p(\tilde{\tilde{g}}(s)) = p(n + \tilde{g}(s)) = p(\tilde{g}(s)) = g(s), \tag{16}$$

so $\tilde{\tilde{g}}(s)$ is a lifting of g with n as an initial point. Then we can define $\tilde{f} * \tilde{\tilde{g}}$ which is the lifting of $f * g$ that begins at 0. This can be seen from the following:

$$p(\tilde{f} * \tilde{\tilde{g}}) = \begin{cases} p(\tilde{f}(2s)),\ s \in [0, \frac{1}{2}] \\ p(\tilde{\tilde{g}}(2s-1)),\ s \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} f(s),\ s \in [0, \frac{1}{2}] \\ g(s),\ s \in [\frac{1}{2}, 1] \end{cases} = f * g \tag{17}$$

The end point of $\tilde{f} * \tilde{\tilde{g}}$ is $\tilde{\tilde{g}}(1) = n + m$. Then

$$\phi([f] * [g]) = \phi([f * g]) = n + m = \phi([f]) + \phi([g]) \tag{18}$$

which proves that ϕ is a homomorphism. □

1.5. **Retractions and fixed points.**

Definition 1.31. If $A \subset X$, a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists, we say that A is a **retract** of X.

From this definition we can see that, for a retraction r, we have $r^2 = r$. This suggests a way of thinking of retractions as the topological analogs of projection operators in other parts of mathematics.

Lemma 1.32. *If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.*

Proof. Let $r : X \to A$ be a retraction of X onto A, then the map $r \circ j : A \to A$ is the identity map of A. From Theorem 1.16, we have that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$ for any $a \in A$, so it is a bijection. This means that j_* must be injective. □

Theorem 1.33. ***(No-retraction theorem)*** *There is no retraction of B^2 onto S^1.*

Proof. We prove that the homomorphism induced by the inclusion $j : S^1 \to B^2$ is not injective. It is easy to see that the fundamental group of B^2 is trivial since all the loops based at the same point in this space are path-homotopic. However, as we saw in the preceding section, the fundamental group of S^1 is isomorphic to the additive group of integers which is not trivial. Therefore, there is no injective mapping between the fundamental group of S^1 and that of B^2, so j_* is not injective. This means that S^1 is not a retract of B^2. □

Lemma 1.34. *Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent:*

- *h is nullhomotopic*
- *h extends to a continuous map $k : B^2 \to X$*
- *h_* is the trivial homomorphism of fundamental groups.*

Proof. (1)⇒(2). Let $H : S^1 \times I \to X$ be a homotopy between h and a constant map. Let $\pi : S^1 \times I \to B^2$ be the map

$$\pi(x, t) = (1 - t)x. \tag{19}$$

Here, we think of x in terms of its coordinates in a plane, for example $(cos\theta, sin\theta)$. Then π is continuous, closed and surjective, so it is a quotient map. We have $\pi(S^1 \times 1) = \{0\}$, and $\pi|(S^1 \times [0, 1))$ is injective. Since H is constant on $S^1 \times 1$, it induces a map $k : B^2 \to X$ such that $k \circ \pi = H$. Since H is continuous, k is also continuous. In order to restrict k to S^1, we must restrict $k \circ \pi$ to $S^1 \times 0$. Then we have

$$k|S^1 = k \circ \pi|(S^1 \times 0) = H|(S^1 \times 0) = h. \tag{20}$$

This proves that k is an extension of h.

(2)⇒(3). If $j : S^1 \to B^2$ is the inclusion map, then $h = k \circ j$. Hence $h_* = k_* \circ j_*$. But the fundamental group of B^2 is trivial, so

$$j_* : \pi_1(S^1, b_0) \to \pi_1(B^2, b_0) \tag{21}$$

is trivial. Therefore, h_* is also trivial.

(3)⇒ (1). Let $p : \mathbb{R} \to S^1$ be the covering map (6), and let $p_0 : I \to S^1$ be its restriction to I. Then p_0 is a loop in S^1. Its lift must satisfy $p \circ \tilde{p}_0 = (cos2\pi x, sin2\pi x)$, for any $x \in I$, so $\bar{p}_0$ begins at 0 and ends at 1. This means that $[p_0]$ generates $\pi_1(S^1, b_0)$.

Let $x_0 = h(b_0)$. Because h_* is trivial, the loop $f = h \circ p_0$ represents the identity element of $\pi_1(X, x_0)$. Therefore, there is a path homotopy F in X between f and the constant path at x_0. The map $p_0 \times id : I \times I \to S^1 \times I$ is a quotient map, being continuous, closed, and surjective. It maps $0 \times t$ and $1 \times t$ to $b_0 \times t$ for each t, but is otherwise injective. The path homotopy F maps $0 \times I$, $1 \times I$ and $I \times 1$ to the point x_0 of X, so it induces a continuous map $H : S^1 \times I \to X$ that is a homotopy between h and a constant map. □

Corollary 1.35. *The inclusion map $j : S^1 \to \mathbb{R}^2 - (0, 0)$ is not nullhomotopic. The identity map $i : S^1 \to S^1$ is not nullhomotopic.*

Proof. $r(x) = x/\|x\|$ is a retraction of $\mathbb{R} - (0, 0)$ onto S^1. Then, by Lemma 1.32, j_* is injective, so it is not trivial. Therefore, it is not nullhomotopic. i_* is the identity homomorphism, so it is also nontrivial. □

1.6. **Deformation retracts and homotopy type.**

Lemma 1.36. *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and if the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then the homomorphisms h_* and k_* are equal.*

Proof. Let $H : X \times I \to Y$ be the homotopy between h and k such that $H(x_0, t) = y_0$ for all t. Let f be a loop in X based at x_0. Then $H \circ (f \times id) : I \times I \to Y$ is a homotopy between $h \circ f$ and $k \circ f$. To see this we compute

$$H \circ (f \times id)(x, 0) = H(f(x), 0) = h(f(x)) = h \circ f(x). \tag{22}$$

Similarly, one obtains $H \circ (f \times id)(x, 1) = k \circ f(x)$. Now, $h \circ f$ and $k \circ f$ are both loops based at y_0, so they are path homotopic. Then we have $[h \circ f] = [k \circ f]$, so $h_*([f]) = k_*([f])$ as homomorphisms from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. □

Theorem 1.37. *The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ induces an isomorphism of fundamental groups.*

Proof. Let $X = \mathbb{R}^{n+1} - \mathbf{0}$ and $b_0 = (1, 0, ..., 0) \in \mathbb{R}^{n+1} - \mathbf{0}$. Let $r : X \to S^n$ be the map $r(x) = x/\|x\|$. Then $r \circ j$ is the identity map of S^n, so that $r_* \circ j_*$ is the identity homomorphism of $\pi_1(S^n, b_0)$.

Now consider $j \circ r : X \to X$. This is homotopic to the identity map. One can prove that $H : X \times I \to X$, given by

$$H(x, t) = (1 - t)x + t\frac{x}{\|x\|} \tag{23}$$

is a homotopy between the identity map of X and the map $j \circ r$. Since $H(b_0, t) = b_0$, we obtain, from Lemma 1.36, that $j_* \circ r_*$ is the identity homomorphism of $\pi_1(X, b_0)$, which is bijective. It follows that j_* is surjective. From Lemma 1.32, we have that j_* is injective. Therefore j_* is bijective. □

Definition 1.38. Let A be a subspace of X. We say that A is a **deformation retract** of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. This means that there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$, for all $x \in X$, and $H(a, t) = a$ for all $a \in A$. The homotopy H is called a **deformation retraction** of X onto A. The map $r : X \to A$ defined by $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$, where $j : A \to X$ is inclusion.

An alternative, more visual definition can be formulated as follows:

Definition 1.39. A **deformation retraction** of a space X onto a subspace A is a family of maps $f_t : X \to X$, with $t \in I$, such that $f_0 = id$, $f_1(X) = A$, and $f_t|A = id$ for all t. The family f_t should be continuous in the sense that the associated map $F : X \times I \to X$, $F(x, t) = f_t(x)$, is continuous.

Theorem 1.40. *Let A be a deformation retract of X, and let $x_0 \in A$. Then the inclusion map*

$$j : (A, x_0) \to (X, x_0) \tag{24}$$

induces an isomorphism of fundamental groups.

Proof. By Lemma 1.32, j_* is injective. Since there is a homotopy between $j \circ r$ and id, by Lemma 1.36, $j_* \circ r_* = id_*$, so j_* is surjective as well. □

Definition 1.41. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are called **homotopy equivalences**, and each is said to be a **homotopy inverse** of the other.

Let $f : X \to Y$ and $h : Y \to Z$ be homotopy equivalences. Then there exist maps $g : Y \to X$ and $l : Z \to Y$ such that $f \circ g$, $g \circ f$, $h \circ l$ and $l \circ h$ are homotopic to the corresponding identity maps. Then we have

$$[(h \circ f) \circ (g \circ l)] = h_* ([(f \circ g) \circ l]) = h_* ((f \circ g)_*([l])) = h_*([l]) = [h \circ l] \tag{25}$$

so $(h \circ f) \circ (g \circ l)$ is homotopic to the identity map of Z. Similarly we can prove that $(g \circ l) \circ (h \circ f)$ is homotopic to the identity map of X. Hence $h \circ f : X \to Z$ is a homotopy equivalence of X with Z. Since every space is homotopic to itself by the identity map, the relation of homotopy equivalence is an equivalence relation. Two spaces that are homotopy equivalent are said to have the same **homotopy type**.

Let A be a deformation retract of X. Let $j : A \to X$ be the inclusion map and $r : X \to A$ be the retraction mapping. Then $r \circ j$ is the identity map of A and, since $j \circ r$ is homotopic to the identity map of X, A has the same homotopy type as X.

Lemma 1.42. *Let $h, k : X \to Y$ be continuous maps. Let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. If $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.*

Proof. Let $f : I \to X$ be a loop in X based at x_0. We must show that $k_*([f]) = \hat{\alpha}(h_*([f]))$ if and only if $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$. Taking the composition on the left with $[\alpha]$, the relation we must prove becomes

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha]. \tag{26}$$

Consider the loops f_0 and f_1 in the space $X \times I$ given by $f_0(s) = (f(s), 0)$ and $f_1(s) = (f(s), 1)$. Also, consider the path $c : I \to X \times I$ given by $c(t) = (x_0, t)$. Then $H \circ f_0(s) = H(f(s), 0) = h(f(s))$. So $H \circ f_0 = h \circ f$ and, similarly, $H \circ f_1 = k \circ f$. Also, $H \circ c = \alpha$.

Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the paths in $I \times I$ $\beta_0(s) = (s, 0)$, $\beta_1(s) = (s, 1)$, $\gamma_0(t) = (0, t)$ and $\gamma_1(t) = (1, t)$. We have $F \circ \beta_0 = f_0$, $F \circ \beta_1 = f_1$, and $F \circ \gamma_0 = F \circ \gamma_1 = c$.

The paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0, 0)$ to $(1, 1)$ and, since $I \times I$ is convex, there is a path homotopy G between them. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. Also, $H \circ (F \circ G)$ is a path homotopy in Y between $(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha$ and $(H \circ c) * (H \circ f_1) = \alpha * (k \circ f)$. Therefore $[(h \circ f) * \alpha] = [\alpha * (k \circ f)]$, so $[h \circ f] * [\alpha] = [\alpha] * [k \circ f]$. □

Since $\hat{\alpha}$ is a group isomorphism, we have the following:

Corollary 1.43. *Let $h, k : X \to Y$ be homotopic continuous maps such that $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is injective, surjective, or trivial, so is k_*.*

Corollary 1.44. *Let $h : X \to Y$ be nullhomotopic. Then h_* is the trivial homomorphism.*

Proof. By the preceding corollary, since the constant map induces the trivial homomorphism, h_* is also trivial. □

Theorem 1.45. *Let $f : X \to Y$ be continuous. Let $f(x_0) = y_0$. If f is a homotopy equivalence, then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \tag{27}$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse for f such that $g(y_0) = x_1$. Let $f(x_1) = y_1$. Then

$$g \circ f : (X, x_0) \to (X, x_1) \tag{28}$$

is homotopic to the identity map of X, so there is a path α in X such that

$$(g \circ f)_* = \hat{\alpha} \circ (id_X)_* = \hat{\alpha}. \tag{29}$$

Since $\hat{\alpha}$ is an isomorphism, $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is also an isomorphism. Therefore g_* is surjective. Similarly, we obtain that $(f_{x_1}) \circ g_*$ is an isomorphism, so g_* is injective. Then g_* is an isomorphism, and we obtain, from (29)

$$(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha} \tag{30}$$

so (f_{x_0}) is also an isomorphism. □

1.7. **The Seifert-van Kampen theorem.**

Theorem 1.46. *Suppose $X = U \cup V$, where U and V are open sets of X. Suppose that $U \cap V$ is path connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms*

$$\begin{aligned} i_* &: \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and} \\ j_* &: \pi_1(V, x_0) \to \pi_1(X, x_0) \end{aligned} \tag{31}$$

generate $\pi_1(X, x_0)$.

Proof. The theorem can be restated as follows: for any loop f in X based at x_0, $[f] = [g_1] * [g_2] * ... * [g_n]$, where g_i are loops based at x_0 that lie either in U or in V. This is what we set out to prove.

First we show there is a subdivision $a_0 < ... < a_n$ of I such that $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i])$ is contained either in U or in V, for each i. First, by Lebesgue number lemma, we choose a subdivision $b_0, ..., b_m$ of $[0, 1]$ such that for each i, the set $f([b_{i-1}, b_i])$ is contained in either U or V. If $f(b_i) \in U \cap V$ for each i, we are done. If not, let i be an index such that $f(b_i) \notin U \cap V$. Each of $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lies either in U or in V. If $f(b_i) \in U$, then both sets must lie in U. If $f(b_i) \in V$, both sets must lie in V. In either case, we may delete b_i, obtaining a subdivision $c_0, ..., c_{m-1}$. After a finite number of repetitions of this process we obtain the desired subdivision.

We prove the theorem. Given f, let $a_0, ..., a_n$ be the subdivision constructed above. Define f_i to be the path in X that equals the positive linear map of $[0, 1]$ onto $[a_{i-1}, a_i]$ followed by f. Then f_i is a path that lies either in U or in V, and by Theorem 1.7

$$[f] = [f_1] * ... * [f_n]. \tag{32}$$

Since $U \cap V$ is path connected, for each i, we can choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$. Also, $f(a_0) = f(a_n) = x_0$, so we can choose α_0 and α_n to be the constant path at x_0.

Now we set

$$g_i = (\alpha_{i-1} * f_i) * \bar{\alpha_i} \tag{33}$$

for each i. Then g_i is a loop in X based at x_0 whose image lies in U or in V. We obtain

$$\begin{aligned} [g_1] * [g_2] * ... * [g_n] &= [\alpha_0 * f_1 * \bar{\alpha_1} * \alpha_1 * f_2 * \bar{\alpha_2} * ... * \alpha_{n-1} * f_n * \bar{\alpha_n}] \\ &= [f_1] * [f_2] * ... * [f_n] = [f]. \end{aligned} \tag{34}$$

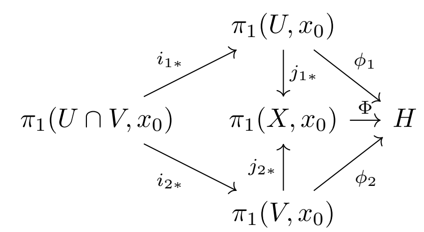
□

Theorem 1.47. ***(Seifert-van Kampen).*** *Let $X = U \cup V$, where U and V are open in X and U, V, and $U \cap V$ are path connected. Let $x_0 \in U \cap V$. Let H be a* *group and let* $\phi_1 : \pi_1(U, x_0) \to H$ *and* $\phi_2 : \pi_1(V, x_0) \to H$ *be homomorphisms. Let* i_{1*}, i_{2*}, j_{1*}, j_{2*} *be homomorphisms induced by inclusion as follows:*

$$\begin{aligned} i_{1*} &: \pi_1(U \cap V, x_0) \to \pi_1(U, x_0), \\ i_{2*} &: \pi_1(U \cap V, x_0) \to \pi_1(V, x_0), \\ j_{1*} &: \pi_1(U, x_0) \to \pi_1(X, x_0), \\ j_{2*} &: \pi_1(V, x_0) \to \pi_1(X, x_0). \end{aligned} \tag{35}$$

If $\phi_1 \circ i_{1*} = \phi_2 \circ i_{2*}$, *then there is a unique homomorphism* $\Phi : \pi_1(X, x_0) \to H$ *such that* $\Phi \circ j_{1*} = \phi_1$ *and* $\Phi \circ j_{2*} = \phi_2$.

The relations (35) can be represented schematically as in the following diagram:



Proof. By Theorem 1.46, $\pi_1(X, x_0)$ is generated by the images of j_{1*} and j_{2*}. Then, if $g \in U$ is a loop based at x_0, $\Phi(j_{1*}(g)) = \phi(g)$. Similarly, for $g \in V$, we have $\Phi(j_{2*}(g)) = \phi(g)$. Hence Φ is uniquely determined by ϕ_1 and ϕ_2. Now we need to prove existence.

First, we introduce some notation: Given a path f in X, we denote its path-homotopy class in X by $[f]$. If f lies in U, we denote by $[f]_U$ its path-homotopy class in U. $[f]_V$ and $[f]_{U \cap V}$ are defined similarly.

Step 1. We define a set map ρ that assigns, to each loop f based at x_0 that lies in U or in V, an element of the group H. We define $\rho(f) = \phi_1([f]_U)$ if f lies in U and $\rho(f) = \phi_2([f]_V)$ if f lies in V. To show that ρ is well-defined, note that if f lies in $U \cap V$, then

$$\phi_1([f]_U) = \phi_1 i_{1*}([f]_{U\cap V}) = \phi_2 i_{2*}([f]_{U\cap V}) = \phi_2([f]_V) \tag{36}$$

by hypothesis. If $[f]_U = [g]_U$ or $[f]_V = [g]_V$, then, by definition, $\rho(f) = \rho(g)$. We also have that if both f and g lie in U (or if both lie in V), then

$$\rho(f * g) = \phi_1([f * g]_U) = \phi_1([f] * [g]) = \phi_1([f]) \cdot \phi_1([g]) = \rho(f) \cdot \rho(g), \tag{37}$$

where we denoted by "$\cdot$" the composition law on H.

Step 2. Now, we extend ρ to a set map σ that assigns, to each **path** f lying in U or V, an element of H, such that if $[f]_U = [g]_U$ or $[f]_V = [g]_V$, then $\sigma(f) = \sigma(g)$, and σ satisfies (37) when $f * g$ is defined.

For each x in X, choose a path α_x from x_0 to x as follows: If $x = x_0$, let α_x be the constant path at x_0. If $x \in U \cap V$, let α_x be a path in $U \cap V$. If x is in U or V but not in $U \cap V$, let α_x be a path in U or V, respectively.

Then, for any path f in U or in V, we define a loop $L(f)$ in U or V, respectively, based at x_0, by the equation

$$L(f) = \alpha_x * (f * \bar{\alpha}_y) \tag{38}$$

where x is the initial point of f and y is the final point of f. Finally, we define

$$\sigma(f) = \rho(L(f)). \tag{39}$$

We prove that σ is an extension of ρ. If f is a loop based at x_0 lying in either U or V, then

$$L(f) = e_{x_0} * (f * e_{x_0}) \tag{40}$$

where e_{x_0} is the constant path at x_0. Then $[L(f)] = [f]$, so $L(f)$ is path homotopic to f in either U or V, so that $\rho(L(f)) = \rho(f)$. Hence $\sigma(f) = \rho(f)$, which proves that σ is an extension of ρ to appropriate paths.

Let f and g be paths that are path homotopic in U or in V. Then the loops $L(f)$ and $L(g)$ are also path homotopic either in U or in V and we have

$$\sigma(f) = \rho(L(f)) = \rho(L(g)) = \sigma(g). \tag{41}$$

To check that σ verifies equation (37), let f and g be arbitrary paths in U or in V such that $f(1) = g(0)$. We have

$$L(f) * L(g) = (\alpha_x * (f * \bar{\alpha}_y)) * (\alpha_y * (g * \bar{\alpha}_z)) \tag{42}$$

where $f(0) = x$, $f(1) = g(0) = y$ and $g(1) = z$. From

$$[L(f) * L(g)]_U = [\alpha_x]_U * [f]_U * [\bar{\alpha}_y * \alpha_y]_U * [g]_U * [\bar{\alpha}_z]_U = [L(f * g)]_U \tag{43}$$

we obtain $L(f) * L(g) \simeq_p L(f * g)$ in U (or V). Then

$$\rho(L(f * g)) = \rho(L(f) * L(g)) = \rho(L(f)) \cdot \rho(L(g)). \tag{44}$$

Hence $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$.

Step 3. We extend σ to a set map τ that assigns to an arbitrary path f of X, an element of H. We want it to satisfy the following conditions:

(1) If $[f] = [g]$, then $\tau(f) = \tau(g)$
(2) $\tau(f * g) = \tau(f) \cdot \tau(g)$ if $f * g$ is defined.

Given f, choose a subdivision $s_0 < ... < s_n$ of $[0, 1]$ such that f maps each of the subintervals $[s_{i-1}, s_i]$ into U or V. Let f_i denote the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$, followed by f. Then f_i is a path in U or in V, and

$$[f] = [f_1] * ... * [f_n]. \tag{45}$$

In order for τ to be an extension of σ and satisfy (1) and (2), we define it as

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) \cdot ... \cdot \sigma(f_n). \tag{46}$$

To show that this definition is independent of the choice of subdivision we show that the value of $\tau(f)$ remains unchanged if we adjoin a single additional point p to the subdivision. Let i be the index such that $s_{i-1} < p < s_i$. Then

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) \cdot ... \cdot \sigma(f_{i-1}) \cdot \sigma(f_i') \cdot \sigma(f_i'') \cdot ... \cdot \sigma(f_n) \tag{47}$$

where f_i' and f_i'' equal the positive linear maps of $[0, 1]$ to $[s_{i-1}, p]$ and to $[p, s_i]$, respectively, followed by f. But f_i is path homotopic to $f_i' * f_i''$ in U or V, so that $\sigma(f_i) = \sigma(f_i' * f_i'') = \sigma(f_i') \cdot \sigma(f_i'')$, and we obtain the same expression for $\tau(f)$. Thus τ is well-defined.

Now, if f already lies in U or V, we use the trivial partition to define $\tau(f)$, and we have, by definition, $\tau(f) = \sigma(f)$. This proves that τ is an extension of σ.

Step 4. We prove condition (1) for τ. First, we consider a special case. Let f and g be paths in X from x to y and let F be a path homotopy between them. Assume that there exists a subdivision $s_0, ..., s_n$ of $[0, 1]$ such that F carries each rectangle $R_i = [s_{i-1}, s_i] \times I$ into either U or V. We show that $\tau(f) = \tau(g)$.

Let f_i and g_i be the linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$ followed by f or g, respectively. The restriction of F to R_i is a homotopy between f_i and g_i in either U or V. We consider the paths traced by the end points during this homotopy. Define the path $\beta_i(t) = F(s_i, t)$. Then β_i is a path in X from $f(s_i)$ to $g(s_i)$. The paths β_0 and β_n are the constant paths at x and y, respectively. We show that

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i, \tag{48}$$

with a path homotopy in U or V.

We have $F([s_{i-1}, s_i] \times 0) = f_i(I)$and $F(s_i \times [0,1]) = \beta_i(s_i)$. Then $F([s_{i-1}, s_i] \times 0) * F(s_i \times [0,1]) = f_i * \beta_i$. Similarly, $F(s_{i-1} \times [0,1]) * F([s_{i-1}, s_i] \times 1) = \beta_{i-1} * g_i$. Since R_i is convex, there is a path homotopy in R_i between the path from $s_{i-1} \times 0$ to $s_i \times 0$ to $s_i \times 1$ and the path from $s_{i-1} \times 0$ to $s_{i-1} \times 1$ to $s_i \times 1$. Then following this path by F we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ that takes place in either U or V. Then

$$\sigma(f_i) \cdot \sigma(\beta_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i), \tag{49}$$

which gives

$$\sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1}. \tag{50}$$

Since $\sigma(\beta_0) = \sigma(\beta_n) = 1$, we have

$$\tau(f) = \sigma(f_1) \cdot ... \cdot \sigma(f_n) = \sigma(g_1) \cdot ... \cdot \sigma(g_n) = \tau(g). \tag{51}$$

So, in this special case, τ satisfies condition (1).

Now we prove that τ satisfies condition (1) in the general case. Given f and g and a path homotopy F between them, choose subdivisions $s_0, ..., s_n$ and $t_0, ..., t_m$ of $[0,1]$ such that F maps each rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into either U or V. Let f_j be paths $f_j(s) = F(s, t_j)$. Then $f_0 = f$ and $f_m = g$. The pair of paths f_{j-1} and f_j satisfy the requirements of our special case, so $\tau(f_{j-1}) = \tau(f_j)$ for each j. Then $\tau(f) = \tau(g)$.

Step 5. Now we prove (2). Let $f * g$ be a path in X, and choose a subdivision $s_0 < ... < s_n$ of $[0,1]$ containing the point $1/2$ such that $f*g$ carries each subinterval into either U or V. Let $s_k = 1/2$.

For $i = 1, ..., k$, the positive linear map of $[0,1]$ to $[s_{i-1}, s_i]$, followed by $f * g$, is the same as the positive linear map of $[0,1]$ to $[2s_{i-1}, 2s_i]$, call it f_i. Similarly, for $i = k+1, ..., n$, the positive linear map of $[0,1]$ to $[s_{i-1}, s_i]$, followed by $f * g$, is the same as the positive linear map of $[0,1]$ to $[2s_{i-1} - 1, 2s_i - 1]$ followed by g, call it g_{i-k}. With the subdivision $s_0, ..., s_n$ for the domain of $f * g$ we have

$$\tau(f * g) = \sigma(f_1) \cdot ... \cdot \sigma(f_k) \cdot \sigma(g_1) \cdot ... \cdot \sigma(g_{n-k}). \tag{52}$$

With the subdivision $2s_0, ..., 2s_k$ for the path f, we have

$$\tau(f) = \sigma(f_1) \cdot ... \cdot \sigma(f_k) \tag{53}$$

and with the subdivision $2s_k - 1, ..., 2s_n - 1$ for the path g, we have

$$\tau(g) = \sigma(g_1) \cdot ... \cdot \sigma(g_{n-k}). \tag{54}$$

Then

$$\tau(f * g) = \tau(f) \cdot \tau(g), \tag{55}$$

so τ satisfies condition (2).

Step 6. We prove the theorem. For each loop f in X based at x_0, we define

$$\Phi([f]) = \tau(f) \tag{56}$$

Condition (1) implies that Φ is well-defined and condition (2) implies that Φ is a homomorphism. We show that $\Phi \circ j_1 = \phi_1$. If f is a loop in U, then

$$\Phi(j_{1*}([f]_U)) = \Phi([f]) = \tau(f) = \rho(f) = \phi_1([f]_U). \tag{57}$$

Similarly, we can show that $\Phi \circ j_2 = \phi_2$. □

Now we give a broader statement of the van Kampen theorem, which makes use of the theory of free products of groups. First we introduce some notation. Let $X = \cup_\alpha A_\alpha$, such that each A_α is open, path-connected, $x_0 \in \cap_\alpha A_\alpha$, and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then, the homomorphisms $j_{\alpha *} : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ induced by inclusions $j_\alpha : A_\alpha \to X$, extend to a homomorphism

$$\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0) \tag{58}$$

given by $\Phi(x_1, ..., x_n) = j_{1*}(x_1) \cdot ... \cdot j_{n*}(x_n)$ for any word $(x_1, ..., x_n)$. We also denote by $i_{\alpha\beta*} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$ the homomorphism induced by the inclusion $i_{\alpha\beta} : A_\alpha \cap A_\beta \to A_\alpha$. With this an extended version of the van Kampen theorem can be formulated as:

Theorem 1.48. *Let $X = \cup_\alpha A_\alpha$, such that each A_α is open, path-connected, $x_0 \in \cap_\alpha A_\alpha$, and each intersection $A_\alpha \cap A_\beta$ is path-connected. Then the homomorphism*

$$\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0) \tag{59}$$

is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta*}(\omega) i^{-1}_{\beta\alpha*}(\omega)$, and so Φ induces an isomorphism $\pi_1(X, x_0) \simeq *_\alpha \pi_1(A_\alpha, x_0)/N$.

Proof. Let $f : I \to X$ be a loop in X based at x_0. Let $0 = s_0 < s_1 < ... < s_m = 1$ be a partition of I such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α. Denote the A_α containing $f([s_{i-1}, s_i])$ by A_i and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$. Then f is the product of the paths $f_1 * ... * f_m$.

Since we assume that $A_i \cap A_{i+1}$ is path-connected, we may choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to the point $f(s_i) \in A_i \cap A_{i+1}$. Consider the loop

$$(f_1 * \bar{g}_1) * (g_1 * f_2 * \bar{g}_2) * (g_2 * f_3 * \bar{g}_3) * ... * (g_{m-1} * f_m). \tag{60}$$

This is homotopic to f, as one can easily see by computing its path homotopy equivalence class. It is also a product of loops based at x_0, each lying in a single A_i. Hence $[f]$ is in the image of Φ, and Φ is surjective.

Before proceeding with the remainder of the proof we introduce some terminology. First, when discussing fundamental groups we shall drop the base point x_0 since all the fundamental groups are with respect to this point.

Now, by a *factorization* of an element $[f] \in \pi_1(X)$ we shall mean a formal product $[f_1]...[f_k]$ where:

- Each f_i is a loop in some A_α at the basepoint x_0, and $[f_i] \in \pi_1(A_\alpha)$ is the homotopy class of f_i.
- The loop f is homotopic to $f_1 * ... * f_k$ in X.

A factorization of $[f]$ is thus a word in $*_\alpha \pi_1(A_\alpha)$ that is mapped to $[f]$ by Φ. Since Φ is surjective, every $[f] \in \pi_1(X)$ has a factorization.

We study uniqueness of factorizations. Call two factorizations of $[f]$ *equivalent* if they are related by a sequence of the following two sorts of moves or their inverses:

- Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i * f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(A_\alpha)$.
- Regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in the group $\pi_1(A_\beta)$ rather than $\pi_1(A_\alpha)$ if f_i is a loop in $A_\alpha \cap A_\beta$.

The first of these does not change the element of $*_\alpha \pi_1(A_\alpha)$ defined by the factorization. The second move does not change the image of this element in the quotient group $Q = *_\alpha \pi_1(A_\alpha)/N$. To see this, let $[g_j] \in \pi_1(A_\alpha \cap A_\beta)$ and let $[f_j] = i_{\alpha\beta*}([g_j]) \in \pi_1(A_\alpha)$ and $[f'_j] = i_{\beta\alpha*}([g_j]) \in \pi_1(A_\beta)$, that is, $[f'_j]$ is obtained by performing the second move on $[f_j]$. Now let $w = u[f_j]^\epsilon v$, with u and v arbitrary words. If w' is the word obtained by performing the second move on $[f_j]$, then we have $w' = u[f'_j]^\epsilon v$. Then

$$w(w')^{-1} = u[f_j]^\epsilon v v^{-1} [f'_j]^{-\epsilon} u^{-1} = u[f_j]^\epsilon [f'_j]^{-\epsilon} u^{-1}. \tag{61}$$

Now, since $[f_j]^\epsilon [f'_j]^{-\epsilon} \in N$ by definition, and N is normal, we also have $w(w')^{-1} \in N$. It is a general property of normal groups that in this case $wN = w'N$. In other words the coset of N by w in $*_\alpha \pi_1(A_\alpha)$ is equal to the coset of N by w' in $*_\alpha \pi_1(A_\alpha)$. So equivalent factorizations give the same element of Q.

If we can show that any two factorizations of $[f]$ are equivalent, then the map $\tilde{\Phi} : Q \to \pi_1(X)$ given by $\tilde{\Phi}(uN) = \Phi(u)$ is injective. Then the kernel of ϕ is equal to N.

Let $[f_1]...[f_k]$ and $[f_1']...[f_l']$ be two factorizations of $[f]$. Then the paths $f_1 * ... * f_k$ and $f_1' * ... f_l'$ are homotopic. Let $F : I \times I \to X$ be a homotopy between them. Let $0 = s_0 < s_1 < ... < s_m = 1$ and $0 = t_0 < t_1 < ... < t_n = 1$ be partitions of I such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into a single A_α, labelled A_{ij}. We assume that the $s-$partition subdivides the partitions that give the products $f_1 * ... * f_k$ and $f_1' * ... * f_l'$. We may perturb the vertical sides of the rectangles R_{ij} such that each point of $I \times I$ lies in at most three R_{ij}'s, and relabel them $R_1, R_2, ..., R_{mn}$ as in the figure.

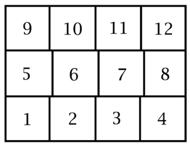


FIGURE 3. Arrangement of rectangles R_{ij}

If γ is a path in $I \times I$ from the left edge to the right edge, Then the restriction $F|\gamma$ is a loop at the basepoint x_0. Let γ_r be the path separating the first r rectangles $R_1, ..., R_r$ from the remaining rectangles. Thus γ_0 is the bottom edge of $I \times I$ and γ_{mn} is the top edge.

Let us call the corners of the R_r's vertices. For each vertex v with $F(v) \neq x_0$, let g_v be a path from x_0 to $F(v)$. We can choose g_v to lie in the intersection of the two or three A_{ij}'s corresponding to the R_r's containing v since we assume the intersection of any two or three A_{ij}'s is path-connected. If we insert into $F|\gamma_r$ the appropriate paths $\bar{g}_v g_v$ at successive vertices, then we obtain a factorization of $[F|\gamma_r]$ by regarding the loop corresponding to a horizontal or vertical segment between adjacent vertices as lying in the A_{ij} for either of the R_s's containing the segment. Pushing γ_r across R_{r+1} to obtain γ_{r+1} changes $F|\gamma_r$ to $F|\gamma_{r+1}$ by a homotopy within the A_{ij} corresponding to R_{r+1}. Since we can choose this A_{ij} for all the segments of γ_r and γ_{r+1} in R_{r+1}, the factorizations associated to successive paths γ_r and γ_{r+1} are equivalent.

We can arrange that the factorization associated to γ_0 is equivalent to the factorization $[f_1]...[f_k]$ by choosing the path g_v for each vertex v along the lower edge of $I \times I$ to lie not just in the two A_{ij}'s corresponding to the R_s's containing v, but also to lie in the A_α for the f_i containing v in its domain. In the case v is the common end point of the domains of two consecutive f_i's we have $F(v) = x_0$, so there is no need to choose a g_v. In similar fashion we may assume that the factorizations associated to all the γ_r's are equivalent, we conclude that the factorizations $[f_1]...[f_k]$ and $[f_1']...[f_l']$ are equivalent. □

Assuming the hypotheses of Theorem 1.47, Theorem 1.48 becomes

Theorem 1.49. *Let*

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0) \tag{62}$$

be the homomorphism of the free product that extends the homomorphisms j_{1*} *and* j_{2*} *induced by inclusion. Then* Φ *is surjective, and its kernel is generated by all elements of the free product of the form* $i_{1*}(g)^{-1} i_{2*}(g)$ *and their conjugates.*

Said differently, the kernel of Φ *is the least normal subgroup* N *of the free product that contains all elements represented by words of the form* $(i_{1*}(g)^{-1}, i_{2*}(g))$, *for* $g \in \pi_1(U \cap V, x_0)$.

1.8. **The fundamental group of S^n.** Theorem 1.46 can also be used to compute the fundamental group of S^n. For this, we first give a corollary of this theorem:

Corollary 1.50. *Suppose $X = U \cup V$, where U and V are open sets of X. Suppose $U \cap V$ is nonempty and path connected. If U and V are simply connected, then X is simply connected.*

Proof. To prove this, note that both fundamental groups of U and V are trivial. Then the fundamental group of X is also trivial. □

Theorem 1.51. *If $n \geq 2$, the n-sphere S^n is simply connected.*

Proof. Let $p = (0, ..., 0, 1) \in \mathbb{R}^{n+1}$ and $q = (0, ..., 0, -1) \in \mathbb{R}^{n+1}$ be the "north pole" and the "south pole" of S^n, respectively.

First, we show that if $n \geq 1$, the punctured sphere $S^n - p$ is homeomorphic to $\mathbb{R}^n$.

Define the **stereographic projection** $f : (S^n - p) \to \mathbb{R}^n$ by

$$f(x) = f(x_1, ..., x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n). \tag{63}$$

It is easy to see that the map $g : \mathbb{R}^n \to (S^n - p)$ given by

$$g(y) = g(y_1, ..., y_n) = (t(y) \cdot y_1, ..., t(y) \cdot y_n, 1 - t(y)) \tag{64}$$

with $t(y) = 2/(1 + \|y\|^2)$, is a right and left inverse for f. Therefore f is a homeomorphism. Since the reflection map $(x_1, ..., x_{n+1}) \to (x_1, ..., x_n, -x_{n+1})$ defines a homeomorphism of $S^n - p$ with $S^n - q$, we have that $S^n - q$ is also homeomorphic to $\mathbb{R}^n$.

We now prove the theorem. Let U and V be the open sets $U = S^n - p$ and $V = S^n - q$ of S^n.

For $n \geq 1$, since U and V are path connected and have the point $(1, 0, ..., 0)$ in common, S^n is also path connected.

For $n \geq 2$, since U and V are homeomorphic to $\mathbb{R}^n$, they are simply connected. Their intersection is $S^n - p - q$, which is homeomorphic under the stereographic projection to $\mathbb{R}^n - \mathbf{0}$, which is path connected. Therefore $U \cap V$ is path connected, and, by Corollary 1.50, S^n is simply connected for $n \geq 2$ and its fundamental group is trivial. □

1.9. **Equivalence of covering spaces.**

Theorem 1.52. *Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$.*

(1) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

(2) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0) \tag{65}$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(3) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

Proof. (1) Suppose $\tilde{h}$ is a loop in E at e_0, and $p_*([\tilde{h}])$ is the identity element. Let F be a path homotopy between $p \circ \tilde{h} = h$ and the constant loop. The constant loop at e_0 is a lifting of the constant loop at b_0, and $\tilde{h}$ is a lifting of h. Then, by Theorem 1.27, if $\tilde{F}$ is the lifting of F to E such that $\tilde{F}(0, 0) = e_0$, then $\tilde{F}$ is a path homotopy between $\tilde{h}$ and the constant loop at e_0.

(2) Given loops f and g in B, let $\tilde{f}$ and $\tilde{g}$ be liftings of them to E that begin at e_0. Then $\phi([f]) = \tilde{f}(1)$ and $\phi([g]) = \tilde{g}(1)$. We show that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$.

Suppose $[f] \in H * [g]$. Then $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Then $\tilde{h} * \tilde{g}$ is defined and is a lifting of $h * g$. Because $[f] = [h * g]$, the liftings $\tilde{f}$ and $\tilde{h} * \tilde{g}$, which begin at e_0, must end at the same point of E. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E, so $\phi([f]) = \phi([g])$.

Now suppose that $\phi([f]) = \phi([g])$. Then $\tilde{f}$ and $\tilde{g}$ end at the same point of E. Let $\tilde{h} = \tilde{f} * \bar{\tilde{g}}$. $\tilde{h}$ is a loop in E based at e_0. Then

$$[\tilde{h} * \tilde{g}] = [\tilde{f} * \bar{\tilde{g}} * \tilde{g}] = [\tilde{f}] * [\bar{\tilde{g}} * \tilde{g}] = [\tilde{f}]. \tag{66}$$

Then, if $\tilde{F}$ is a path homotopy in E between the loops $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy in B between $h * g$ and f, where $h = p \circ \tilde{h} \in H$. Thus $[f] \in H * [g]$.

If E is path connected, by Theorem 1.29, ϕ is surjective, so Φ is also surjective.

(3) Injectivity of Φ means that $\phi([f]) = \phi([g])$ if and only if $[f] \in H * [g]$. If g is a constant loop, then $\phi([f]) = e_0$ if and only if $[f] \in H$. If g is a constant loop we have $H * [g] = H$ and $\phi([g]) = e_0$. Therefore, in this case, $\phi([f]) = e_0$ if and only if $[f] \in H$. But $\phi([f]) = e_0$ means $\tilde{f}(1) = e_0$. Since $\tilde{f}(0) = e_0$, we have that $[f] \in H$ if and only if $\tilde{f}$ is a loop in E based at e_0. □

Throughout the rest of this chapter we shall use the following convention: the statement that $p : E \to B$ is a covering map will include the assumption that E and B are locally path connected and path connected, unless specifically stated otherwise.

By Theorem 1.52, if $p : E \to B$ is a covering map with $p(e_0) = b_0$, then the induced homomorphism p_* is injective. Then

$$H_0 = p_*(\pi_1(E, e_0)) \tag{67}$$

is a subgroup of $\pi_1(B, b_0)$ isomorphic to $\pi_1(E, e_0)$.

Definition 1.53. Let $p : E \to B$ and $p' : E' \to B$ be covering maps. They are said to be **equivalent** if there exists a homeomorphism $h : E \to E'$ such that $p = p' \circ h$. The homeomorphism h is called an **equivalence of covering maps** or an **equivalence of covering spaces**.

Lemma 1.54. ***(The general lifting lemma).*** *Let $p : E \to B$ be a covering map such that $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)). \tag{68}$$

Furthermore, if such a lifting exists, it is unique.

Proof. ("⇒") If the lifting $\tilde{f}$ exists, then, since $f = p \circ \tilde{f}$, we have

$$f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0))) \subset p_*(\pi_1(E, e_0)). \tag{69}$$

Now we prove uniqueness. Let $y_1 \in Y$ and choose a path α in Y from y_0 to y_1. Take the path $f \circ \alpha$ in B and lift it to a path γ in E beginning at e_0. If a lifting $\tilde{f}$ of f exists, then $\tilde{f}(y_1) = \gamma(1)$. Since the lifting of a path is unique, $\tilde{f}$ is unique.

("⇐") Given $y_1 \in Y$, choose a path α in Y from y_0 to y_1. Lift the path $f \circ \alpha$ to a path γ in E beginning at e_0, and define $\tilde{f}(y_1) = \gamma(1)$. To prove continuity of $\tilde{f}$ at a point y_1 of Y, we show that, given a neighborhood N of $\tilde{f}(y_1)$, there is a neighborhood W of y_1 such that $\tilde{f}(W) \subset N$.

Choose a path-connected neighborhood U of $f(y_1)$ that is evenly covered by p. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices and let V_0 be the slice that contains the point $\tilde{f}(y_1)$. We can assume $V_0 \subset N$, since, if this is not the case, we can always choose a smaller neighborhood of $f(y_1)$ such that the slice of the partition corresponding to it through p, satisfies this condition. Let $p_0 : V_0 \to U$ be obtained by restricting p. Then p_0 is a homeomorphism. Since f is continuous at y_1 and Y is locally path connected, we can find a path-connected neighborhood W of y_1 such that $f(W) \subset U$. We show that $\tilde{f}(W) \subset V_0$.

Given $y \in W$, choose a path β in W from y_1 to y. Since $\tilde{f}$ is well defined (we will prove this later), we can obtain $\tilde{f}(y)$ by taking the path $\alpha * \beta$ from y_0 to y, lifting $f \circ (\alpha * \beta)$ to a path in E beginning at e_0, and letting $\tilde{f}(y)$ be the end point of this lifted path. Since $f \circ \beta$ is a path in U, the path $\delta = p_0^{-1} \circ f \circ \beta$ is a lifting of it that begins at $\tilde{f}(y_1)$. Then

$$p \circ (\gamma * \delta) = (p \circ \gamma) * (p \circ \delta) = (f \circ \alpha) * (f \circ \beta) = f \circ (\alpha * \beta), \tag{70}$$

so $\gamma * \delta$ is a lifting of $f \circ (\alpha * \beta)$ that begins at e_0 and ends at $\tilde{f}(y) = \delta(1) \in V_0$. Hence $\tilde{f}(W) \subset V_0$.

To show that $\tilde{f}$ is well defined let α and β be two paths in Y from y_0 to y_1. We show that the liftings of $f \circ \alpha$ and $f \circ \beta$ to paths in E beginning at e_0 end at the same point in E.

Lift $f \circ \alpha$ to a path γ in E beginning at e_0 and lift $f \circ \bar{\beta}$ to a path δ in E beginning at $\gamma(1)$. Then $\gamma * \delta$ is a lifting of the loop $f \circ (\alpha * \bar{\beta})$. Since, by hypothesis, $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$, $[f \circ (\alpha * \bar{\beta})]$ is in the image of p_*. Then, by Theorem 1.52, its lift $\gamma * \delta$ is a loop in E. This means that the initial point of γ is equal to the initial point of $\bar{\delta}$. But $\bar{\delta}$ is a lifting of $f \circ \beta$ that begins at e_0 and γ is a lifting of $f \circ \alpha$ that begins at e_0, and both liftings end at the same point of E. Then $\tilde{f}$ is well defined. □

Theorem 1.55. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps such that $p(e_0) = p'(e_0') = b_0$. There is an equivalence $h : E \to E'$ such that $f(e_0) = e_0'$ if and only if the groups*

$$H_0 = p_*(\pi_1(E, e_0)) \text{ and } H_0' = p_*'(\pi_1(E', e_0'))$$

are equal. If h exists, it is unique.

Proof. Suppose h exists. Since it is a homeomorphism, its induced homomorphism is an isomorphism. So,

$$h_*(\pi_1(E, e_0)) = \pi_1(E', e_0'). \tag{71}$$

Since $p' \circ h = p$, we have

$$H_0 = p_*(\pi_1(E, e_0)) = p_*' h_*(\pi_1(E, e_0)) = p_*'(\pi_1(E', e_0')) = H_0'. \tag{72}$$

Now we assume that $H_0 = H_0'$ and show that h exists. Since p' is a covering map and E is path connected and locally path connected, there exists a map $h : E \to E'$ with $h(e_0) = e_0'$ that is a lifting of p, i.e. $p' \circ h = p$. Similarly, there is a map $k : E' \to E$ with $k(e_0') = e_0$ such that $p \circ k = p'$.

Now, $p \circ k \circ h = p' \circ h = p$, so $k \circ h : E \to E$ is a lifting of p with $k \circ h(e_0) = e_0$. The identity id_E is another lifting of p with $id_E(e_0) = e_0$. By the uniqueness part of Lemma 1.54, $k \circ h = id_E$. Similarly, $h \circ k = id_{E'}$. This proves that h is a homeomorphism with inverse k, and thus it is an equivalence. □

Lemma 1.56. *Let $p : E \to B$ be a covering map. Let e_0 and e_1 be points of $p^{-1}(b_0)$, and let $H_i = p_*(\pi_1(E, e_i))$.*

(1) If γ is a path in E from e_0 to e_1, and α is the loop $p \circ \gamma$ in B, then

$$[\alpha] * H_1 * [\alpha]^{-1} = H_0. \tag{73}$$

Hence H_0 and H_1 are conjugate.

(2) Given e_0 and given a subgroup H of $\pi_1(B, b_0)$ conjugate to H_0, there exists a point e_1 of $p^{-1}(b_0)$ such that $H_1 = H$.

Proof. (1). Given $[h] \in H_1$, we have $[h] = p_*([\tilde{h}])$ for some loop $\tilde{h}$ in E based at e_1. Let $\tilde{k} = (\gamma * \tilde{h}) * \bar{\gamma}$. It is a loop in E based at e_0, and

$$\begin{aligned} p_*([\tilde{k}]) = [p \circ \tilde{k}] = [p \circ ((\gamma * \tilde{h}) * \bar{\gamma})] = [((p \circ \gamma) * (p \circ \tilde{h})) * (p \circ \bar{\gamma})] \\ = [(\alpha * h) * \bar{\alpha}] = [\alpha] * [h] * [\alpha]^{-1}. \end{aligned} \tag{74}$$

Since $p_*([\tilde{k}]) \in H_0$, we have $[\alpha] * H_1 * [\alpha]^{-1} \subset H_0$.

Now, let $[h'] \in H_0$. Then $[h] = p_*([\tilde{h}'])$, for some loop $\tilde{h}$ in E based at e_0. Let $\tilde{k}' = (\bar{\gamma} * \tilde{h}') * \gamma$. It is a loop in E based at e_1. By a similar procedure as before, noting that $\bar{\alpha} = p \circ \bar{\gamma}$, we obtain

$$[\bar{\alpha}] * H_0 * [\bar{\alpha}]^{-1} \subset H_1, \tag{75}$$

which implies $H_0 \subset [\alpha] * H_1 * [\alpha]^{-1}$. This proves that $[\alpha] * H_1 * [\alpha]^{-1} = H_0$.

(2). Let e_0 be given and let H be conjugate to H_0. Then $H_0 = [\alpha] * H * [\alpha]^{-1}$ for some loop α in B based at b_0. Let γ be the lifting of α to a path in E beginning at e_0, and let $e_1 = \gamma(1)$. Then (1) implies that $H_0 = [\alpha] * H_1 * [\alpha]^{-1}$. Then $H = H_1$. □

Theorem 1.57. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps and let $p(e_0) = p'(e_0') = b_0$. The covering maps p and p' are equivalent if and only if the subgroups*

$$H_0 = p_*(\pi_1(E, e_0)) \text{ and } H_0' = p_*'(\pi_1(E', e_0'))$$

of $\pi_1(B, b_0)$ are conjugate.

Proof. Suppose p and p' are equivalent and let $h : E \to E'$ be an equivalence between them. Let $e_1' = h(e_0)$, and let $H_1' = p_*(\pi_1(E', e_1'))$. By Theorem 1.55, $H_0 = H_1'$. By Lemma 1.56, H_1' is conjugate to H_0'. So H_0 and H_0' are conjugate.

Now, suppose H_0 and H_0' are conjugate. By Lemma 1.56 there is a point e_1' of E' such that $H_1' = H_0$. Using again Theorem 1.55, there exists an equivalence $h : E \to E'$ such that $h(e_0) = e_1'$. □

This theorem states that there exists an injective correspondence from equivalence classes of coverings of B to conjugacy classes of subgroups of $\pi_1(B, b_0)$. In general, this correspondence is not surjective.

1.10. **The universal covering space.**

Definition 1.58. Suppose $p : E \to B$ is a covering map, with $p(e_0) = b_0$. If E is simply connected, then E is called a **universal covering space** of B. Since $\pi_1(E, e_0)$ is trivial, the covering space E corresponds to the trivial subgroup of $\pi_1(B, b_0)$ under the correspondence given by $H = p_*(\pi_1(E, e_0))$. Here H is the trivial subgroup of $\pi_1(B, b_0)$.

By Theorem 1.57, any two universal covering spaces of B are equivalent. This is why we often speak of "the" universal covering space of a given space B. In what follows we will assume that B has a universal covering space and derive some results. However, as we will see, this is not always the case.

Lemma 1.59. *Let B be path connected and locally path connected. Let $p : E \to B$ be a covering map (here we do not require E to be path connected). If E_0 is a path component of E, then the map $p_0 : E_0 \to B$ obtained by restricting p is a covering map.*

Proof. Since E is locally homeomorphic to B, it is also locally path connected. Therefore E_0 is open in E. Then $p(E_0)$ is open in B. We show that $p(E_0)$ is also closed in B, so that $p(E_0) = B$.

Let x be a point of B belonging to the closure of $p(E_0)$. Let U be a path-connected neighborhood of x that is evenly covered by p. Since U contains a point of $p(E_0)$, some slice V_α of $p^{-1}(U)$ must intersect E_0. Since V_α is homeomorphic to U, it is path connected. Therefore it must be contained in E_0. Then $p(V_\alpha) = U$ is contained in $p(E_0)$, so $x \in p(E_0)$. So $p(E_0)$ is closed.

Now we show that $p_0 : E_0 \to B$ is a covering map. Given $x \in B$, choose a path-connected neighborhood U of x that is evenly covered by p. If V_α is a slice of $p^{-1}(U)$, then V_α is path connected. If it intersects E_0, it lies in E_0. Therefore, $p_0^{-1}(U)$ is the union of those slices V_α of $p^{-1}(U)$ that intersect E_0. Each of these is open in E_0 and is mapped homeomorphically by p_0 onto U. Thus U is evenly covered by p_0, so $p_0 : E_0 \to B$ is a covering map. □

Lemma 1.60. *Let $p : X \to Z$, $q : X \to Y$, and $r : Y \to Z$ be continuous maps with $p = r \circ q$, and X, Y, and Z be path connected and locally path connected. If p and r are covering maps, so is q.*

Proof. Let $x_0 \in X$. Set $y_0 = q(x_0)$ and $z_0 = p(x_0)$. We first show that q is surjective.

Given $y \in Y$, choose a path $\tilde{\alpha}$ in Y from y_0 to y. Then $\alpha = r \circ \tilde{\alpha}$ is a path in Z beginning at z_0, so $\tilde{\alpha}$ is a lifting of α. Let $\tilde{\tilde{\alpha}}$ be a lifting of α to a path in X beginning at x_0. Then $q \circ \tilde{\tilde{\alpha}}$ is a lifting of α to Y that begins at y_0. Then $q \circ \tilde{\tilde{\alpha}}$ is a lifting of α to Y that begins at y_0. By uniqueness of path liftings, $\tilde{\alpha} = q \circ \tilde{\tilde{\alpha}}$. Then q maps the end point of $\tilde{\tilde{\alpha}}$ to the end point y of $\tilde{\alpha}$. Thus q is surjective.

Given $y \in Y$, we find a neighborhood of y that is evenly covered by q. Let $z = r(y)$. Since p and r are covering maps, we can find a path-connected neighborhood U of z that is evenly covered by both p and r. Let V be the slice of $r^{-1}(U)$ that contains the point y. We show that V is evenly covered by q. Let $\{U_\alpha\}$ be the collection of slices of $p^{-1}(U)$. Now, since $p = r \circ q$, q maps each U_α into $r^{-1}(U)$. Because U_α is connected it must be mapped by q into a single one of the slices of $r^{-1}(U)$. Therefore, $q^{-1}(V)$ equals the union of those slices U_α that are mapped by q into V. Therefore, $q^{-1}(V)$ equals the union of those slices U_α that are mapped by q into V. Let $p_0 : U_\alpha \to U$, $q_0 : U_\alpha \to V$, and $r_0 : V \to U$ be the mappings obtained by restriction. Since p_0 and r_0 are homeomorphisms, so is $q_0 = r_0^{-1} \circ p_0$. Thus V is evenly covered by q. □

Theorem 1.61. *Let $p : E \to B$ be a covering map, with E simply connected. Given any covering map $r : Y \to B$, there is a covering map $q : E \to Y$ such that $r \circ q = p$.*

Proof. Let $b_0 \in B$ and choose e_0 and y_0 such that $p(e_0) = b_0$ and $r(y_0) = b_0$. Since E is simply connected, its fundamental group is trivial. So we have

$$p_*(\pi_1(E, e_0)) \subset r_*(\pi_1(Y, y_0)). \tag{76}$$

Since r is a covering map, by Lemma 1.54, there exists a map $q : E \to Y$ such that $r \circ q = p$ and $q(e_0) = y_0$. Then, by Lemma 1.60, q is a covering map. □

This is why E is called a universal covering space of B; it covers every other covering space of B.

Lemma 1.62. *Let $p : E \to B$ be a covering map such that $p(e_0) = b_0$. If E is simply connected, then b_0 has a neighborhood U such that the inclusion $i : U \to B$ induces the trivial homomorphism*

$$i_* : \pi_1(U, b_0) \to \pi_1(B, b_0). \tag{77}$$

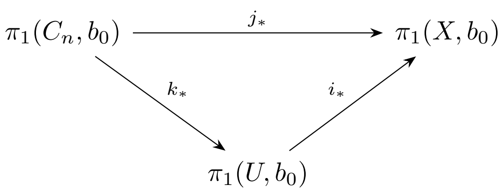
Proof. Let U be a neighborhood of b_0 that is evenly covered by p. Then $p^{-1}(U)$ can be partitioned into disjoint slices. Let U_α be the slice containing e_0. Let f be a loop in U based at b_0. Since p maps U_α homeomorphically onto U, the lifting $\tilde{f}$

of f in U_α beginning at e_0 is a loop based at e_0. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and a constant loop. Then $p \circ \tilde{F}$ is a path homotopy in B between f and a constant loop. Then $\pi_1(U, b_0)$ is trivial. □

Now we give an example of a space that has no universal covering space.

Example 1.63. Let C_n be the circle of radius $1/n$ in the plane with center at the point $(1/n, 0)$. We consider the space X given by the union of the circles C_n. Let b_0 be the origin. We show that if U is any neighborhood of b_0 in X, then the homomorphism of fundamental groups induced by inclusion $i : U \to X$ is not trivial.

Given n, there is a retraction $r : X \to C_n$ obtained by letting r map each circle C_i for $i \neq n$ to the point b_0. Choose n large enough that C_n lies in U. Then in the following diagram of homomorphisms induced by inclusion, j_* is injective, hence i_* cannot be trivial.



Then, even though X is path connected and locally path connected, it has no universal covering space.

1.11. **Existence of covering spaces.**

Definition 1.64. A space B is said to be **semilocally simply connected** if for each $b \in B$, there is a neighborhood U of b such that the homomorphism

$$i_* : \pi_1(U, b) \to \pi_1(B, b) \tag{78}$$

induced by inclusion is trivial.

By Lemma 1.62, semilocally simply connectedness is a necessary condition for E to be simply connected. If this is not the case, then there is no covering space corresponding to the trivial subgroup of $\pi_1(B, b_0)$. Then, semilocal simple connectedness of B is a necessary condition for there to exist, for every conjugacy class of subgroups of $\pi_1(B, b_0)$ a corresponding covering space of B. We now prove that this condition is also sufficient.

Theorem 1.65. *Let B be path connected, locally path connected and semilocally simply connected. Let $b_0 \in B$. Given a subgroup H of $\pi_1(B, b_0)$, there exists a covering map $p : E \to B$ and a point $e_0 \in p^{-1}(b_0)$ such that*

$$p_*(\pi_1(E, e_0)) = H. \tag{79}$$

Proof. Step 1. Construction of E. Let $\mathcal{P}$ denote the set of all paths in B beginning at b_0. We define an equivalence relation on $\mathcal{P}$ by setting $\alpha \sim \beta$ if α and β end at the same point of B and

$$[\alpha * \bar{\beta}] \in H. \tag{80}$$

We show that this is an equivalence relation:

- Reflexivity: Since $[\alpha * \bar{\alpha}]$ is the equivalence class of the constant loop based at b_0, it is an element of H.
- Symmetry: Let $\alpha \sim \beta$. Then $[\alpha * \bar{\beta}] \in H$ and its inverse is $[\beta * \bar{\alpha}]$. So $\beta \sim \alpha$.

- Transitivity: Let $\alpha \sim \beta$ and $\beta \sim \delta$. Then $[\alpha * \bar{\beta}] * [\beta * \bar{\delta}]$ is an element of H. But $[\alpha * \bar{\beta}] * [\beta * \bar{\delta}] = [\alpha * \bar{\delta}]$, so $\alpha \sim \delta$.

We denote the equivalence class of the path α by $\alpha^{\#}$. Let E denote the collection of equivalence classes, and define $p : E \to B$ by

$$p(\alpha^{\#}) = \alpha(1). \tag{81}$$

Since B is path connected there is a path from b_0 to every point of B. Then p is surjective.

If $[\alpha] = [\beta]$, then $[\alpha * \bar{\beta}] = [\alpha] * [\bar{\alpha}]$, which is the identity element, which belongs to H. Then $\alpha^{\#} = \beta^{\#}$. Now, $\alpha * \delta$ and $\beta * \delta$ end at the same point of B, and

$$[(\alpha * \delta) * \overline{(\beta * \delta)}] = [(\alpha * \delta) * (\bar{\delta} * \bar{\beta})] = [\alpha * \bar{\beta}] \in H. \tag{82}$$

So, if $\alpha^{\#} = \beta^{\#}$, then $(\alpha * \delta)^{\#} = (\beta * \delta)^{\#}$ for any path δ in B beginning at $\alpha(1)$.

Step 2. Topologizing E. Let α be any element of $\mathcal{P}$, and let U be any path-connected neighborhood of $\alpha(1)$. Define

$$B(U, \alpha) = \{(\alpha * \delta)^{\#} | \delta \text{ is a path in } U \text{ beginning at } \alpha(1)\}. \tag{83}$$

$\alpha^{\#}$ is an element of $B(U, \alpha)$.

If $\beta^{\#} \in B(U, \alpha)$, then $\beta^{\#} = (\alpha * \delta)^{\#}$ for some path δ in U. Then, by (2), $(\beta * \bar{\delta})^{\#} = ((\alpha * \delta) * \bar{\delta})^{\#}$. But $[(\alpha * \delta) * \bar{\delta}] = [\alpha]$, so, by (1), $(\beta * \bar{\delta})^{\#} = \alpha^{\#}$. Then $\alpha^{\#} \in B(U, \beta)$. We now show that $B(U, \beta) \subset B(U, \alpha)$. Let $(\beta * \gamma)^{\#}$ be an element of $B(U, \beta)$. Since $\beta^{\#} = (\alpha * \delta)^{\#}$, and $[(\alpha * \delta) * \gamma] = [\alpha * (\delta * \gamma)]$ we have

$$(\beta * \gamma)^{\#} = ((\alpha * \delta) * \gamma)^{\#} = (\alpha * (\delta * \gamma))^{\#} \in B(U, \alpha). \tag{84}$$

So $B(U, \beta) \subset B(U, \alpha)$. Similarly, we can prove $B(U, \alpha) \subset B(U, \beta)$, so $B(U, \alpha) = B(U, \beta)$, whenever $\beta^{\#} \in B(U, \alpha)$.

Now we show that the sets $B(U, \alpha)$ form a basis for a topology on E. If $\beta^{\#}$ belongs to the intersection $B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$, then we choose a path-connected neighborhood V of $\beta(1)$ contained in $U_1 \cap U_2$. Then

$$B(V, \beta) \subset B(U_1, \beta) \cap B(U_2, \beta) = B(U_1, \alpha_1) \cap B(U_2, \alpha_2). \tag{85}$$

Step 3. The map p is continuous and open. Since $p(\alpha^{\#}) = \alpha(1)$, $p(B(U, \alpha)) \in U$. Given $x \in U$, choose a path δ in U from $\alpha(1)$ to x. Then $(\alpha * \delta)^{\#} \in B(U, \alpha)$ and $p((\alpha * \delta)^{\#}) = x$. Then p maps the open basis element $B(U, \alpha)$ onto the open subset U of B, so p is open.

To show that p is continuous, take an element $\alpha^{\#}$ of E and a neighborhood W of $p(\alpha^{\#})$. Choose a path-connected neighborhood U of the point $p(\alpha^{\#}) = \alpha(1)$ in W. Then $B(U, \alpha)$ is a neighborhood of $\alpha^{\#}$ that is mapped by p into W. Thus p is continuous.

Step 4. Every point of B has a neighborhood that is evenly covered by p. Given $b_1 \in B$, choose U to be a path-connected neighborhood of b_1 that satisfies the condition that the homomorphism $\pi_1(U, b_1) \to \pi_1(B, b_1)$ induced by inclusion is trivial (this is possible because B is semilocally simply connected). We want to show that U is evenly covered by p.

First, we show that $p^{-1}(U)$ equals the union of the sets $B(U, \alpha)$, as α ranges over all paths in B from b_0 to b_1. Since p maps each set $B(U, \alpha)$ onto U, then $p^{-1}(U)$ contains this union. If $\beta^{\#}$ belongs to $p^{-1}(U)$, then $\beta(1) \in U$. Choose a path δ in U from b_1 to $\beta(1)$ and let α be the path $\beta * \bar{\delta}$ from b_0 to b_1. Then $[\beta] = [\alpha * \delta]$, so $\beta^{\#} = (\alpha * \delta)^{\#} \in B(U, \alpha)$. Thus $p^{-1}(U)$ is contained in the union of the sets $B(U, \alpha)$, so $p^{-1}(U)$ is equal to this union.

Second, if $\beta^{\#} \in B(U, \alpha_1) \cap B(U, \alpha_2)$, then $B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2)$. So the sets $B(U, \alpha)$ are disjoint.

Third, we show that p defines a bijective map of $B(U,\alpha)$ with U. From step 3 we know that p is surjective. To prove injectivity, suppose that

$$p((\alpha * \delta_1)^{\#}) = p((\alpha * \delta_2)^{\#}), \tag{86}$$

where δ_1 and δ_2 are paths in U. Then $\delta_1(1) = \delta_2(1)$. Because B is semilocally simply connected, the homomorphism $\pi_1(U, b_1) \to \pi_1(B, b_1)$ induced by inclusion is trivial. Then, $\delta_1 * \bar{\delta}_2$ is path homotopic in B to the constant loop. Then $[\alpha * \delta_1] = [\alpha * \delta_2]$, so $(\alpha * \delta_1)^{\#} = (\alpha * \delta_2)^{\#}$. Thus p is bijective. Since it is also continuous and open, $p|B(U,\alpha)$ is a homeomorphism, and thus p is a covering map. We also show that E is path connected.

Step 5. Lifting a path in B. Let e_0 denote the equivalence class of the constant path at b_0. Then, by definition, $p(e_0) = b_0$. Given a path α in B beginning at b_0, we calculate its lift to a path in E beginning at e_0 and show that this lift ends at $\alpha^{\#}$.

Given $c \in [0,1]$, let $\alpha_c : I \to B$ denote the path defined by

$$\alpha_c(t) = \alpha(tc) \text{ for } 0 \le t \le 1. \tag{87}$$

Then α_c is the "portion" of α that runs from $\alpha(0)$ to $\alpha(c)$. We define $\tilde{\alpha} : I \to E$ by

$$\tilde{\alpha}(c) = (\alpha_c)^{\#} \tag{88}$$

and we show that it is continuous. For this we introduce the following notation. Given $0 \le c < d \le 1$, let $\delta_{c,d}$ denote the path that equals the positive linear map of I onto $[c,d]$ followed by α. Then the paths α_d and $\alpha_c * \delta_{c,d}$ are path homotopic because one is just a reparametrization of the other.

To verify the continuity of $\tilde{\alpha}$ at the point $c \in [0,1]$, let W be a basis element in E about the point $\tilde{\alpha}(c)$. Then $W = B(U, \alpha_c)$ for some path-connected neighborhood U of $\alpha_c(1) = \alpha(c)$. Choose $\epsilon > 0$ such that for $|c-t| < \epsilon$, the point $\alpha(t)$ lies in U. We show that if d is a point of $[0,1]$ with $|c-d| < \epsilon$, then $\tilde{\alpha}(d) \in W$. This proves continuity of $\tilde{\alpha}$ at c.

Suppose $|c-d| < \epsilon$, and $d > c$. Set $\delta = \delta_{c,d}$. Then, since $[\alpha_d] = [\alpha_c * \delta]$, we have

$$\tilde{\alpha}(d) = (\alpha_d)^{\#} = (\alpha_c * \delta)^{\#}. \tag{89}$$

Since δ lies in U, $\tilde{\alpha}(d) \in B(U, \alpha_c)$. If $d < c$, set $\delta = \delta_{d,c}$. Then the proof is similar.

Step 6. E is path connected. If $\alpha^{\#}$ is any point of E, then the lift $\tilde{\alpha}$ of α is a path in E from e_0 to $\alpha^{\#}$.

Step 7. $H = p_*(\pi_1(E, e_0))$. Let α be a loop in B at b_0. Let $\tilde{\alpha}$ be its lift to E beginning at e_0. By Theorem 1.52, $[\alpha] \in p_*(\pi_1(E, e_0))$ if and only if $\tilde{\alpha}$ is a loop in E. The final point of $\tilde{\alpha}$ is the point $\alpha^{\#}$, and $\alpha^{\#} = e_0$ if and only if α is equivalent to the constant path at b_0, that is, if and only if $[\alpha * \bar{e}_{b_0}] \in H$, which means $[\alpha] \in H$. □

Lemma 1.62 and Theorem 1.65 give the following:

Corollary 1.66. *The space B has a universal covering space if and only if B is path connected, locally path connected, and semilocally simply connected.*

2. The higher homotopy groups of a topological space and their basic properties

2.1. **Homotopy groups.** In this chapter we shall use the notation f_t for homotopies between two maps, say $f_0(t) = f(t)$ and $f_1(t) = f'(t)$.

Let I^n be the $n-$dimensional unit cube. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1. We define higher homotopy groups as follows:

Definition 2.1. For a topological space X with a base point x_0, the $\mathbf{n^{th}}$ **homotopy group** $\pi_n(X, x_0)$ is defined as the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t, together with an operation that we define next.

We generalize the product of paths from the previous chapter to an operation between maps as

$$(f+g)(s_1, s_2, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n), & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, ..., s_n), & s_1 \in [\frac{1}{2}, 1] \end{cases}. \tag{90}$$

This operation induces a well-defined operation between the homotopy classes we consider, given by

$$[f] + [g] = [f + g]. \tag{91}$$

Note that the requirement that $f_t(\partial I^n) = x_0$ for all t makes possible defining this operation on all the elements of a higher homotopy group.

The identity element of $\pi_n(X, x_0)$ is the constant map sending I^n to x_0, and the inverse of an element $[f]$ is given by $[-f]$, where $-f(s_1, s_2, ..., s_n) = f(1 - s_1, s_2, ..., s_n)$.

The reason we use the additive notation for the group operation is because $\pi_n(X, x_0)$ is abelian for $n \geq 2$. We can see that $f + g \simeq g + f$ by the homotopy indicated in the figure:

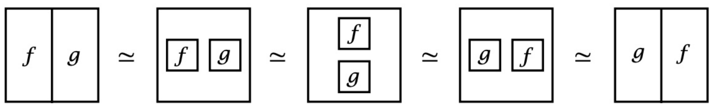


Figure 4. Homotopy between $f + g$ and $g + f$

Another way of defining higher order homotopy groups is by noting that maps from $(I^n, \partial I^n)$ to (X, x_0) are the same as maps of the quotient $I^n/\partial I^n = S^n$ to X taking the basepoint $s_0 = \partial I^n/\partial I^n$ to x_0. So we can view $\pi_n(X, x_0)$ as homotopy classes of these maps, where homotopies are through maps of the same form $(S^n, s_0) \to (X, x_0)$. The wedge sum of two topological spaces $X \vee Y$ is obtained by identifying $x_0 \in X$ and $y_0 \in Y$ as a single point and taking the union of X and Y. With this, the sum $f + g$, in this interpretation of homotopy groups, is the composition $c \circ f \vee g : S^n \to X$ where $c : S^n \to S^n \vee S^n$ collapses the equator S^{n-1} in S^n to a point and we choose the base point s_0 to lie in this S^{n-1}.

Proposition 2.2. *If X is path-connected, different choices of the base point x_0 produce isomorphic groups $\pi_n(X, x_0)$.*

This justifies writing $\pi_n(X)$ instead of $\pi_n(X, x_0)$ in these cases. We already saw that this is true for the fundamental homotopy group in Corollary 1.11. Now we prove for the case $n > 1$.

Proof. Let $\gamma : I \to X$ be a path in X from $\gamma(0) = x_0$ to $\gamma(1) = x_1$. To each map $f : (I^n, \partial I^n) \to (X, x_1)$ we associate a map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n.

A homotopy of γ or f through maps fixing ∂I or ∂I^n, respectively, yields a homotopy of γf through maps $(I^n, \partial I^n) \to (X, x_0)$. We also have the following:

(1) $\gamma(f + g) \simeq \gamma f + \gamma g$.
(2) $(\gamma\eta)f \simeq \gamma(\eta f)$.
(3) $1f \simeq f$, where 1 denotes the constant path.

In order to obtain the homotopy for (1), we deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we call $f + 0$ and $0 + g$, then we excise progressively wider symmetric middle slabs of $\gamma(f + 0) + \gamma(0 + g)$ until we obtain $\gamma(f + g)$. This process is shown in the figure:

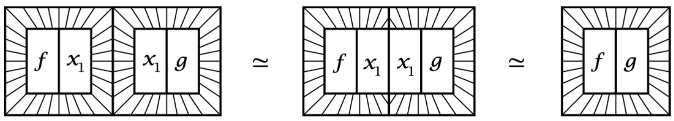


FIGURE 5. Homotopy between $\gamma(f + g)$ and $\gamma f + \gamma g$

We define a change of base point transformation $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ by $\beta_\gamma([f]) = [\gamma f]$. By (1), we have

$$\beta_\gamma([f] + [g]) = \beta_\gamma([f + g]) = [\gamma(f + g)] = \beta_\gamma([f]) + \beta_\gamma([g]), \tag{92}$$

so β_γ is a homomorphism. Let $\bar{\gamma}(s) = \gamma(1 - s)$. Then $\beta_{\bar{\gamma}}$ is an inverse of β_γ. This can be seen from

$$\beta_{\bar{\gamma}}(\beta_\gamma([f])) = [(\bar{\gamma}\gamma)f] = [f]. \tag{93}$$

So β_γ is a group isomorphism, and different choices of base points of a path-connected space give isomorphic homotopy groups. □

Definition 2.3. Let γ be a loop based at x_0. Since $\beta_{\gamma\eta} = \beta_\gamma \beta_\eta$, the association $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1(X, x_0)$ to $Aut(\pi_n(X, x_0))$, the group of automorphisms of $\pi_n(X, x_0)$. We call this homomorphism the **action of π_1 on π_n**.

We see that each element γ of π_1 acts as an automorphism $[f] \mapsto [\gamma f]$ of π_n. For $n = 1$ this is the action of π_1 on itself by inner automorphisms.

Definition 2.4. A space with trivial π_1 action on π_n is called **n-simple**. If a space is n-simple for all n it is called **simple**.

A map $\phi : (X, x_0) \to (Y, y_0)$ induces a well-defined map $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ given by $\phi_*([f]) = [\phi \circ f]$. We have

$$\phi_*([f] + [g]) = \phi_*([f + g]) = \phi_*([f]) + \phi_*([g]) \tag{94}$$

so ϕ_* is a homomorphism. Also $(\phi\psi)_*([f]) = [\phi\psi f] = \phi_*\psi_*([f])$, and $id_* = id$. Giving the action of π_n on ϕ by $\pi_n(\phi) = \phi_*$, we observe that π_n is a functor $\pi_n : \mathbf{Top}_* \to \mathbf{Grp}$.

Proposition 2.5. *A covering map $p : (E, e_0) \to (B, b_0)$ induces isomorphisms $p_* : \pi_n(E, e_0) \to \pi_n(B, b_0)$ for all $n \geq 2$.*

Proof. For $n \geq 2$, $\pi_1(S^n, s_0)$ is trivial. Let $[f] \in \pi_n(B, b_0)$, then f is a map $(S^n, s_0) \to (B, b_0)$. We have $f_*(\pi_1(S^n, s_0)) = 0$ so $f_*(\pi_1(S^n, s_0)) \subset p_*(\pi_1(E, e_0))$.

By Lemma 1.54, there exists $\tilde{f} : (S^n, s_0) \to (B, b_0)$ such that $p_*([\tilde{f}]) = [p\tilde{f}] = [f]$, proving surjectivity.

The proof of injectivity is similar to that of Theorem 1.52, point (1). □

Proposition 2.6. *For a product $\prod_\alpha X_\alpha$ of an arbitrary collection of path connected spaces X_α there are isomorphisms $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for all n.*

Proof. Any map $f : S^n \to \prod_\alpha X_\alpha$ can be thought of as a collection of paths $f_\alpha : S^n \to X_\alpha$. The correspondence between f and the collection f_α is obviously one-to-one and onto. We have $[f] \in \pi_n(\prod_\alpha X_\alpha)$ and $\prod_\alpha [f_\alpha] \in \prod_\alpha \pi_n(X_\alpha)$, so the two groups are isomorphic. □

Definition 2.7. Let X be a topological space, $A \subset X$ a subspace and $x_0 \in A$. Let I^{n-1} be the face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$, that is, the union of the remaining faces. We define the **relative homotopy groups** $\pi_n(X, A, x_0)$ for $n \geq 1$, as the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ with homotopies through maps of the same form.

The sum operation in $\pi_n(X, A, x_0)$ is defined by the same formulas as for $\pi_n(X, x_0)$, except that the coordinate s_n does not enter this operation. For $n = 1$, $I^1 = [0, 1]$, $I^0 = \{0\}$, and $J^0 = \{1\}$, so $\pi_1(X, A, x_0)$ is the set of homotopy classes of paths in X from a varying point in A to the fixed base point $x_0 \in A$. By the operation on homotopy classes we defined in the previous chapter, $\pi_1(X, A, x_0)$, in general, is not a group.

Since collapsing J^{n-1} to a point converts $(I^n, \partial I^n, J^{n-1})$ into (D^n, S^{n-1}, s_0), we can give an alternative definition for the relative homotopy groups, just as we did for general homotopy groups.

Definition 2.8. $\pi_n(X, A, x_0)$ is the set of homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. From this viewpoint, addition of mappings is done via the map $c : D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Definition 2.9. A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is called a **homotopy relative to** A, or more concisely, a homotopy rel A.

Proposition 2.10. ***(Compression criterion).*** *A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ iff it is homotopic rel S^{n-1} to a map with image contained in A.*

Proof. Let g be a map with image contained in A that is homotopic rel A to f. Then $[f] = [g]$ in $\pi_n(X, A, x_0)$. Let r be a deformation retraction of D^n to s_0. Then $g \simeq g \circ r$, which is the constant map at x_0, so $[g] = 0$.

Now, let $[f] = 0$ via a homotopy $F : D^n \times I \to X$. Then, by restricting F to a family of cylinders with the top cover, $D^n \times \{t\} \cup S^{n-1} \times [0, t]$ for $t \in [0, 1]$, we obtain a homotopy from f to a map into A, which is stationary on S^{n-1}. □

A map $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces maps $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms for $n \geq 2$ and have the properties: $(\phi\psi)_* = \phi_*\psi_*$, $id_* = id$, and $\phi_* = \psi_*$ if $\phi \simeq \psi$ through maps $(X, A, x_0) \to (Y, B, y_0)$.

Now consider the sequence

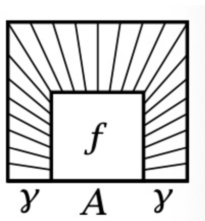
$$... \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to ... \to \pi_0(X, x_0),$$

where i and j are inclusions $(A, x_0) \to (X, x_0)$ and $(X, x_0, x_0) \to (X, A, x_0)$, respectively, and the map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to S^{n-1}. ∂ is called the boundary map and is a homomorphism when $n > 1$.

Theorem 2.11. *This sequence is exact.*

Example 2.12. Let CX be the cone on a path connected space X, that is, the quotient space of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point. We can view X as the subspace $X \times \{1\} \subset CX$. Since CX is contractible, all of its homotopy groups are trivial and the long exact sequence of homotopy groups for the pair (CX, X) gives isomorphisms $\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0)$ for all $n \geq 1$. Then, any group G can be realized as a relative π_2 group by choosing X such that $\pi_1(X) \cong G$.

Similar as before, we can construct isomorphisms that change the basepoint for relative homotopy groups. We will also denote them by β_γ. To do this, start with a path γ in $A \subset X$ from x_0 to x_1. Then $\beta_\gamma : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$ is defined by $\beta_\gamma([f]) = [\gamma f]$ where γf is defined by restricting the domain of f to a smaller cube with its face I^{n-1} centered in the corresponding face of the larger cube, and with γ inserted on the segments joining the larger cube to the smaller one, as in the picture. The isomorphisms β_γ show that $\pi_n(X, A, x_0)$ is independent of x_0 when A is path-connected. In such cases we write $\pi_n(X, A)$ instead of $\pi_n(X, A, x_0)$.



Restricting to loops at the base point, the association $\gamma \mapsto \beta_\gamma$ defines an action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$ analogous to the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Definition 2.13. A space X with base point x_0 is said to be **n-connected** if $\pi_i(X, x_0) = 0$ for all $i \leq n$.

Thus a 0-connected space is path connected and a 1-connected space is simply connected. The following are equivalent

(1) Every map $S^i \to X$ is homotopic to a constant map.
(2) Every map $S^i \to X$ extends to a map $D^{i+1} \to X$.
(3) $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.

Thus, the condition of a space being $n-$connected can be expressed without mention of a base point.

Similarly, in the case of relative homotopy groups, for $i > 0$, the following are equivalent:

(1) Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic rel ∂D^i to a map $D^i \to A$.
(2) Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$.
(3) Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$.
(4) $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

The pair (X, A) is called n-connected if (1)-(4) hold for all $i \leq n$, $i > 0$, and $(1) - (3)$ hold for $i = 0$. Note that X is n-connected iff (X, x_0) is n-connected for some x_0 and hence for all x_0.

2.2. **Cellular approximation.**

Definition 2.14. A pair (X, A) consisting of a topological space and a subset has the **homotopy extension property** with respect to a space Y if for every map $f : X \to Y$ and homotopy $h : A \times I \to Y$ such that $h(a, 0) = f(a)$, there is a homotopy $H : X \times I \to Y$ such that $H(X, 0) = f(x)$ and $H(a, t) = h(a, t)$ for $(a, t) \in A \times I$.

Proposition 2.15. *If the pair* (X, A) *satisfies the homotopy extension property and* A *is contractible, then the quotient map* $q : X \to X/A$ *is a homotopy equivalence.*

Definition 2.16. A **CW complex** is a space X constructed in the following way:

(1) Start with a discrete set X^0, the 0-cells of X.
(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$.
(3) $X = \cup_n X^n$ with the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

A **subcomplex** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X. A pair (X, A) consisting of a cell complex X and a subcomplex A is called a **CW pair**.

The letters "C" and "W" in CW complex stand for closure finiteness and weak topology, properties of these complexes. Closure finiteness means that the closure of each cell meets only finitely many other cells.

Lemma 2.17. ***(Extension)****. Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

If X and Y are CW complexes, then their product space $X \times Y$ is a CW complex. We also state the following results without proof:

Theorem 2.18. *Let A be a subcomplex of a CW complex X and let V be a neighborhood of A. Then there is an open set $U \subset X$ such that $A \subset U \subset V$ and A is a deformation retract of U.*

Lemma 2.19. *Let (X, A) be a pair such that A is closed in X and $X \times I$ is a normal space, that is, it is Hausdorff and its disjoint closed sets can be separated by disjoint neighborhoods. If there is a neighborhood U of which $(X \times \{0\}) \cup (A \times I)$ is a retract, then any map $H' : (X \times \{0\}) \cup (A \times I) \to Y$ extends over $X \times I$.*

Lemma 2.20. *Let V be open in $\mathbb{R}^n$ with $\bar{V}$ compact. Let $g : \bar{V} \to D^m$ be a continuous map such that $g^{-1}(D^m - \partial D^m) \subset V$. Then if $n < m$, g is homotopic to a map g' relative to $\partial V = \bar{V} - V$ which omits an interior point of D^m.*

Theorem 2.21. ***(Homotopy extension theorem)****. Let (X, A) be a CW pair. Then (X, A) has the homotopy extension property with respect to any space Y.*

Proof. Suppose that $f : X \to Y$ is a map and $h : A \times I \to Y$ is a homotopy such that $h(a, 0) = f(a)$. Define $H' : (X \times \{0\}) \cup (A \times I) \to Y$ by $H'|(A \times I) = h$ and $H'(x, 0) = f(x)$. Since I is a CW complex composed of two 0-cells and one 1-cell, $X \times I$ is also a CW complex. Then by Theorem 2.18, the subcomplex $(X \times \{0\}) \cup (A \times I)$ has a neighborhood of which it is a deformation retract. By Lemma 2.19, H' extends to a map $H : X \times I \to Y$. □

Definition 2.22. Let X and Y be cell complexes. A map $f : X \to Y$ is said to be **cellular** provided that, for each n, $f(X^n) \subset Y^n$.

Theorem 2.23. ***(Cellular approximation theorem)****. Let X and Y be CW complexes, let A be a subcomplex of X and let $f : X \to Y$ be a continuous map such that $f|A$ is cellular. Then there is a cellular map $g : X \to Y$ and a homotopy H of f and g such that $H(a, t) = f(a) = g(a)$ for $a \in A$.*

Proof. We will construct g and the homotopy H of f with g inductively over the subcomplexes $A \cup X^n$. For $n = 0$, define $g^0|A = f|A$, and for any 0-cell $\sigma \subset X - A$, define $g^0(\sigma)$ to be any 0-cell in the path component of $f(\sigma)$. Then g^0 is cellular. Define the homotopy H^0 of $f|(A \cup X^0)$ with g^0 to be $H(a, t) = f(a)$ for $a \in A$, and $H^0|(\sigma \times I)$ to be any path between $f(\sigma)$ and $g^0(\sigma)$ for σ a 0-cell in $X - A$.

Suppose we have constructed $g^{n-1} : A \cup X^{n-1} \to Y$ so that g^{n-1} is cellular, and we have constructed a homotopy H^{n-1} of $f|(A \cup X^{n-1})$ with g^{n-1} such that $H^{n-1}(a,t) = f(a) = g^{n-1}(a)$ for $a \in A$. Define

$$H' : ((A \cup X^{n-1}) \times I) \cup (X^n \times \{0\}) \to Y \tag{95}$$

by

$$H'|((A \cup X^{n-1}) \times I) = H^{n-1}, \qquad H'|(X^n \times \{0\}) = f. \tag{96}$$

By the homotopy extension theorem this extends to

$$\bar{H} : (A \cup X^n) \times I \to Y. \tag{97}$$

Then $\bar{H}|((A \cup X^n) \times \{1\}) = g'$ is a map which extends g^{n-1} over $A \cup X^n$, but need not map n-cells into n-cells.

Let σ be an n-cell with interior in $X - A$, let g' denote $g'|\sigma$, and suppose g' maps $(\sigma, \partial\sigma)$ into (Y, Y^{n-1}). Since $g'(\sigma)$ is compact, its carrier is a finite complex, and thus we may assume that Y is the carrier of $g'(\sigma)$ and Y is a finite complex. Let τ be any cell of top dimension in Y, where we assume that the dimension of τ is greater than that of σ.

Since $g'(\partial\sigma) \subset Y^{n-1}$, it follows that $(g')^{-1}(\tau - \partial\tau) \subset \sigma - \partial\sigma$. Thus if $y \in g'(\sigma) \cap (\tau - \partial\tau)$, then $(g')^{-1}(y)$ is closed and compact in σ. Since it is a subset of $\sigma - \partial\sigma$ it is compact in $\sigma - \partial\sigma$. Let D^m be a set containing y and that is contained in $\tau - \partial\tau$ homeomorphic to a closed disc. Let $V = (g')^{-1}(D^m - \partial D^m)$. Then $\bar{V} \subset \sigma$, and $g' : \bar{V} \to D^m$. If $n < m$, by Lemma 2.20 we conclude that g' is homotopic relative to $\sigma - V$ to a map which omits a point of $\tau - \partial\tau$. Thus g' is homotopic relative to $\sigma - V$ to a map which does not intersect $\tau - \partial\tau$. □

The n-sphere can be given a CW structure with exactly two cells in each dimension, obtained inductively by attaching two n-dimensional hemispheres to the $(n-1)$-sphere regarded as the equator in the n-sphere. With this we have the following corollary to the cellular approximation theorem:

Corollary 2.24. $\pi_n(S^k) = 0$ *for* $n < k$.

Proof. Considering the 0-cells of S^n and S^k as base points, every continuous map $S^n \to S^k$ preserving the base point has a homotopy to a cellular map which is constant if $n < k$. □

2.3. Whitehead's theorem.

Lemma 2.25. ***(Compression lemma).*** *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \varnothing$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.*

Theorem 2.26. ***(Whitehead's theorem).*** *If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, then X is a deformation retract of Y.*

Proof. If f is the inclusion of a subcomplex, consider the long exact sequence of homotopy groups for the pair (Y, X). Since X and Y are connected CW complexes they are path-connected, so $\pi_0(X) = 0$ and $\pi_0(Y) = 0$. Since f induces isomorphisms on all homotopy groups, the relative groups $\pi_n(Y, X)$ are all zero. By the preceding lemma, the identity map $(Y, X) \to (Y, X)$ is homotopic to a deformation retraction.

Let M_f be the mapping cylinder of the map $f : X \to Y$, that is, the quotient space of the disjoint union of $X \times I$ and Y under the identifications $(x, 1) \sim f(x)$.

Thus M_f contains both $X = X \times \{0\}$ and Y as subspaces, and M_f deformation retracts onto Y. The map f becomes the composition of the inclusion $X \hookrightarrow M_f$ with the retraction $M_f \to Y$. This retraction is a homotopy equivalence. We show that M_f deformation retracts onto X if the relative groups $\pi_n(M_f, X)$ are all zero.

If f is cellular, then (M_f, X) is a CW pair and by an argument similar to the one in the first paragraph of the proof, if $\pi_n(M_f, X) = 0$ for all n, then M_f deformation retracts onto X. If f is not cellular, by the preceding lemma we have a homotopy rel X of the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to a map into X. Since the pair $(M_f, X \cup Y)$ has the homotopy extension property, it extends to a homotopy from the identity map of M_f to a map $g : M_f \to M_f$ taking $X \cup Y$ into X. Applying the preceding lemma again to the composition $(X \times I \sqcup Y, X \times \partial I \sqcup Y) \to (M_f, X \cup Y) \xrightarrow{g} (M_f, X)$ gives a deformation retraction of M_f onto X. Then the composition of the inclusion $X \hookrightarrow M_f$ with the retraction $M_f \to Y$, which is equal to f, is a homotopy equivalence. □

2.4. **CW approximation.**

Definition 2.27. A map $f : X \to Y$ is called a **weak homotopy equivalence** if it induces isomorphisms $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all choices of base point x_0.

With this we can restate Whitehead's theorem as saying that a weak homotopy equivalence between CW complexes is a homotopy equivalence. This also holds for spaces homotopy equivalent to CW complexes since homotopy equivalence is an equivalence relation.

Definition 2.28. Given a pair (X, A) where the subspace $A \subset X$ is a nonempty CW complex, an **n-connected CW model** for (X, A) is an n-connected CW pair (Z, A) and a map $f : Z \to X$ with $f|A$ the identity, such that $f_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$, for all choices of basepoint.

Since (Z, A) is n-connected, the map $\pi_i(A) \to \pi_i(Z)$ is an isomorphism for $i < n$ and a surjection for $i = n$. The maps $A \hookrightarrow Z \xrightarrow{f} X$ induce a composition $\pi_n(A) \to \pi_n(Z) \to \pi_n(X)$ factoring the map $\pi_n(A) \to \pi_n(X)$ as a surjection followed by an injection. Taking $n = 0$ we obtain a **CW approximation** as a weak homotopy equivalence for a space X, $f : Z \to X$, where Z is a CW complex.

Proposition 2.29. *For every pair (X, A) with A a nonempty CW complex there exist n-connected CW models $f : (Z, A) \to (X, A)$ for all $n \geq 0$, and these models can be chosen to have the additional property that Z is obtained from A by attaching cells of dimension greater than n.*

Example 2.30. When X is path-connected and A is a point, the construction of a 0-connected CW model for (X, A) gives a CW approximation to X with a single 0-cell and all higher cells attached by basepoint-preserving maps. In particular, any connected CW complex is homotopy equivalent to a CW complex with these properties.

Corollary 2.31. *If (X, A) is an n-connected CW pair, then there exists a CW pair $(Z, A) \simeq (X, A)$ rel A such that all cells of $Z - A$ have dimension greater than n.*

Proof. Let $f : (Z, A) \to (X, A)$ be an n-connected CW approximation given by the preceding proposition. By definition, for $i > n$ f induces isomorphisms $\pi_i(Z) \cong \pi_i(X)$. For $i < n$, since both inclusions $A \hookrightarrow Z$ and $A \hookrightarrow X$ induce isomorphisms on homotopy groups, we also have $\pi_i(Z) \cong \pi_i(X)$. For $i = n$, f induces an injection on π_n by definition, and since the inclusion $A \hookrightarrow X$ induces a surjection on π_n, so

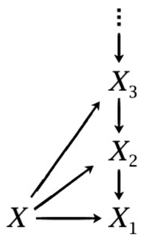


FIGURE 6. Postnikov tower

does f via the composition $\pi_n(A) \to \pi_n(Z) \to \pi_n(X)$. So f induces isomorphisms $\pi_i(Z) \cong \pi_i(X)$ for all i and is a homotopy equivalence.

Let W be a quotient space of the mapping cylinder M_f formed by collapsing each segment $\{a\} \times I$ to a point, for $a \in A$. Assuming f has been made cellular, W is a CW complex containing X and Z as subcomplexes, and W deformation retracts to X just as M_f does. Also, $\pi_i(W, Z) = 0$ for all i since f induces isomorphisms on all homotopy groups, so W deformation retracts onto Z. These two deformation retractions of W onto X and Z are stationary on A, hence they give a homotopy equivalence $X \simeq Z$ rel A. □

Example 2.32. **Postnikov Towers.** Let X be a connected CW complex. We construct a sequence of spaces X_n such that $\pi_i(X_n) \cong \pi_i(X)$ for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$. Choose cellular maps $\varphi_\alpha : S^{n+1} \to X$ generating π_{n+1} and we use these to attach cells e_α^{n+2} to X, forming a CW complex Y. By cellular approximation the inclusion $X \hookrightarrow Y$ induces isomorphisms on π_i for $i \leq n$, and $\pi_{n+1}(Y) = 0$ since any element of $\pi_{n+1}(Y)$ is represented by a map to X by cellular approximation, and such maps are nullhomotopic in Y by construction. We can repeat the process with Y in place of X and n replaced by $n+1$ to make a space with π_{n+2} zero as well as π_{n+1}, by attaching $(n+3)$-cells. After infinitely many iterations we have enlarged X to a CW complex X_n such that the inclusion $X \hookrightarrow X_n$ induces an isomorphism on π_i for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$. This is a special case of the construction of CW models, with (X_n, X) an $(n+1)$-connected CW model for (CX, X) with CX the cone on X.

Since X_{n+1} is obtained from X by attaching cells of dimension $n+3$ and greater, and $\pi_i(X_n) = 0$, by the Extension Lemma 2.17 for $i > n$ the inclusion $X \hookrightarrow X_n$ extends to a map $X_{n+1} \to X_n$. This construction is called a Postnikov tower for X and is represented schematically in the figure.

Proposition 2.33. *Suppose we are given:*

(1) an n-connected CW model $f : (Z, A) \to (X, A)$,
(2) an n'-connected CW model $f' : (Z', A') \to (X', A')$,
(3) a map $g : (X, A) \to (X', A')$.

Then if $n \geq n'$, there is a map $h : Z \to Z'$ such that $h|A = g$ and $gf \simeq f'h$ rel A, so that the following diagram is commutative up to homotopy rel A.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{f'} & X' \end{array}$$

Furthermore, such a map h is unique up to homotopy rel A.

Corollary 2.34. *An n-connected CW model for (X, A) is unique up to homotopy equivalence rel A. In particular, CW approximations to spaces are unique up to homotopy equivalence.*

Proof. Given two n-connected CW models (Z, A) and (Z', A) for (X, A), we apply the proposition twice with g the identity map to obtain maps $h : Z \to Z'$ and $h' : Z' \to Z$. The uniqueness statement gives homotopies $hh' \simeq id$ and $h'h \simeq id$ rel A. □

Let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$ and let $\langle X, Y\rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$. We have the following:

Proposition 2.35. *A weak homotopy equivalence $f : Y \to Z$ induces bijections $[X, Y] \to [X, Z]$ and $\langle X, Y\rangle \to \langle X, Z\rangle$ for all CW complexes X.*

Proof. Replacing Z by the mapping cylinder M_f we may assume f is an inclusion. The groups $\pi_n(Z, Y, y_0)$ are then zero for all n and all basepoints $y_0 \in Y$. Then, by the compression lemma, any map $X \to Z$ can be homotoped to have image in Y. This gives surjectivity of $[X, Y] \to [X, Z]$. The fact that we can deform a homotopy $(X \times I, X \times \partial I) \to (Z, Y)$ to have image in Y leads to the proof of injectivity in the case $[X, Y] \to [X, Z]$.

Consider now $\langle X, Y\rangle \to \langle X, Z\rangle$. We use the same argument as before, but replacing M_f by the reduced mapping cylinder, the quotient of M_f obtained by collapsing the segment $\{y_0\}\times I$ to a point, for y_0 the basepoint of Y. This collapsed segment then serves as the common basepoint of Y, Z, and the reduced mapping cylinder. The reduced mapping cylinder deformation retracts to Z just as M_f, but in this case the basepoint does not move. □

2.5. **Excision for homotopy groups.**

Definition 2.36. For a space X, the **suspension** SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

In the case of maps, we say that $f : X \to Y$ suspends to $Sf : SX \to SY$, the quotient map of $f \times id : X \times I \to Y \times I$.

Theorem 2.37. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A\cap B$. If (A, C) is m-connected and (B, C) is n-connected, with $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.*

This gives the following:

Corollary 2.38. ***(Freudenthal suspension theorem).*** *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$. More generally this holds for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ whenever X is an $(n - 1)$-connected CW complex.*

Proof. Consider the suspension SX as the union of two cones C_+X and C_-X intersecting in a copy of X. The suspension map is the same as the map

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX) \tag{98}$$

where the two isomorphisms come from long exact sequences of pairs and the map is induced by inclusion. From the long exact sequence of the pair $(C_\pm X, X)$ we see that this pair is n-connected if X is $(n - 1)$-connected. By the preceding theorem, the middle map is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$. □

Corollary 2.39. *$\pi_n(S^n) \cong \mathbb{Z}$, generated by the identity map, for all $n \geq 1$.*

Proof. From the Freudenthal suspension theorem we have that in the suspension sequence

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to ... \tag{99}$$

the first map is surjective and all the subsequent maps are isomorphisms. Since $\pi_1(S^1)$ is $\mathbb{Z}$ generated by the identity map, it follows that $\pi_n(S^n)$ for $n \geq 2$ is a finite or infinite cyclic group independent of n, generated by the identity map. The proof that this cyclic group is infinite can be done via homology theory, or using the Hopf bundle, and we shall not pursue it here. □

Proposition 2.40. *If a CW pair (X, A) is r-connected and A is s-connected, with $r, s \geq 0$, then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r + s$ and a surjection for $i = r + s + 1$.*

Proof. Consider $X \cup CA$, the complex obtained from X by attaching a cone CA along $A \subset X$. Since CA is a contractible subcomplex of $X \cup CA$, the quotient map $X \cup CA \to (X \cup CA)/CA = X/A$ is a homotopy equivalence by Proposition 2.15. Then we have a commutative diagram where the vertical isomorphism comes from a long exact sequence.

$$\begin{array}{ccccc}
\pi_i(X, A) & \longrightarrow & \pi_i(X \cup CA, CA) & \longrightarrow & \pi_i(X \cup CA/CA) = \pi_i(X/A) \\
 & & \cong \uparrow & \overset{\cong}{\nearrow} & \\
 & & \pi_i(X \cup CA) & &
\end{array}$$

From the exact sequence for the pair (CA, A) we obtain that (CA, A) is $(s+1)$-connected if A is s-connected. By the excision theorem we obtain the desired result. □

Definition 2.41. A space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ is called an **Eilenberg-MacLane space** $K(G, n)$.

We can build a CW complex $K(G, n)$ for arbitrary G and n, assuming G is abelian if $n > 1$, in the following way. Let X be an $(n-1)$-connected CW complex of dimension $n + 1$ such that $\pi_n(X) \cong G$. Then we attach higher-dimensional cells to X to make π_i trivial for $i > n$ without affecting π_n or the lower homotopy groups as in Example 2.32.

By taking products of $K(G, n)$'s for varying n we can then realize any sequence of groups G_n, abelian for $n > 1$, as the homotopy groups π_n of a space.

Lemma 2.42. *Let X be a CW complex of the form $(\vee_\alpha S^n_\alpha) \cup_\beta e^{n+1}_\beta$ for some $n \geq 1$. Then for every homomorphism $\psi : \pi_n(X) \to \pi_n(Y)$ with Y path-connected there exists a map $f : X \to Y$ with $f_* = \psi$.*

Proposition 2.43. *The homotopy type of a CW complex $K(G, n)$ is uniquely determined by G and n.*

Proof. Suppose K and K' are $K(G, n)$ CW complexes. Since homotopy equivalence is an equivalence relation we may assume K is a particular $K(G, n)$, namely one constructed from a space X as in the lemma by attaching cells of dimension $n + 2$ and greater. Then there is a map $f : X \to K'$ inducing an isomorphism on π_n. To extend this f over K we proceed inductively. For each cell e^{n+2}, the composition of its attaching map with f is nullhomotopic in K' since $\pi_{n+1}(K') = 0$, so f extends over this cell. We apply the same argument for all the higher-dimensional cells. The resulting $f : K \to K'$ is a homotopy equivalence since it induces isomorphisms on all homotopy groups. □

2.6. **Fiber bundles.**

Definition 2.44. A **fiber bundle** is a short exact sequence $F \to E \xrightarrow{p} B$ such that all the subspaces $p^{-1}(b) \subset E$, which are called **fibers**, are homeomorphic.

Example 2.45. A trivial example is given by $E = F \times B$ with $p : E \to B$ the projection.

Definition 2.46. A map $p : E \to B$ is said to have the **homotopy lifting property** with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, so $p \circ \tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

Definition 2.47. A **fibration** is a map $p : E \to B$ having the homotopy lifting property with respect to all spaces X.

Example 2.48. A projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\tilde{g}_t(x) = (g_t(x), h(x))$ where $\tilde{g}_0(x) = (g_0(x), h(x))$.

Example 2.49. A fiber bundle with fiber a discrete space is a covering space. Conversely, a covering space whose fibers all have the same cardinality, for example a covering space over a connected base space, is a fiber bundle with discrete fiber.

Definition 2.50. The map $p : E \to B$ is said to have the **homotopy lifting property for a pair** (X, A) if each homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$.

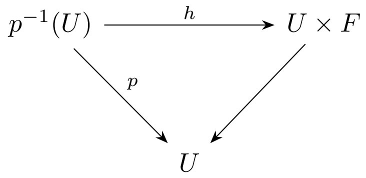
Theorem 2.51. *Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence if B is path connected, there is a long exact sequence*

$$... \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to ... \to \pi_0(E, x_0) \to 0.$$

Proof. First we show that p_* is onto. Let $f : (I^n, \partial I^n) \to (B, b_0)$ represent an element of $\pi_n(B, b_0)$. The constant map to x_0 provides a lift of f to E over the subspace $J^{n-1} \subset I^n$, so the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\tilde{f} : I^n \to E$, and this lift satisfies $\tilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b_0$. Then $\tilde{f}$ represents an element of $\pi_n(E, F, x_0)$ with $p_*([\tilde{f}]) = [f]$ since $p\tilde{f} = f$.

Now we show that p_* is one-to-one. Given $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p_*([\tilde{f}_0]) = p_*([\tilde{f}_1])$, let $G : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. Let $\tilde{G}$ be given by $\tilde{f}_0$ on $I^n \times \{0\}$, $\tilde{f}_1$ on $I^n \times \{1\}$, and the constant map to x_0 on $J^{n-1} \times I$. The relative homotopy lifting property gives an extension of $\tilde{G}$ to a lift $\tilde{G} : I^n \times I \to E$. This is a homotopy $\tilde{f}_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ from $\tilde{f}_0$ to $\tilde{f}_1$, so p_* is injective. □

Definition 2.52. A **fiber bundle** structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ such that the following diagram commutes.



h is called a **local trivialization** of the bundle. The space B is called the **base space** of the bundle, and E is the **total space**.

In the above definition the unlabeled map is the projection onto the first factor. Commutativity of the diagram means that h carries each fiber $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\} \times F$ of F. Since the first coordinate of h is just p, h is determined by its second coordinate, a map $p^{-1}(U) \to F$ which is a homeomorphism on each fiber F_b.