Geometric Cosmology

October 15, 2025

Abstract

This report was prepared as part of my research traineeship in Romania at the Horia Hulubei National Institute for Physics and Nuclear Engineering (IFIN-HH), focusing on the theme of *Geometric Cosmology*.

The exposition begins with a brief review of basic mathematical and differential geometric notions and then moves on to geometric mechanics, presenting the symplectic formulation of Hamiltonian systems as a precursor to the contact-geometric framework developed later.

Subsequent sections transition from symplectic to contact geometry, exploring integrable distributions, the Cartan and contact structures of jet bundles, and their role in the geometric formulation of the Hamilton–Jacobi equation, both in its classical and dissipative forms.

Applications are discussed for contact Hamiltonian models of dissipative and thermodynamical systems, including the damped parametric oscillator and geometric integration of Herglotz-type dynamics. The resulting formalism highlights the geometric encoding of entropy production, thermodynamic irreversibility, and energy-preserving discretizations and concludes with a brief overview of the pertaining aspects found in the work of Lazaroiu, Slupic and Babalic in geometric cosmology.

1 Mathematical Preliminaries

This section was compiled from notes taken during various undergraduate and graduate courses I attended last year in Greece. It is included here both to provide a hopefully more accessible introduction for readers without an extensive mathematical background and as an opportunity to organize and revise my own understanding of the material.

1.1 Multilinear maps

Let $V_1, \dots, V_n$ be finite-dimensional vector spaces over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A map

$$F : V_1 \times \cdots \times V_n \longrightarrow W$$

is *multilinear* if it is linear in each argument separately.

1.2 Component representation

Fix bases $\{e^{(k)}_{i_k}\}_{i_k=1}^{m_k}$ of V_k and let $v_k = \sum_{i_k} x^{(k)}_{i_k} e^{(k)}_{i_k}$. For a multilinear form $F : V_1 \times \cdots \times V_n \to \mathbb{K}$ set

$$a_{i_1 \cdots i_n} := F\Big(e^{(1)}_{i_1}, \dots, e^{(n)}_{i_n}\Big) \in \mathbb{K}.$$

By multilinearity,

$$F(v_1, \dots, v_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} a_{i_1 \cdots i_n}\, x^{(1)}_{i_1} \cdots x^{(n)}_{i_n}.$$

1.3 Construction of the tensor product as a quotient

Let $\mathbb{F}(V_1, \dots, V_n)$ be the free vector space generated by the set $V_1 \times \cdots \times V_n$. Let $R(V_1, \dots, V_n)$ be the subspace spanned by the *multilinearity relations*

$$(v_1, \dots, v_j + v_j', \dots, v_n) - (v_1, \dots, v_j, \dots, v_n) - (v_1, \dots, v_j', \dots, v_n),$$
$$(v_1, \dots, \lambda v_j, \dots, v_n) - \lambda (v_1, \dots, v_j, \dots, v_n),$$

for all $j \in \{1, \dots, n\}$, $v_j, v_j' \in V_j$, and $\lambda \in \mathbb{K}$.

Definition. The *tensor product* of $V_1, \dots, V_n$ is the quotient space

$$V_1 \otimes \cdots \otimes V_n := \mathbb{F}(V_1, \dots, V_n) \Big/ R(V_1, \dots, V_n),$$

and we write $v_1 \otimes \cdots \otimes v_n$ for the class of $(v_1, \dots, v_n)$.

1.4 Elementary tensors and coordinates

An *elementary tensor* is $v_1 \otimes \cdots \otimes v_n$. Every element is a finite sum

$$T = \sum_{i_1,\ldots,i_n} a_{i_1\cdots i_n}\, v^{(1)}_{i_1} \otimes \cdots \otimes v^{(n)}_{i_n}.$$

Given bases $\{e^{(k)}_{i_k}\}$ of V_k, the set $\{e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}\}$ is a basis of $V_1 \otimes \cdots \otimes V_n$, so every tensor has a unique expansion

$$T = \sum_{i_1,\ldots,i_n} a_{i_1\cdots i_n}\, e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}.$$

1.5 Algebraic properties of the tensor product

The tensor product $V_1 \otimes \cdots \otimes V_n$ satisfies the following algebraic rules, which reflect the multilinearity relations used in its construction.

(a) Additivity. For any $v_j, v_j' \in V_j$,

$$v_1 \otimes \cdots \otimes (v_j + v_j') \otimes \cdots \otimes v_n = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j' \otimes \cdots \otimes v_n.$$

(b) Homogeneity. For any scalar $\lambda \in \mathbb{K}$,

$$v_1 \otimes \cdots \otimes (\lambda v_j) \otimes \cdots \otimes v_n = \lambda\,(v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n).$$

1.6 Remarks

(1) The tensor product $V_1 \otimes \cdots \otimes V_n$ is itself a vector space. It is not a subset of the Cartesian product $V_1 \times \cdots \times V_n$, even though its elements are formally built from n-tuples of vectors. Each elementary tensor $v_1 \otimes \cdots \otimes v_n$ represents a new object in a different vector space.

(2) The tensor product is *not* the same as the direct sum:

$$V_1 \oplus \cdots \oplus V_n \;\neq\; V_1 \otimes \cdots \otimes V_n.$$

For example, in the direct sum, addition is componentwise:

$$(v_1, \ldots, v_j + v_j', \ldots, v_n) = (v_1, \ldots, v_j, \ldots, v_n) + (0, \ldots, v_j', \ldots, 0),$$

whereas in the tensor product we have the multilinearity rule

$$v_1 \otimes \cdots \otimes (v_j + v_j') \otimes \cdots \otimes v_n = v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j' \otimes \cdots \otimes v_n.$$

Thus, while $\oplus$ expresses independent combination of components, $\otimes$ encodes multilinear interaction among them.

1.7 Universal property of the tensor product

The tensor product $V_1 \otimes \cdots \otimes V_n$ is characterized by the following universal mapping property.

Definition 1 (Universal property)**.** *There exists a canonical multilinear map*

$$\varphi : V_1 \times \cdots \times V_n \longrightarrow V_1 \otimes \cdots \otimes V_n, \qquad \varphi(v_1, \ldots, v_n) = v_1 \otimes \cdots \otimes v_n,$$

such that for every multilinear map

$$f : V_1 \times \cdots \times V_n \longrightarrow W$$

to a vector space W, there exists a unique linear map

$$\widehat{f} : V_1 \otimes \cdots \otimes V_n \longrightarrow W$$

satisfying

$$f = \widehat{f} \circ \varphi.$$

Diagrammatically,

$$\begin{array}{ccc} V_1 \times \cdots \times V_n & \xrightarrow{\varphi} & V_1 \otimes \cdots \otimes V_n \\ & \searrow^{f} & \downarrow{\scriptstyle \widehat{f}} \\ & & W \end{array}$$

commutes. This property uniquely determines $V_1 \otimes \cdots \otimes V_n$ up to canonical isomorphism.

1.8 Basic identities of the tensor product

The tensor product satisfies the following canonical isomorphisms.

(1) Hom–tensor duality:

$$\mathrm{Hom}(V, W) \;\cong\; V^* \otimes W,$$

where $V^* = \mathrm{Hom}(V, \mathbb{K})$ is the dual space.

(2) Symmetry (commutativity):

$$\tau : V \otimes W \;\xrightarrow{\sim}\; W \otimes V, \qquad \tau(v \otimes w) = w \otimes v.$$

(3) Distributivity over direct sums: For any family $\{V_i\}_{i\in I}$,

$$\left(\bigoplus_{i\in I} V_i\right) \otimes W \;\cong\; \bigoplus_{i\in I}(V_i \otimes W),$$

with the canonical isomorphism sending

$$(v_i)_{i\in I} \otimes w \;\longmapsto\; (v_i \otimes w)_{i\in I}.$$

(4) Endomorphism spaces:

$$\mathrm{End}(V) = \mathrm{Hom}(V, V) \cong V^* \otimes V.$$

(5) Dual tensor product:

$$V^* \otimes W^* \;\cong\; (V \otimes W)^*, \qquad (\alpha_1 \otimes \alpha_2)(v_1 \otimes v_2) = \alpha_1(v_1)\,\alpha_2(v_2),$$

where $\alpha_1 \in V^*$, $\alpha_2 \in W^*$, and the isomorphism is canonical.

(6) Tensoring with the base field:

$$V \otimes \mathbb{K} \;\cong\; V, \qquad v \otimes \lambda \;\longmapsto\; \lambda v.$$

(7) Bases of tensor products: If $\{v_1, \dots, v_m\}$ is a basis of V and $\{w_1, \dots, w_n\}$ a basis of W, then

$$\mathcal{B} = \{\, v_i \otimes w_j : i = 1, \dots, m,\ j = 1, \dots, n \,\}$$

is a basis of $V \otimes W$.

(8) Spaces of multilinear maps:

$$\mathcal{L}(V_1, \dots, V_n; W) = \{\, f : V_1 \times \cdots \times V_n \to W \mid f \text{ multilinear} \,\}.$$

This is the space of all multilinear maps from $V_1 \times \cdots \times V_n$ to W. For $n = 1$, $\mathcal{L}(V_1; W) = \mathrm{Hom}(V_1, W) \cong V_1^* \otimes W$. In general,

$$\mathcal{L}(V_1, \dots, V_n; W) \;\cong\; V_1^* \otimes \cdots \otimes V_n^* \otimes W.$$

1.9 Tensors of Type (r, s)

Definition 2. *Let V be a finite-dimensional vector space over a field $\mathbb{K}$. For nonnegative integers r, s, the space of* tensors of type (r, s) *on V is defined as*

$$\mathcal{T}_s^r(V) := \underbrace{V \otimes \cdots \otimes V}_{r \text{ factors}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ factors}}.$$

Elements of $\mathcal{T}_s^r(V)$ are called tensors of type (r, s).

In particular,

$$\mathcal{T}_0^0(V) = \mathbb{K},$$

that is, tensors of type $(0, 0)$ are simply scalars.

1.10 Tensor Algebra and Components

Definition 3 (Tensor Algebra)**.** *For a finite-dimensional vector space V over $\mathbb{K}$, the* tensor algebra *of V is the graded direct sum*

$$\mathcal{T}(V) = \bigoplus_{r,s \geq 0} \mathcal{T}_s^r(V),$$

where each $\mathcal{T}_s^r(V)$ is the space of tensors of type (r, s).

Given $\alpha \in \mathcal{T}_s^r(V)$ and $\beta \in \mathcal{T}_{s'}^{r'}(V)$, their tensor product

$$\alpha \otimes \beta \in \mathcal{T}_{s+s'}^{r+r'}(V)$$

is defined by concatenating tensor factors.
In particular, if

$$\alpha = (0, \ldots, 0, \alpha, 0, \ldots) \in \mathcal{T}_s^r(V), \quad \beta = (0, \ldots, 0, \beta, 0, \ldots) \in \mathcal{T}_{s'}^{r'}(V),$$

then $\alpha \otimes \beta = (0, \ldots, 0, \alpha \otimes \beta, 0, \ldots)$ belongs to $\mathcal{T}_{s+s'}^{r+r'}(V)$.

1.11 Component Representation

Let $T \in \mathcal{T}_s^r(V)$. Assume V has a basis $\{v_1, \ldots, v_n\}$ and dual basis $\{v^1, \ldots, v^n\}$. Then T can be expressed as

$$T = T_{j_1 \ldots j_s}^{i_1 \ldots i_r} \, v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s},$$

where the coefficients

$$T_{j_1 \ldots j_s}^{i_1 \ldots i_r} = T(v^{j_1}, \ldots, v^{j_s}, v_{i_1}, \ldots, v_{i_r})$$

are the *components* of T in this basis.

The set

$$\left\{v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s} \;:\; 1 \leq i_k, j_\ell \leq n\right\}$$

forms a basis of $\mathcal{T}^r_s(V)$. Hence, if $\dim V = n$, then

$$\dim \mathcal{T}^r_s(V) = n^{r+s}.$$

1.12 Example: Tensors of Type $(1,1)$

Tensors of type $(1,1)$ correspond naturally to endomorphisms:

$$\mathcal{T}^1_1(V) = V \otimes V^* \cong \mathrm{Hom}(V,V).$$

In coordinates,

$$A = a_i^{\,j}\, v_i \otimes v^j,$$

where $(a_i^{\,j})$ are the matrix entries of A relative to the basis $\{v_1, \ldots, v_n\}$ of V.

1.13 Operations with Tensors

1. **Addition.** Tensors of the same type can be added componentwise:

$$\alpha, \beta \in \mathcal{T}^r_s(V) \quad \Rightarrow \quad \alpha + \beta \in \mathcal{T}^r_s(V).$$

   The space $\mathcal{T}^r_s(V)$ is therefore a vector space.

2. **Tensor product.** If $\alpha \in \mathcal{T}^r_s(V)$ and $\beta \in \mathcal{T}^{r'}_{s'}(V)$, their tensor product is

$$\alpha \otimes \beta \in \mathcal{T}^{r+r'}_{s+s'}(V),$$

   obtained by concatenating tensor factors.

1.14 Contraction of Tensors

Contraction lowers one contravariant and one covariant index:

$$C : \mathcal{T}^r_s(V) \;\longrightarrow\; \mathcal{T}^{r-1}_{s-1}(V).$$

If

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s}\, v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s},$$

then the contraction with respect to i_ℓ and j_t is defined by

$$T' = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \, v_{i_1} \otimes \cdots \widehat{v_{i_\ell}} \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \widehat{v^{j_t}} \cdots \otimes v^{j_s} \, \delta^{j_t}_{i_\ell},$$

where the hat indicates omission of the contracted indices. This defines a tensor of type $(r-1, s-1)$.

1.15 Example: The Trace

A common example of contraction is the *trace*. For endomorphisms, we have

$$\mathrm{Hom}(V,V) \cong V^* \otimes V = \mathcal{T}^1_1(V).$$

If $A = a^j_i \, v_j \otimes v^i$, then contraction of A yields

$$\mathrm{Tr}(A) = a^i_i \;\in\; \mathcal{T}^0_0(V) = \mathbb{K}.$$

Hence, the trace is the contraction of a tensor of type $(1,1)$.

1.16 Symmetric and Antisymmetric Tensors

Let $r > 0$ be an integer. We denote by

$$\mathcal{T}^r_0(V) = \mathcal{T}^r(V) = \underbrace{V \otimes \cdots \otimes V}_{r \text{ factors}}, \qquad \mathcal{T}^0_r(V) = \mathcal{T}_r(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{r \text{ factors}}.$$

Let S_r denote the symmetric group on r letters, i.e.

$$S_r = \{\sigma : \{1,\ldots,r\} \to \{1,\ldots,r\} \mid \sigma \text{ bijective}\}, \qquad |S_r| = r!.$$

Each permutation $\sigma \in S_r$ acts linearly on $\mathcal{T}^r(V)$ by permuting tensor factors:

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_r) := v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)}.$$

The sign of σ, denoted $\mathrm{sgn}(\sigma)$, is defined by

$$\mathrm{sgn}(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is even}, \\ -1, & \text{if } \sigma \text{ is odd}. \end{cases}$$

1.17 Symmetrization and Antisymmetrization

Let S_r act on $\mathcal{T}^r(V)$ by permutation of tensor factors as before.

Definition 4 (Symmetrization and Antisymmetrization Operators)**.** *Define linear maps*

$$S_r, A_r : \mathcal{T}^r(V) \longrightarrow \mathcal{T}^r(V)$$

by

$$S_r(\alpha) = \frac{1}{r!}\sum_{\sigma\in S_r} \sigma(\alpha), \qquad A_r(\alpha) = \frac{1}{r!}\sum_{\sigma\in S_r} \operatorname{sgn}(\sigma)\, \sigma(\alpha), \quad \alpha \in \mathcal{T}^r(V).$$

For instance,

$$A_3(v_1\otimes v_2\otimes v_3) = \frac{1}{3!}(v_1\otimes v_2\otimes v_3 - v_1\otimes v_3\otimes v_2 - v_2\otimes v_1\otimes v_3 + v_2\otimes v_3\otimes v_1 + v_3\otimes v_1\otimes v_2 - v_3\otimes v_2\otimes v_1).$$

Similarly,

$$S_2(v_1 \otimes v_2) = \frac{1}{2!}(v_1 \otimes v_2 + v_2 \otimes v_1).$$

The maps S_r and A_r are projection operators on $\mathcal{T}^r(V)$:

$$A_r^2 = A_r, \qquad S_r^2 = S_r, \qquad A_r \circ S_r = S_r \circ A_r = 0.$$

Definition 5 (Symmetric and Antisymmetric Tensors)**.** *Let* $\alpha \in \mathcal{T}^r(V)$*. We say:*

1. α *is* symmetric *if* $S_r(\alpha) = \alpha$,
2. α *is* antisymmetric *if* $A_r(\alpha) = \alpha$.

Definition 6 (Symmetric and Exterior Powers)**.** *The subspace of symmetric tensors of degree* r *is denoted*

$$S_r(\mathcal{T}^r(V)) =: \operatorname{Sym}^r(V),$$

and the subspace of antisymmetric tensors (alternating tensors) is denoted

$$A_r(\mathcal{T}^r(V)) =: \Lambda^r(V).$$

These are respectively called the symmetric power *and the* exterior power *of* V*.*

1.18 Exterior Algebra of Antisymmetric Tensors

Let $\Lambda^r V$ denote the space of alternating (antisymmetric) tensors of degree r. We define a bilinear map, called the *wedge product*,

$$\Lambda^r V \times \Lambda^s V \longrightarrow \Lambda^{r+s} V, \qquad (\alpha, \beta) \longmapsto \alpha \wedge \beta,$$

given by

$$\alpha \wedge \beta = \frac{(r+s)!}{r!s!}\, A_{r+s}(\alpha \otimes \beta),$$

where A_{r+s} denotes the antisymmetrization operator.

This product makes

$$\Lambda V := \bigoplus_{r \geq 0} \Lambda^r V$$

into a *graded algebra*, called the **exterior algebra** of V.

1. **Associativity:** For all $\alpha, \beta, \gamma \in \Lambda V$,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

   Hence ΛV is a graded associative algebra.

2. **Anticommutativity:** For $\alpha \in \Lambda^r V$, $\beta \in \Lambda^s V$,

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha.$$

   In particular, for $\alpha \in \Lambda^1 V = V$,

$$\alpha \wedge \alpha = (-1)^1 \alpha \wedge \alpha = 0.$$

   Indeed,

$$\alpha \wedge \alpha = \frac{2!}{1!1!} A_2(\alpha \otimes \alpha) = (\alpha \otimes \alpha - \alpha \otimes \alpha) = 0.$$

3. **Bilinearity (distributivity).** For all scalars λ,

$$(\alpha + \alpha') \wedge \beta = \alpha \wedge \beta + \alpha' \wedge \beta, \quad \alpha \wedge (\beta + \beta') = \alpha \wedge \beta + \alpha \wedge \beta', \quad (\lambda \alpha) \wedge \beta = \lambda(\alpha \wedge \beta).$$

4. **Nilpotence by degree.** If $\dim V = n$, then $\Lambda^r V = \{0\}$ for $r > n$. Hence $\alpha \wedge \beta = 0$ whenever $\deg \alpha + \deg \beta > n$.

5. **Basis and dimension.** If $(e_1, \ldots, e_n)$ is a basis of V, then

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_r} \ : \ 1 \leq i_1 < \cdots < i_r \leq n \}$$

   is a basis of $\Lambda^r V$ and $\dim \Lambda^r V = \binom{n}{r}$.

6. **Linear–independence test.** Vectors $v_1, \ldots, v_r$ are linearly independent $\iff$ $v_1 \wedge \cdots \wedge v_r \neq 0$.

7. **Determinant identity (covectors).** For $\alpha^1, \ldots, \alpha^r \in V^*$ and $v_1, \ldots, v_r \in V$,

$$(\alpha^1 \wedge \cdots \wedge \alpha^r)(v_1, \ldots, v_r) = \det\Big(\alpha^i(v_j)\Big)_{1 \leq i,j \leq r}.$$

8. **Functoriality.** A linear map $L : V \to W$ induces

$$\Lambda^r L : \Lambda^r V \to \Lambda^r W, \qquad \Lambda^r L(v_1 \wedge \cdots \wedge v_r) = Lv_1 \wedge \cdots \wedge Lv_r,$$

and $\Lambda L(\alpha \wedge \beta) = (\Lambda L\, \alpha) \wedge (\Lambda L\, \beta)$.

1.19 Interior product and pairings

Let V be a vector space and $\Lambda^r(V^*)$ the r-forms on V.

Definition 7 (Interior product)**.** *For $u \in V$, the* interior product *(or contraction)* $i_u : \Lambda^r(V^*) \to \Lambda^{r-1}(V^*)$ *is*

$$(i_u\omega)(v_1, \ldots, v_{r-1}) := \omega(u, v_1, \ldots, v_{r-1}), \qquad \omega \in \Lambda^r(V^*).$$

Basic identities. For $\alpha \in \Lambda^p(V^*)$, $\beta \in \Lambda^q(V^*)$:

$$\begin{aligned}
&\text{(Degree)} \quad i_u : \Lambda^r \to \Lambda^{r-1} \quad (\text{degree } -1).\\
&\text{(Nilpotence)} \quad i_u \circ i_u = 0.\\
&\text{(Graded Leibniz rule)} \quad i_u(\alpha \wedge \beta) = (i_u\alpha) \wedge \beta + (-1)^p\, \alpha \wedge (i_u\beta).
\end{aligned}$$

Coordinate rule. If (e_i) is a basis of V with dual basis (e^i), then for $\omega = e^{i_1} \wedge \cdots \wedge e^{i_r}$ and $e_j \in V$,

$$i_{e_j}\, \omega = \sum_{k=1}^{r} (-1)^{k-1}\, \delta_j^{i_k}\; e^{i_1} \wedge \cdots \wedge \widehat{e^{i_k}} \wedge \cdots \wedge e^{i_r}.$$

Natural pairing. There is a canonical evaluation pairing

$$\langle\, \cdot\,, \cdot\, \rangle : \Lambda^r(V^*) \times \Lambda^r(V) \longrightarrow \mathbb{K}, \qquad \langle \alpha,\, v_1 \wedge \cdots \wedge v_r \rangle := \alpha(v_1, \ldots, v_r).$$

This yields the basis–free isomorphism

$$\Lambda^r(V^*) \;\xrightarrow{\;\sim\;}\; \big(\Lambda^r(V)\big)^*, \qquad \alpha \longmapsto \big[v \mapsto \langle \alpha, v \rangle\big].$$

1.20 Smooth manifolds

Definition and differentiable structure. A (second countable, Hausdorff) topological space M of dimension n is a *smooth manifold* if it admits an atlas $\{(U_\alpha, \varphi_\alpha)\}$

with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ such that all transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are C^∞. A *smooth structure* on M is a maximal C^∞-atlas.

Smooth maps between manifolds. A function $f : M \to N$ is *smooth* if for every pair of charts (U, φ) on M and (V, ψ) on N with $f(U) \subset V$, the coordinate representative

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \longrightarrow \psi(V) \subset \mathbb{R}^n$$

is a C^∞ map. We write $C^\infty(M, N)$ for the set of such maps.

Immersions, submersions, and embeddings. Let $F : M \to N$ be a smooth map between smooth manifolds. At each point $p \in M$, the differential

$$dF_p : T_pM \to T_{F(p)}N$$

is a linear map between tangent spaces.

- F is an *immersion* at p if dF_p is injective.
- F is a *submersion* at p if dF_p is surjective.
- F is an *embedding* if it is an immersion which is also a homeomorphism onto its image (with the subspace topology).

An immersion is locally an embedding, but not necessarily globally. Submersions locally look like projections $\mathbb{R}^m \to \mathbb{R}^n$, while immersions locally look like inclusions $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$.

Example. The inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$ is an embedding. The projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ onto the first n coordinates is a submersion. The map $t \mapsto (\cos t, \sin t)$ from $\mathbb{R} \to S^1$ is an immersion but not an embedding, since it is not injective.

Definition 8 (Closed embedding (topological spaces))**.** *Let $f \colon X \to Y$ be a continuous map between topological spaces. We say that f is an* embedding *if it is a homeomorphism of X onto its image $f(X)$ endowed with the subspace topology from Y. It is a* closed embedding *if, in addition, $f(X)$ is a closed subset of Y. Equivalently: f is a topological embedding and $f(X)$ is closed in Y.*

Definition 9 (Closed embedding (smooth manifolds))**.** *Let $f\colon M \to N$ be a smooth map between smooth manifolds. We say that f is a* closed smooth embedding *if it is an immersion and a homeomorphism of M onto its image $f(M)$ with the subspace topology if $f(M)$ is a closed subset of N.*

Tangent vectors and tangent spaces. For $p \in M$, a *tangent vector at p* is a derivation $X_p : C^\infty(M) \to \mathbb{R}$ at p:

$$X_p(af + bg) = a\,X_p(f) + b\,X_p(g), \qquad X_p(fg) = f(p)\,X_p(g) + g(p)\,X_p(f).$$

The set T_pM of all such derivations is an n-dimensional vector space, the *tangent space at p*. In coordinates $x = (x^1, \dots, x^n)$, the classes of coordinate curves give a basis $\{\partial_{x^1}|_p, \dots, \partial_{x^n}|_p\}$ with

$$\partial_{x^i}\Big|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\Bigg|_{\varphi(p)}.$$

Equivalently, a tangent vector is the velocity class of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$.

Tangent bundle TM. The disjoint union $TM := \bigsqcup_{p\in M} T_pM$ carries a canonical smooth structure making the projection $\pi : TM \to M$, $\pi(p, v) = p$, a smooth rank-n vector bundle. In a chart (U, φ), the induced coordinates on $\pi^{-1}(U)$ are

$$(x^1, \dots, x^n; v^1, \dots, v^n), \qquad v = v^i\,\partial_{x^i}\Big|_p, \quad p \in U,$$

giving a local trivialization $\pi^{-1}(U) \cong \varphi(U) \times \mathbb{R}^n$. A *vector field* is a smooth section $X \in \Gamma(TM)$.

Differential of a smooth map. If $f \in C^\infty(M, N)$ and $p \in M$, the *differential*

$$df_p \equiv f_{*p} : T_pM \longrightarrow T_{f(p)}N$$

is the unique linear map satisfying

$$(df_p X_p)(g) = X_p(g \circ f) \qquad \text{for all } g \in C^\infty(N).$$

In coordinates x on M, y on N, this is the Jacobian:

$$df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial(y^j \circ f)}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{f(p)}.$$

The differential is smooth in (p, v) and obeys the chain rule $d(g \circ f)_p = dg_{f(p)} \circ df_p$.

1.21 Tensor and Vector Fields on Manifolds

Let M be a smooth n–manifold. For $r, s \geq 0$ the *tensor bundle of type* (r, s) is

$$T^r_s M := \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ times}}.$$

A *tensor field of type* (r, s) is a smooth section $T \in \Gamma(T^r_s M)$; pointwise $T_p \in (T_p M)^{\otimes r} \otimes (T^*_p M)^{\otimes s}$.

Definition 10 (Smooth vector bundle)**.** *Let M be a smooth manifold and $n \in \mathbb{N}$. A* smooth vector bundle of rank n over M *is a smooth manifold E together with a smooth surjective map*

$$\pi : E \longrightarrow M$$

such that:

(a) For each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ has the structure of an n-dimensional real vector space.

(b) For every $p \in M$, there exists an open neighborhood $U \subset M$ and a diffeomorphism (called a local trivialization*)*

$$\Phi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

satisfying

$$\pi = \mathrm{pr}_1 \circ \Phi_U,$$

and such that for each $q \in U$, the restriction

$$\Phi_U|_{E_q} : E_q \longrightarrow \{q\} \times \mathbb{R}^n$$

is a linear isomorphism between vector spaces.

(c) On overlaps $U \cap V \neq \varnothing$, the transition map

$$\Phi_V \circ \Phi_U^{-1} : (U \cap V) \times \mathbb{R}^n \longrightarrow (U \cap V) \times \mathbb{R}^n$$

has the form

$$\Phi_V \circ \Phi_U^{-1}(p, v) = (p, g_{VU}(p)v),$$

where $g_{VU} : U \cap V \to GL_n(\mathbb{R})$ is a smooth map (the transition function*).*

A smooth section of $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The collection of all smooth sections is denoted $\Gamma(E)$.

A section of the tangent bundle $\pi : TM \to M$ is called a *vector field*, and one of the cotangent bundle T^*M is a *1-form*.

Examples.

- Vector fields $\mathfrak{X}(M) = \Gamma(TM)$ (type $(1,0)$), 1-forms $\Omega^1(M) = \Gamma(T^*M)$ (type $(0,1)$).
- Differential k-forms $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$: antisymmetric covariant tensors (type $(0,k)$).
- Endomorphism fields $A \in \Gamma(TM \otimes T^*M)$ (type $(1,1)$), Riemannian metrics g (symmetric type $(0,2)$), symplectic forms ω (closed, nondegenerate type $(0,2)$).

Local expression and transformation. In a chart $x = (x^1, \dots, x^n)$ with frame $\{\partial_i\}$ and dual $\{dx^i\}$,

$$T = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$

Under a change of coordinates $x \mapsto y$, the components transform by the tensorial rule

$$\tilde{T}^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} = \frac{\partial y^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{a_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{b_1}} \cdots \frac{\partial x^{j_s}}{\partial y^{b_s}}\, T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}.$$

Symmetric, antisymmetric, and mixed tensors. On each fiber $T^r_{p\,s}M$ we can symmetrize/antisymmetrize any block of indices. For covariant indices, define the symmetrization S and antisymmetrization A by

$$(ST)_{j_1 \cdots j_s} = \frac{1}{s!} \sum_{\sigma \in S_s} T_{j_{\sigma(1)} \cdots j_{\sigma(s)}}, \qquad (AT)_{j_1 \cdots j_s} = \frac{1}{s!} \sum_{\sigma \in S_s} \mathrm{sgn}(\sigma)\, T_{j_{\sigma(1)} \cdots j_{\sigma(s)}},$$

and analogously for contravariant indices. A field is *symmetric* (resp. *antisymmetric*) in a chosen block if it equals its S- (resp. A-) projection. Mixed-type tensors have a prescribed symmetry pattern in specified index blocks.

Operations. If $S \in \Gamma(T^r_s M)$ and $T \in \Gamma(T^{r'}_{s'} M)$,

$$S \otimes T \in \Gamma\Big(T^{r+r'}_{s+s'} M\Big), \qquad \mathrm{contr}^k_\ell(T) \in \Gamma\Big(T^{r-1}_{s-1} M\Big)$$

are defined pointwise by the usual tensor product and the pairing $T_pM \otimes T_p^*M \to \mathbb{R}$. For $\alpha \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, the interior product $i_X\alpha$ is the contraction in one covariant slot.

Pullback/pushforward (remarks). For a smooth $f : M \to N$:

- Covariant tensors (in particular forms) pull back: $f^* : \Gamma(T^0_sN) \to \Gamma(T^0_sM)$.
- Contravariant tensors push forward along *df* where meaningful: $f_* : \Gamma(T^r_0M) \to \Gamma(T^r_0N)$.
- For diffeomorphisms, one combines f^* on covariant and f_* on contravariant indices to act on general (r, s) tensors.

1.22 Integral curves, flows, and Lie derivative

Let $X \in \mathfrak{X}(M)$.

Integral curves. A C^1 curve $\gamma : I \to M$ is an integral curve of X if $\dot\gamma(t) = X_{\gamma(t)}$. In a chart (U, x) with $X = \sum_i X^i\partial_{x^i}$, the components satisfy $\dot x^i(t) = X^i(x(t))$.

Existence–uniqueness; maximality For each $p \in M$ there exists an open interval $I_p \ni 0$ and a unique maximal integral curve $\gamma_p : I_p \to M$ with $\gamma_p(0) = p$. The map $(t, p) \mapsto \gamma_p(t)$ is smooth on its open domain.

Flow of X Define

$$\Phi : D \subset \mathbb{R} \times M \to M, \qquad \Phi(t, p) := \gamma_p(t),$$

on $D := \bigcup_{p\in M} I_p \times \{p\}$. Write $\Phi_t(p) := \Phi(t, p)$. Then

$$\Phi_0 = \mathrm{id}_M, \qquad \Phi_{t+s} = \Phi_t \circ \Phi_s \text{ (where defined)}, \qquad \left.\frac{d}{dt}\right|_{t=0} \Phi_t(p) = X_p,$$

and each Φ_t is a local diffeomorphism. X is *complete* iff $D = \mathbb{R} \times M$ (global flow). Every smooth vector field on a compact manifold is complete.

Proposition 1 (Differentiate along integral curves)**.** *For $f \in C^\infty(M)$ and γ_p,*

$$\frac{d}{dt} f(\gamma_p(t)) = (Xf)(\gamma_p(t)).$$

Definition 11 (Exponential map)**.** *If X is* complete *(e.g. M is compact), the flow is defined for all $t \in \mathbb{R}$; we write*

$$\exp(tX) := \Phi_t \in \mathrm{Diff}(M).$$

It is the unique smooth one–parameter group satisfying

$$\exp(0 \cdot X) = \mathrm{id}_M, \qquad \frac{d}{dt}\exp(tX) = X \circ \exp(tX), \qquad \exp((t+s)X) = \exp(tX) \circ \exp(sX).$$

Lie derivative For any smooth tensor field T,

$$\mathcal{L}_X T := \frac{d}{dt}\Big|_{t=0} \Phi_t^* T,$$

a derivation on tensor fields (Leibniz rule). For $\omega \in \Omega^r(M)$ (Cartan):

$$\mathcal{L}_X \omega = d\, i_X \omega + i_X d\omega.$$

Lie bracket. For $X, Y \in \Gamma(TM)$, $[X, Y] := \mathcal{L}_X Y$; this makes $(\Gamma(TM), [\cdot, \cdot])$ a Lie algebra (defined later).

1.23 Isotopies and time–dependent flows

Definition 12 (Isotopy of diffeomorphisms)**.** *A smooth map $\sigma : [0,1] \times M \to M$ is an* isotopy *if $\sigma_t := \sigma(t, \cdot) \in \mathrm{Diff}(M)$ for each t and $\sigma_0 = \mathrm{id}_M$. Its* generator *is the time–dependent vector field $V_t \in \Gamma(TM)$ defined by*

$$V_t(p) = \frac{d}{ds}\sigma_s(p)\Big|_{s=t}, \qquad \textit{i.e.} \qquad \frac{d}{dt}\sigma_t = V_t \circ \sigma_t.$$

Proposition 2 (Converses / existence)**.** *Given a smooth family $V_t \in \Gamma(TM)$, if M is compact (or each V_t is complete), there exists a unique isotopy σ_t solving $\frac{d}{dt}\sigma_t = V_t \circ \sigma_t$ with $\sigma_0 = \mathrm{id}_M$. Thus isotopies of M are in bijection with time–dependent vector fields via the flow equation.*

Proposition 3 (Evolution (pullback) formula)**.** *For any smooth tensor field T on M and the isotopy σ_t generated by V_t,*

$$\frac{d}{dt}\sigma_t^* T = \sigma_t^*\big(\mathcal{L}_{V_t} T\big).$$

In particular, for $f \in C^\infty(M)$,

$$\frac{d}{dt} f\Big(\sigma_t(p)\Big) = (V_t f)\Big(\sigma_t(p)\Big).$$

Definition 13 (Time–independent case)**.** *If* $V_t \equiv X$ *is constant in* t*, the isotopy is the* flow of X*:* $\sigma_t = \exp(tX)$*, satisfying*

$$\exp(0 \cdot X) = \mathrm{id}_M, \qquad \frac{d}{dt}\exp(tX) = X \circ \exp(tX), \qquad \exp((t+s)X) = \exp(tX) \circ \exp(sX).$$

1.24 Cotangent Bundle and Differential Forms

Cotangent bundle. The *cotangent bundle* of a smooth n–manifold M is

$$T^*M := \bigsqcup_{p \in M} T_p^*M,$$

a rank–n smooth vector bundle dual to TM. In a chart (U, x) with frame $\{\partial_{x^i}\}$, the dual coframe is $\{dx^i\}$.

k**-forms.** For each $k \geq 0$, the bundle of exterior covectors is $\Lambda^k T^*M$, and its smooth sections

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M)$$

are the *differential* k*-forms* on M. Locally,

$$\omega \in \Omega^k(U) \quad \Longleftrightarrow \quad \omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad \omega_{i_1 \cdots i_k} \in C^\infty(U).$$

Wedge product. The exterior product

$$\wedge : \Omega^p(M) \times \Omega^q(M) \longrightarrow \Omega^{p+q}(M)$$

is bilinear, associative, and graded-commutative: $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ for $\alpha \in \Omega^p$, $\beta \in \Omega^q$.

1.25 Exterior Derivative

Homogeneous forms and grading. The graded algebra of smooth differential forms on a manifold M is

$$\Omega^\bullet(M) := \bigoplus_{k=0}^{\dim M} \Omega^k(M),$$

where $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$ are the smooth k-forms. An element $\alpha \in \Omega^k(M)$ is called *homogeneous of degree* k, and we write $\deg \alpha = k$. A general form $\omega = \omega_0 + \cdots + \omega_n$ may have components of different degrees.

Germs and locality. Let $\alpha, \beta \in \Omega^k(M)$ and $p \in M$. We say α and β *have the same germ at* p if there exists an open neighborhood $U \ni p$ such that $\alpha|_U = \beta|_U$. The equivalence class $[\alpha]_p$ is the *germ of* α *at* p. An operator $D : \Omega^k(M) \to \Omega^{k+1}(M)$ is called *local* if $D\alpha(p)$ depends only on the germ $[\alpha]_p$, i.e. $\alpha|_U = \beta|_U$ near p implies $D\alpha|_U = D\beta|_U$. The exterior derivative d will have precisely this locality property.

Definition 14 (Exterior derivative)**.** *For a smooth manifold M, the* exterior derivative *is the unique $\mathbb{R}$–linear operator of degree $+1$*

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \qquad (k \geq 0)$$

satisfying:

(i) ***On functions:*** *for $f \in C^\infty(M)$, df is the usual differential, locally*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, dx^i.$$

(ii) ***Graded Leibniz rule:*** *for homogeneous forms $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$,*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta.$$

(iii) ***Locality:*** *d depends only on the germ of its input and agrees in each coordinate chart with the standard Euclidean expression for d.*

Remark 1 (Local coordinate formula)**.** *If on a chart $(U; x^1, \ldots, x^n)$ a k-form is*

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^{n} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

and one reorders to increasing indices, adjusting signs.

Proposition 4 (Nilpotence)**.** *The operator d is* nilpotent*: $d^2 = 0$.*

Idea. For $f \in C^\infty(M)$, $d(df) = 0$ follows from $\partial_i \partial_j f = \partial_j \partial_i f$ and antisymmetry of the wedge. The general case follows by the graded Leibniz rule and locality. $\square$

Proposition 5 (Naturality)**.** *For any smooth map $f : M \to N$ and $\eta \in \Omega^\bullet(N)$,*

$$f^*(d\eta) = d(f^*\eta).$$

Sketch. For 0–forms g, $f^*(dg) = d(g \circ f) = d(f^*g)$ by the chain rule. For coordinate 1–forms dy^j on N, $f^*(dy^j) = d(y^j \circ f)$, hence $f^*(d\,dy^j) = 0 = d(f^*dy^j)$. Extend to arbitrary forms locally by the Leibniz rule and glue. □

[Low degree examples]

(a) For $f \in C^\infty(M)$, $df = \sum_i (\partial_i f)\,dx^i$.

(b) For $\alpha = \sum_i a_i\,dx^i$, $d\alpha = \sum_{i<j} (\partial_i a_j - \partial_j a_i)\,dx^i \wedge dx^j$.

(c) For $\beta = \sum_{i<j} b_{ij}\,dx^i \wedge dx^j$, $d\beta = \sum_{i<j<k} (\partial_i b_{jk} + \partial_j b_{ki} + \partial_k b_{ij})\,dx^i \wedge dx^j \wedge dx^k$.

Remark 2 (Cartan calculus)**.** *For any vector field $X \in \mathfrak{X}(M)$, the Lie derivative satisfies*

$$\boxed{\mathcal{L}_X = d\,i_X + i_X\,d},$$

where i_X is the interior product. This is Cartan's magic formula, expressing infinitesimal flow action on forms.

1.26 Pullback of Forms

Definition. For a smooth map $f : M \to N$ and $\eta \in \Omega^k(N)$, the *pullback* $f^*\eta \in \Omega^k(M)$ is defined by

$$(f^*\eta)_p(v_1, \ldots, v_k) := \eta_{f(p)}\Big(df_p v_1, \ldots, df_p v_k\Big), \qquad p \in M,\ v_i \in T_pM.$$

Equivalently, in local coordinates $y = (y^1, \ldots, y^m)$ on N,

$$f^*(dy^{i_1} \wedge \cdots \wedge dy^{i_k}) = d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f).$$

Functorial properties.

$$(\mathrm{id}_M)^* = \mathrm{id}, \qquad (g \circ f)^* = f^* \circ g^*.$$

Compatibility. For homogeneous forms α, β,

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta, \qquad f^*d = d\,f^*.$$

(Thus f^* is a morphism of differential graded algebras $(\Omega^\bullet, d)$.)

1.27 Integration of Forms in $\mathbb{R}^n$

Support of a function. For a continuous or smooth function $\rho : M \to \mathbb{R}$, the *support* of ρ, denoted $\operatorname{supp}\rho$, is defined as

$$\operatorname{supp}\rho := \overline{\{\, p \in M \mid \rho(p) \neq 0 \,\}}.$$

It is a closed subset of M. We say that ρ has *compact support* if $\operatorname{supp}\rho$ is a compact subset of M.

Let $U \subseteq \mathbb{R}^n$ be an open set with the standard coordinates $(x^1, \dots, x^n)$. An *n-form* on U is a differential form of the type

$$\omega = f(x)\, dx^1 \wedge \cdots \wedge dx^n, \qquad f \in C_c^\infty(U),$$

where $C_c^\infty(U)$ denotes the space of smooth functions with compact support in U.

Definition. The *integral of* ω *over* U is defined by

$$\int_U \omega := \int_U f(x)\, dx^1 \cdots dx^n,$$

where the right-hand side is the usual Lebesgue integral in $\mathbb{R}^n$.

Change of variables. If $\varphi : U \to V$ is an orientation-preserving diffeomorphism between open subsets of $\mathbb{R}^n$, then for every $\omega \in \Omega_c^n(V)$,

$$\int_V \omega = \int_U \varphi^*\omega.$$

In coordinates, if $\omega = g(y)\, dy^1 \wedge \cdots \wedge dy^n$ and $y = \varphi(x)$, then

$$\varphi^*\omega = g(\varphi(x)) \det(D\varphi(x))\, dx^1 \wedge \cdots \wedge dx^n,$$

so the formula above is equivalent to the usual change-of-variables theorem from multivariable calculus.

1.28 Orientations and Volume Forms

Orientation of $\mathbb{R}^n$. The standard ordered basis $(\partial_{x^1}, \dots, \partial_{x^n})$ determines the *standard orientation* on $\mathbb{R}^n$. Two ordered bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ of $T_p\mathbb{R}^n$ are said to have the same orientation if $\det\big(v_i^j\big)$ and $\det\big(w_i^j\big)$ have the same sign. The space of orientations has two elements: "positive" and "negative."

Orientation on a manifold. Let M be a smooth n-manifold. An *orientation* on M is an equivalence class of atlases $\{(U_i, \varphi_i)\}$ such that, whenever $U_i \cap U_j \neq \varnothing$, the transition map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

has a Jacobian determinant that is *positive* everywhere. A manifold is called *orientable* if it admits such an atlas.

Volume forms. On an oriented manifold M, a *volume form* is a nowhere-vanishing smooth n-form $\mu \in \Omega^n(M)$ that induces the given orientation. Concretely, in an oriented chart (U, φ) with coordinates $(x^1, \dots, x^n)$,

$$\mu|_U = f(x)\, dx^1 \wedge \cdots \wedge dx^n, \qquad f(x) > 0.$$

Two volume forms μ and ν define the same orientation if and only if $\nu = g\mu$ with $g : M \to (0, \infty)$ smooth.

Relation to orientation. A nowhere-vanishing n-form μ defines an orientation on M by declaring that a local chart (U, φ) is *positively oriented* if

$$\mu\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) > 0 \quad \text{on } U.$$

This definition is invariant under multiplication of μ by a positive smooth function, i.e. orientation depends only on the *sign class* of μ, not on its scale.

1.29 Integration over simplices

Standard n-simplex. Let $\overline{\sigma}_n = \langle p_0 p_1 \cdots p_n \rangle \subset \mathbb{R}^n$ denote the (oriented) standard n-simplex, i.e.

$$\operatorname{conv}\{p_0, \dots, p_n\} = \left\{ x = \sum_{j=0}^{n} p_j\, x^j \;\middle|\; \sum_{j=0}^{n} x^j = 1,\ x^j \geq 0 \right\}.$$

A standard choice of vertices is

$$p_0 = (0, \dots, 0), \quad p_1 = (1, 0, \dots, 0),\ \dots,\ p_n = (0, \dots, 0, 1).$$

Orientations. Write $\langle p_0 \cdots p_k \rangle$ for the oriented k-simplex with ordered vertices $(p_0, \dots, p_k)$. If π is a permutation of $\{0, 1, \dots, k\}$, then

$$\langle p_{\pi(0)} \cdots p_{\pi(k)} \rangle \;=\; \operatorname{sgn}(\pi)\, \langle p_0 \cdots p_k \rangle.$$

In particular, for $k = 1$, $\langle p_0 p_1 \rangle$ is oriented from p_0 to p_1, and $-\langle p_0 p_1 \rangle = \langle p_1 p_0 \rangle$.

Boundaries of geometric simplices. For a 0-simplex (p_0) set $\partial_0(p_0) = 0$. For a 1-simplex,

$$\partial_1 \langle p_0 p_1 \rangle = (p_1) - (p_0).$$

In general, for an oriented k-simplex,

$$\partial_k \langle p_0 \cdots p_k \rangle = \sum_{j=0}^{k} (-1)^j \, \langle p_0 \cdots \widehat{p_j} \cdots p_k \rangle,$$

where the hat $\widehat{p_j}$ means the vertex p_j is omitted.

Integrating a top form over $\overline{\sigma}_n$. If $\omega = a(x)\, dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, define

$$\int_{\overline{\sigma}_n} \omega \;:=\; \int_{\overline{\sigma}_n} a(x)\, dx^1 \cdots dx^n.$$

Example $(n = 2)$: for $\omega = dx \wedge dy$ and $\overline{\sigma}_2 = \{x \geq 0,\ y \geq 0,\ x + y \leq 1\}$,

$$\int_{\overline{\sigma}_2} \omega = \int_0^1 \int_0^{1-x} dy\, dx = \tfrac{1}{2}.$$

Singular simplices in a manifold. Let $\sigma_k \subset \mathbb{R}^n$ be a geometric k-simplex and $f : U \to M$ a smooth map from an open set $U \supset \sigma_k$ into a smooth manifold M. The restriction $s_k := f|_{\sigma_k} : \sigma_k \to M$ is a *singular k-simplex in M* (a parametrized k-simplex in M).

k-chains. A *k-chain* is a finite formal linear combination

$$c = \sum_i a_i \, s_{k,i}, \qquad a_i \in \mathbb{R}, \;\; a_i = 0 \text{ for all but finitely many } i.$$

The group of k-chains is the direct sum

$$C_k(M) = \bigoplus_i \mathbb{R}\, s_{k,i}.$$

(Replace $\mathbb{R}$ by $\mathbb{Z}$ if you prefer integral coefficients.)

Boundary of singular simplices and chains. If $s_k = f|_{\sigma_k}$ with $\sigma_k = \langle p_0 \cdots p_k \rangle$, set

$$\partial_k s_k := f(\partial_k \sigma_k) = \sum_{j=0}^{k} (-1)^j \left(f\Big|_{\langle p_0 \cdots \widehat{p_j} \cdots p_k \rangle} \right),$$

a finite linear combination of singular $(k-1)$-simplices in M. Extend by linearity:

$$\partial_k\Big(\sum_i a_i s_{k,i}\Big) = \sum_i a_i\, \partial_k s_{k,i}, \qquad \partial_k : C_k(M) \to C_{k-1}(M).$$

Moreover,

$$\partial_{k-1} \circ \partial_k = 0 \qquad (\text{i.e., } \partial^2 = 0),$$

since each $(k-2)$-face appears twice with opposite sign.

Integration over a general chain. For a k-form ω on M and a finite chain $c = \sum_i a_i s_{k,i}$, define

$$\int_c \omega := \sum_i a_i \int_{s_{k,i}} \omega.$$

Orientation matters: $\displaystyle\int_{-s_k} \omega = -\int_{s_k} \omega$, and the map $(c, \omega) \mapsto \int_c \omega$ is linear in both arguments.

1.30 Integration of Forms on Manifolds

Locally finite covers and atlases. An open cover $\{U_i\}_{i \in I}$ of a manifold M is called *locally finite* if every point $p \in M$ has a neighborhood intersecting only finitely many U_i. An *atlas* $\{(U_i, \varphi_i)\}$ is locally finite if the corresponding open sets $\{U_i\}$ form a locally finite cover of M. Local finiteness ensures that sums or constructions involving all indices i remain finite in any neighborhood.

Paracompact manifolds. A topological space M is called *paracompact* if every open cover $\{U_i\}$ of M admits a locally finite open refinement $\{V_j\}$, i.e., each $V_j \subset U_i$ for some i and every point of M intersects only finitely many V_j. Every smooth manifold that is Hausdorff and second countable is paracompact. This property guarantees the existence of smooth partitions of unity.

Partitions of unity. Let $\{U_i\}_{i \in I}$ be a locally finite open cover of a smooth manifold M. A *smooth partition of unity subordinate to* $\{U_i\}$ is a collection of smooth functions $\{\rho_i\}_{i \in I}$ such that:

$$\rho_i \in C^\infty(M), \qquad 0 \le \rho_i \le 1, \qquad \operatorname{supp} \rho_i \subset U_i, \qquad \sum_{i \in I} \rho_i = 1.$$

The sum is locally finite, meaning that for each $p \in M$ only finitely many $\rho_i(p)$ are nonzero. The existence of such partitions is a fundamental consequence of paracompactness.

Compactly supported top-degree forms. Let M be an oriented n-manifold. Denote by $\Omega_c^n(M)$ the space of smooth n-forms with compact support, i.e. forms ω such that $\operatorname{supp}(\omega) = \overline{\{p \in M : \omega_p \neq 0\}}$ is compact in M.

Definition of the integral. Given $\omega \in \Omega_c^n(M)$, choose a locally finite oriented atlas $\{(U_i, \varphi_i)\}$ and a smooth partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$. Each $\varphi_i : U_i \to \mathbb{R}^n$ is an oriented coordinate chart. Define

$$\int_M \omega \ := \ \sum_{i \in I} \int_{\varphi_i(U_i)} (\varphi_i^{-1})^*(\rho_i \, \omega),$$

where the integrals on the right are ordinary integrals in $\mathbb{R}^n$ with respect to the Euclidean orientation. Because of local finiteness, the sum has only finitely many nonzero terms in any neighborhood of M, so the integral is well defined. The value of $\int_M \omega$ does not depend on the choice of atlas or partition.

Properties. For $\omega, \eta \in \Omega_c^n(M)$ and $a, b \in \mathbb{R}$:

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

If $F : M \to N$ is an orientation-preserving diffeomorphism between oriented n-manifolds, then

$$\int_M F^*\tau = \int_N \tau, \qquad \forall\, \tau \in \Omega_c^n(N).$$

1.31 Manifolds with Boundary and Stokes' Theorem

Manifolds with boundary. An n-dimensional smooth manifold with boundary is a second countable Hausdorff space M with an atlas of charts (U, φ) where $\varphi : U \to \mathbb{H}^n$ is a homeomorphism onto an open subset of the closed half-space

$$\mathbb{H}^n := \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

The *boundary* ∂M is the set of points that map to $\{x^n = 0\}$ in some (hence any) boundary chart. It is a smooth $(n-1)$–manifold (without boundary) embedded in M. We write $(M, \partial M)$ when we wish to emphasize the boundary.

If M is a smooth manifold with boundary, then for each point $p \in \partial M$ we can choose a local coordinate chart $(x_1, \ldots, x_n)$ such that

$$M = \{x_n \geq 0\}, \quad \text{and} \quad \partial M = \{x_n = 0\}.$$

In this chart, the vector $\frac{\partial}{\partial x_n}$ points *inward* to M, so the *outward-pointing normal* vector ν is taken to be

$$\nu = -\frac{\partial}{\partial x_n}.$$

More generally, if ρ is a smooth function satisfying $M = \{\rho \geq 0\}$ and $\partial M = \{\rho = 0\}$, then the outward-pointing normal along ∂M is given by

$$\nu = -\frac{\nabla \rho}{\|\nabla \rho\|}.$$

With this convention, we can now describe how the orientation of M induces an orientation on its boundary.

Orientation of the boundary. If M is oriented, then ∂M inherits the *outward-normal-first* orientation: a basis $(v_1, \ldots, v_{n-1})$ of $T_p(\partial M)$ is positive iff $(\nu, v_1, \ldots, v_{n-1})$ is a positive basis of T_pM, where ν is the outward-pointing normal.

Pullback to the boundary. If $\iota : \partial M \hookrightarrow M$ denotes the inclusion, then for any $(n-1)$–form $\omega \in \Omega^{n-1}(M)$, its *restriction to the boundary* is the pullback $\iota^*\omega \in \Omega^{n-1}(\partial M)$.

Stokes' theorem Let M be an oriented smooth n–manifold with boundary, and let $\omega \in \Omega_c^{n-1}(M)$ be a compactly supported $(n-1)$–form. Then

$$\int_M d\omega \;=\; \int_{\partial M} \iota^*\omega.$$

Naturality (functoriality). If $F : (M, \partial M) \to (N, \partial N)$ is a smooth map sending boundary to boundary, orientation-preserving on interiors and on boundaries, then for every $\eta \in \Omega_c^{n-1}(N)$,

$$\int_M d(F^*\eta) \;=\; \int_{\partial M} F^*\eta, \qquad \text{equivalently} \qquad \int_M F^*(d\eta) \;=\; \int_{\partial M} F^*\eta.$$

Local form (Green–Gauss). In an oriented boundary chart (U, φ), Stokes reduces to the classical divergence theorem: if $\omega = \sum_{i=1}^n (-1)^{i-1} f_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge$

$\cdots \wedge dx^n$, then

$$d\omega = \Big(\sum_{i=1}^{n} \partial_{x^i} f_i\Big)\, dx^1 \wedge \cdots \wedge dx^n, \qquad \int_U d\omega \;=\; \int_{\partial U} \omega.$$

1.32 Closed and exact forms

For a smooth manifold M, a k–form $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$ and *exact* if there exists $\eta \in \Omega^{k-1}(M)$ with $\omega = d\eta$. Write

$$Z^k(M) := \ker\Big(d : \Omega^k(M) \to \Omega^{k+1}(M)\Big), \qquad B^k(M) := \operatorname{im}\Big(d : \Omega^{k-1}(M) \to \Omega^k(M)\Big).$$

Since $d^2 = 0$, we have $B^k(M) \subset Z^k(M)$.

1.33 The de Rham complex $(\Omega^\bullet(M), d)$

The graded vector space

$$\Omega^\bullet(M) := \bigoplus_{k \geq 0} \Omega^k(M)$$

together with the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ forms a *cochain complex*:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0, \qquad d \circ d = 0.$$

The wedge product makes $\Omega^\bullet(M)$ into a graded-commutative differential algebra:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

1.34 de Rham cohomology groups

The k–th de Rham cohomology of M is the quotient

$$H^k_{\mathrm{dR}}(M) \;:=\; \frac{Z^k(M)}{B^k(M)} \;=\; \frac{\ker(d : \Omega^k \to \Omega^{k+1})}{\operatorname{im}(d : \Omega^{k-1} \to \Omega^k)}.$$

A smooth map $f : M \to N$ induces a pullback $f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ with $f^* \circ d = d \circ f^*$, hence a well-defined map in cohomology

$$f^* : H^k_{\mathrm{dR}}(N) \longrightarrow H^k_{\mathrm{dR}}(M).$$

Thus $H^\bullet_{\mathrm{dR}}$ is a contravariant functor from smooth manifolds to graded algebras.

1.35 Poincaré Lemma, Contractible Sets, and Homotopies

Definition 15 (Homotopy of maps)**.** *Let X, Y be topological spaces. A* homotopy *between continuous maps $f, g : X \to Y$ is a continuous map*

$$H : X \times [0, 1] \to Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. We write $f \simeq g$.

Let U be a topological space. A *contraction* of U to a point $q \in U$ is a homotopy

$$H : U \times [0, 1] \to U, \qquad H_0 = id_U, \quad H_1 = \text{const.}$$

If such H exists, U is called *contractible.* Equivalently, U is homotopy equivalent to a point.

Open balls in $\mathbb{R}^n$ and star-shaped open sets in $\mathbb{R}^n$ are contractible. Non-examples include the punctured plane $\mathbb{R}^2 \setminus \{0\}$ and the circle S^1.

Remark 3. *If U is contractible, then all de Rham cohomology groups in positive degrees vanish.*

Poincaré Lemma (Euclidean (star-shaped) version) Let $U \subset \mathbb{R}^n$ be star-shaped (hence contractible), and let $\omega \in \Omega^k(U)$ be smooth with $k \geq 1$. If $\mathrm{d}\omega = 0$, then there exists $\eta \in \Omega^{k-1}(U)$ such that

$$\omega = \mathrm{d}\eta.$$

Equivalently, $H^k(U) = 0$ for all $k \geq 1$, while $H^0(U) \cong \mathbb{R}$ (the constants).

Poincaré Lemma (manifold version (local form)) Let M be a smooth manifold and $p \in M$. For each $k \geq 1$ there exists a coordinate neighborhood $U \ni p$ such that every closed k-form $\omega \in \Omega^k(U)$ is exact. In particular, for any contractible open $U \subset M$,

$$\mathrm{d}\omega = 0 \;\Rightarrow\; \omega = \mathrm{d}\eta \quad \text{on } U \quad (k \geq 1).$$

Remark 4 (Endpoints and $k = 0$)**.** *The lemma fails for $k = 0$: closed 0-forms are locally constant, not necessarily exact. In a bounded de Rham complex*

$$0 \longrightarrow \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(U) \longrightarrow 0,$$

the 0's are the zero objects and make the endpoint cohomology conventions explicit.

Proposition 6 (Homotopy invariance; chain homotopy formula)**.** *Let M, N be smooth manifolds and let $F : M \times [0,1] \to N$ be a smooth homotopy with $f_t(\cdot) = F(\cdot, t)$, so $f_0, f_1 : M \to N$ are smoothly homotopic. Define the degree -1 operator*

$$K : \Omega^k(N) \longrightarrow \Omega^{k-1}(M), \qquad K\omega \;=\; \int_0^1 \iota_{\partial_t}\Big(F^*\omega\Big)\,dt \;=\; \int_0^1 f_t^*\Big(\iota_{V_t}\omega\Big)\,dt,$$

where V_t is the time–t velocity field along the homotopy, i.e. $\partial_t F = V_t \circ F$. Then the chain homotopy identity

$$\mathrm{d}K + K\mathrm{d} \;=\; f_1^* - f_0^*$$

holds. In particular, for every closed form $\omega \in \Omega^k(N)$ ($k \geq 1$),

$$f_1^*\omega - f_0^*\omega \;=\; \mathrm{d}\Big(K\omega\Big)$$

is exact. Consequently, f_0^* and f_1^* induce the same map on de Rham cohomology:

$$H^k_{\mathrm{dR}}(N) \xrightarrow{f_0^*} H^k_{\mathrm{dR}}(M) \;=\; H^k_{\mathrm{dR}}(N) \xrightarrow{f_1^*} H^k_{\mathrm{dR}}(M).$$

Proof sketch. Differentiate $F^*\omega$ in t:

$$\frac{d}{dt}F^*\omega \;=\; F^*(\mathcal{L}_{\partial_t}\omega) \;=\; F^*\Big(\mathrm{d}\,\iota_{\partial_t}\omega + \iota_{\partial_t}\mathrm{d}\omega\Big) \quad \text{(Cartan's formula).}$$

Integrate from 0 to 1 and use $F(\cdot, 0) = f_0$, $F(\cdot, 1) = f_1$ to obtain $f_1^* - f_0^* = \mathrm{d}K + K\mathrm{d}$. □

2 Symplectic Geometry and Geometric Mechanics

This section was compiled largely following lecture notes on Symplectic Geometry and Classical Mechanics by Tobias Osborne. The material closely follows the geometric formulation of Hamiltonian mechanics presented in those lectures, with minor adaptations for notation and emphasis to maintain consistency with the rest of this report.

2.1 Lagrangian dynamical systems

Let M be a differentiable manifold, TM its tangent bundle, and let $L : TM \to \mathbb{R}$ be a smooth Lagrangian. A map $\gamma : \mathbb{R} \to M$ is called a *motion* in the Lagrangian system with configuration manifold M and Lagrangian function L if γ is an extremal of the action functional

$$\Phi[\gamma] \;\coloneqq\; \int_{t_0}^{t_1} L\Big(\gamma(t), \dot{\gamma}(t)\Big)\,dt,$$

where $\dot{\gamma}(t) \in T_{\gamma(t)}M$. [Euler–Lagrange equations] Let $\varphi : U \subset M \to \mathbb{R}^n$ be a coordinate chart and write $x^\mu(t) \coloneqq (\varphi \circ \gamma)^\mu(t)$. Then the evolution of the coordinates of a point in motion satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}, \qquad \mu = 1, \ldots, n.$$

In particular, the velocity vector has the local expression

$$\dot\gamma(t) = \dot{x}^\mu(t)\,\frac{\partial}{\partial x^\mu} = v^\mu(t)\,e_\mu,$$

where $e_\mu \coloneqq \frac{\partial}{\partial x^\mu}$ and $v^\mu \coloneqq \dot{x}^\mu$.

Definition 16 (Symplectic manifold)**.** *A* symplectic form *on a smooth manifold M^{2n} is a 2–form $\omega \in \Omega^2(M)$ that is* closed ($\mathrm{d}\omega = 0$) *and* nondegenerate*, i.e. the bundle map*

$$\omega^\flat : TM \to T^*M, \qquad (\omega^\flat)_p(v) = \iota_v\omega_p$$

is an isomorphism for every $p \in M$. Then (M, ω) is a symplectic manifold*.*

Remark 5. *Nondegeneracy forces* $\dim M$ *to be even. The* Liouville volume form *is*

$$\mathrm{vol}_\omega \;=\; \frac{\omega^n}{n!} \in \Omega^{2n}(M).$$

Definition 17 (Symplectomorphism)**.** *A diffeomorphism $\Phi : (M, \omega) \to (N, \eta)$ is a* symplectomorphism *if $\Phi^*\eta = \omega$. Equivalently, Φ preserves all wedge powers ω^k and in particular the Liouville volume.*

Darboux Theorem and Coordinates. Let (M, ω) be a $2n$-dimensional symplectic manifold. For every $p \in M$, there exist local coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ near p such that

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i.$$

Such a coordinate system is called a *Darboux chart* (or *Darboux coordinates*). In these coordinates, the matrix of ω has the standard symplectic form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and all symplectic manifolds are locally symplectomorphic to $(\mathbb{R}^{2n}, \sum_i dx^i \wedge dy_i)$.

Remark 6. *Darboux coordinates are not unique: any two such systems are related by a local* symplectomorphism*, i.e. a diffeomorphism ϕ with $\phi^*\omega = \omega$. In the canonical example $M = T^*Q$, the standard Darboux coordinates are (q^i, p_i) with $\theta = p_i\,dq^i$ and $\omega = d\theta = dq^i \wedge dp_i$.*

Tautological and canonical forms

Let X be a smooth manifold and set $M := T^*X$ with projection $\pi : M \to X$. For a chart $(U, x^1, \dots, x^n)$ on X, the induced coordinates on $T^*U \subset M$ are $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$, where a covector $\xi \in T_x^*X$ is written $\xi = \sum_{j=1}^n \xi_j \, dx_x^j$.

Local formulas.

$$\alpha \;=\; \sum_{j=1}^{n} \xi_j \, dx^j \qquad \text{(tautological/Liouville 1-form)},$$

$$d\alpha \;=\; \sum_{j=1}^{n} d\xi_j \wedge dx^j \;=\; -\sum_{j=1}^{n} dx^j \wedge d\xi_j,$$

$$\omega \;:=\; -\,d\alpha \;=\; \sum_{j=1}^{n} dx^j \wedge d\xi_j \qquad \text{(canonical symplectic form)}.$$

Coordinate-free definition of α. For $p = (x, \xi) \in M$ and $v_p \in T_pM$,

$$\alpha_p(v_p) \;:=\; \xi(d\pi_p(v_p)) \;=\; \Big((d\pi_p)^*\xi\Big)(v_p) \in \mathbb{R}.$$

Equivalently, $\alpha_p = (d\pi_p)^*(\xi) \in T_p^*M$. This intrinsic definition is coordinate-independent, and $\omega := -d\alpha$.

Pushforward / pullback at $p = (x, \xi)$.

$$d\pi_p : \; T_pM \longrightarrow T_xX, \qquad (d\pi_p)^* : \; T_x^*X \longrightarrow T_p^*M.$$

Examples

1. $X = \mathbb{R}$. $M = T^*\mathbb{R} \cong \mathbb{R}^2$ with coordinates (x, y) (identify $\xi = y\,dx$). Then $\pi(x, y) = x$ and
$$\alpha = y\,dx, \qquad \omega = -d\alpha = -d(y\,dx) = dx \wedge dy.$$

2. $X = S^1$ with angular coordinate θ. $M = T^*S^1 \cong S^1 \times \mathbb{R}$ with coordinates (θ, y). Then
$$\alpha = y\,d\theta, \qquad \omega = d\theta \wedge dy.$$

Definition 18 (Symplectic and Hamiltonian vector fields)**.** *A vector field X on (M, ω) is* symplectic *if ${}_X\omega = 0$ (i.e. its flow preserves ω). Using Cartan's formula ${}_X\omega = \mathrm{d}(\iota_X\omega) + \iota_X \mathrm{d}\omega$ and $\mathrm{d}\omega = 0$, this is equivalent to $\mathrm{d}(\iota_X\omega) = 0$.*

If there exists $H \in C^\infty(M)$ with

$$\iota_{X_H}\omega = \mathrm{d}H,$$

then X_H *is a* Hamiltonian vector field *with* Hamiltonian H*. Every Hamiltonian field is symplectic. On a contractible chart (or when* $H^1_{\mathrm{dR}}(M) = 0$*), every symplectic vector field is Hamiltonian.*

Proposition 7 (Hamilton's equations in Darboux coordinates)**.** *In Darboux coordinates* (x^i, y^i)*,*

$$X_H = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial y^i} \Big).$$

Thus integral curves $(x^i(t), y^i(t))$ *of* X_H *satisfy Hamilton's ODEs*

$$\dot{x}^i = \frac{\partial H}{\partial y^i}, \qquad \dot{y}^i = -\frac{\partial H}{\partial x^i} \quad (i = 1, \dots, n).$$

Proposition 8 (Energy conservation)**.** *If* X_H *is Hamiltonian, then the Hamiltonian function* H *is constant along the integral curves of* X_H*. Indeed,*

$$\mathcal{L}_{X_H} H = i_{X_H}(dH) = i_{X_H}(i_{X_H}\omega) = 0.$$

Thus H *is preserved by the flow:*

$$H(x) = (\sigma_t^* H)(x) = H(\sigma_t(x)) \qquad \forall\, t.$$

Proposition 9 (Characterization of symplectic vector fields)**.** *For a smooth vector field* $X \in \mathfrak{X}(M)$ *on a symplectic manifold* (M, ω)*, the following are equivalent:*

1. X *is* symplectic*, i.e. its flow* $\sigma_t = \exp(tX)$ *preserves* ω *for all* t*:*

$$\sigma_t^* \omega = \omega;$$

2. $\mathcal{L}_X \omega = 0$;
3. $i_X \omega$ *is a closed* 1*-form.*

Remark 7. *Locally (on contractible open sets), every symplectic vector field is Hamiltonian. That is, if* $i_X\omega$ *is closed, then locally* $i_X\omega = dH$ *for some smooth function* H*.*

Definition 19 (Musical maps of a symplectic form)**.** *Let* (M^{2n}, ω) *be symplectic. The* flat *map*

$$\omega^\flat : TM \longrightarrow T^*M, \qquad \omega^\flat(v) := \iota_v \omega$$

is a bundle isomorphism (by nondegeneracy). Its inverse is the sharp *map*

$$\omega^\sharp : T^*M \longrightarrow TM, \qquad \textit{characterized by } \omega\Big(\omega^\sharp(\alpha), \cdot\Big) = \alpha(\cdot) \quad (\alpha \in T^*M).$$

Definition 20 (Hamiltonian vector field and Poisson bracket)**.** *For $H \in C^\infty(M)$, the* Hamiltonian vector field *is*

$$X_H := \omega^\sharp(\mathrm{d}H), \qquad \textit{i.e.} \quad \iota_{X_H}\omega = \mathrm{d}H.$$

For $f, g \in C^\infty(M)$, the Poisson bracket *is*

$$\{f, g\} := \omega(X_f, X_g) = \mathrm{d}g\Big(\omega^\sharp(\mathrm{d}f)\Big) = -\,\mathrm{d}f\Big(\omega^\sharp(\mathrm{d}g)\Big).$$

Then $[X_f, X_g] = X_{\{f,g\}}$ and $\{\cdot,\cdot\}$ satisfies bilinearity, antisymmetry, Leibniz, and Jacobi.

Remark 8 (Coordinate/matrix form)**.** *In local coordinates $x = (x^1, \ldots, x^{2n})$ write*

$$\omega = \frac{1}{2}\,\Omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j, \qquad \Omega = -\Omega^\top.$$

Identify vectors and covectors by column vectors relative to the bases $\{\partial_{x^i}\}$ and $\{\mathrm{d}x^i\}$. Then

$$\omega^\flat \;\leftrightarrow\; \Omega, \qquad \omega^\sharp \;\leftrightarrow\; \Omega^{-1},$$

so for a function H, with gradient column $\nabla H = (\partial_i H)$,

$$X_H \;\leftrightarrow\; \Omega^{-1}\,\nabla H.$$

In Darboux coordinates (x^i, y^i), $\Omega = J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ and

$$X_H = \sum_{i=1}^n \Big(\frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}\Big), \quad \{f, g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}\Big).$$

Proposition 10 (Conservation and divergence)**.** *Let Φ_t be the flow of X_H. Then*

$$\Phi_t^*\omega = \omega, \qquad \Phi_t^* H = H, \qquad {}_{X_H}\Big(\omega^n\Big) = 0.$$

In particular, H is conserved along trajectories and X_H is divergence–free with respect to the Liouville volume $\omega^n/n!$ (Liouville's theorem).

Constant form on ^{2n} and symplectic matrices On ^{2n} with $\omega_0 = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}y^i$, write $z = (x, y) \in^{2n}$ and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Then $\omega_0(u, v) = u^\top J v$, and a linear map A is symplectic iff $A^\top J A = J$ (i.e. $A \in \mathrm{Sp}(2n,))$.

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Symplectic Preliminaries: Hamiltonian Fields and Lie Algebra Facts

Phase space and the Hamiltonian vector field

Consider Euclidean phase space $\mathbb{R}^{2n}$ with coordinates

$$(q_1, \ldots, q_n;\, p_1, \ldots, p_n)$$

and canonical symplectic form

$$\omega = \sum_{j=1}^{n} dq_j \wedge dp_j.$$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian and let X_H be its corresponding vector field. Then

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right).$$

Check: $i_{X_H}\omega = dH$

Using $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta - \alpha \wedge (i_X\beta)$ for 1-forms,

$$\begin{aligned} i_{X_H}\omega &= \sum_{j=1}^{n} i_{X_H}(dq_j \wedge dp_j) \\ &= \sum_{j=1}^{n} \Big((i_{X_H} dq_j) \wedge dp_j - dq_j \wedge (i_{X_H} dp_j) \Big) \\ &= \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j}\, dp_j + \frac{\partial H}{\partial q_j}\, dq_j \right) \\ &= dH. \end{aligned}$$

Hamilton's equations and integral curves

If $(q(t), p(t)) = (q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ is an integral curve of X_H, then for each j,

$$\frac{dq_j(t)}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j(t)}{dt} = -\frac{\partial H}{\partial q_j}.$$

Equivalently,

$$\frac{d}{dt}\,(q(t), p(t)) = X_H\Big(q(t), p(t)\Big).$$

Brackets

Definition 21. *A* real Lie algebra *is a real vector space* $\mathfrak{g}$ *together with a bilinear map* $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *(the Lie bracket) such that for all* $X, Y, Z \in \mathfrak{g}$*:*

$$[X, Y] = -[Y, X] \quad \textit{and} \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Definition 22. *For a symplectic manifold* (M, ω)*, write*

$$\mathfrak{X}(M) = \{\textit{smooth vector fields on } M\}, \quad \mathfrak{X}^{\text{symp}}(M) = \{X \in \mathfrak{X}(M) : \mathcal{L}_X \omega = 0\},$$

$$\mathfrak{X}^{\text{ham}}(M) = \{X \in \mathfrak{X}(M) : \exists H \textit{ with } i_X \omega = dH\}.$$

Proposition 11. *If* X, Y *are symplectic vector fields on* (M, ω)*, then* $[X, Y]$ *is Hamiltonian with*

$$i_{[X,Y]}\omega = d(\omega(Y, X)) \, .$$

Hence $[X, Y]$ *has Hamiltonian function* $-\omega(X, Y)$ *(since* $\omega(Y, X) = -\omega(X, Y)$*).*

Proof. Use $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$ and Cartan's formula $\mathcal{L}_X = di_X + i_X d$:

$$\begin{aligned} i_{[X,Y]}\omega &= (\mathcal{L}_X i_Y - i_Y \mathcal{L}_X)\omega \\ &= (di_X + i_X d)\, i_Y \omega - i_Y\, (di_X + i_X d)\omega \\ &= d(i_X i_Y \omega) + i_X d(i_Y \omega) - i_Y d(i_X \omega) - i_Y i_X d\omega. \end{aligned}$$

Because $d\omega = 0$ and $\mathcal{L}_X \omega = \mathcal{L}_Y \omega = 0$, the middle terms vanish, leaving

$$i_{[X,Y]}\omega = d\Big(\omega(Y, X)\Big).$$

□

The inclusions

$$\Big(\mathfrak{X}^{\text{ham}}(M), [\,\cdot, \cdot\,]\Big) \subset \Big(\mathfrak{X}^{\text{symp}}(M), [\,\cdot, \cdot\,]\Big) \subset \Big(\mathfrak{X}(M), [\,\cdot, \cdot\,]\Big)$$

are inclusions of real Lie algebras (each is a Lie subalgebra of the next under the usual Lie bracket of vector fields).

Remark 9 (Exact symplectic manifolds)**.** *If* $\omega = \mathrm{d}\lambda$ *for some* 1*–form* λ *(a* Liouville form*), then* $\iota_{X_H}\omega = \mathrm{d}H$ *implies* ${}_{X_H}\lambda = \mathrm{d}(\lambda(X_H)) - \iota_{X_H}\mathrm{d}\lambda = \mathrm{d}(\lambda(X_H) - H)$*, so* ${}_{X_H}\lambda$ *is exact; this is frequently used in Hamiltonian mechanics and symplectic topology.*

Definition 23 (Poisson algebra)**.** *A* Poisson algebra $(B, \cdot, \{\ ,\ \})$ *is a commutative associative algebra* $(B, \cdot)$ *equipped with a bilinear bracket* $\{\ ,\ \}$ *such that*

- $(B, \{\ ,\ \})$ *is a Lie algebra (antisymmetry and Jacobi);*
- *(Leibniz) for all* $f, g, h \in B$*,*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Proposition 12. *If (M,ω) is a symplectic manifold, then $\big((M),\{\ ,\ \}\big)$ is a Poisson algebra, where for $H \in (M)$ the Hamiltonian vector field X_H is defined by*

$$ {}_{X_H}\,\omega = \mathrm{d}H, $$

and the Poisson bracket is

$$ \{f,g\} = \omega(X_f, X_g). $$

$$ C^\infty(M) \longrightarrow \mathfrak{X}(M) $$

$$ H \longmapsto X_H $$

$$ \{\,\cdot\,,\,\cdot\,\} \longmapsto -[\,\cdot\,,\,\cdot\,] $$

(So $H \mapsto X_H$ is a Lie algebra *anti*-homomorphism.)

Definition. A submanifold of M is a manifold X together with a *closed embedding* $i\colon X \hookrightarrow M$.

Note. We regard $i\colon X \hookrightarrow M$ as an inclusion and identify p with $i(p)$; then

$$ T_pX = di_p\,(T_pX) \subset T_pM. $$

Lagrangian submanifolds. Let (M,ω) be a $2n$-dimensional symplectic manifold. A submanifold $Y \subset M$ is *Lagrangian* if for each $p \in Y$, T_pY is a Lagrangian subspace of T_pM, i.e.

$$ \omega\Big|_{T_pY} = 0 \qquad \text{and} \qquad \dim T_pY = \tfrac{1}{2}\dim M. $$

Equivalently, for the inclusion $i\colon Y \hookrightarrow M$,

$$ i^*\omega = 0 \qquad \text{and} \qquad \dim T_pY = \tfrac{1}{2}\dim T_pM \quad \text{for all } p \in Y. $$

Integrable systems **Definition.** A Hamiltonian system is a triple (M,ω,H), where (M,ω) is a symplectic manifold and

$$ H \in \mathcal{F}(M) = C^\infty(M,\mathbb{R}) $$

is the Hamiltonian function.

Theorem. If $\{f,H\} = 0$, then f is constant along the integral curves of X_H.

Proof. Let $\sigma_t = \exp(tX_H)$ be the flow of X_H. Then

$$ \frac{d}{dt}\Big(f \circ \sigma_t\Big) = \sigma_t^*\,\mathcal{L}_{X_H} f = \sigma_t^*\,\{f,H\} = 0. $$

Such an f is called an *integral of motion* (or *constant of motion*, or *first integral*).

Definition (independence). Functions $f_1, \ldots, f_n$ on M are *independent* if, for p in some dense subset of M,

$$(df_1)_p, \ldots, (df_n)_p \text{ are linearly independent.}$$

Definition (complete integrability). A Hamiltonian system (M, ω, H) is (completely) integrable if it possesses

$$n = \tfrac{1}{2} \dim M$$

independent integrals of motion

$$f_1 = H,\ f_2, \ldots, f_n$$

which pairwise Poisson commute:

$$\{f_i, f_j\} = 0 \qquad \text{for all } i, j.$$

Remark. If f_j are commuting integrals of motion, then at each $p \in M$ the vector fields X_{f_i} span an isotropic subspace of T_pM, because $\omega(X_{f_i}, X_{f_k}) = \{f_i, f_k\} = 0$. If f_i are independent, necessarily $n \leq \frac{1}{2} \dim M$.

Examples/Exercises. The simple harmonic oscillator and the simple pendulum are completely integrable. When $\dim M = 4$, (M, ω, H) is integrable whenever there is one integral of motion independent of H (e.g. the spherical pendulum).

Lagrangian level sets. Let (M, ω, H) be an integrable system with $\dim M = 2n$ and integrals $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of

$$f = (f_1, \ldots, f_n).$$

Then the level set $f^{-1}(c) \subset M$ is a Lagrangian submanifold of M.

Lemma. If the Hamiltonian vector fields X_{f_j} are complete on $f^{-1}(c)$, then each connected component of $f^{-1}(c)$ is diffeomorphic to

$$\mathbb{R}^{n-k} \times \mathbb{T}^k \quad \text{for some } 0 \leq k \leq n,$$

where $\mathbb{T}^k$ is the k–torus. *(Sketch: use flows* $\exp(tX_{f_i})$ *and commutativity of the Poisson subalgebra.)*

Corollary (Liouville torus). Any compact component of $f^{-1}(c)$ is a torus, called a *Liouville torus*.

Theorem (Arnold–Liouville). Let (M, ω, H) be completely integrable with $\dim M = 2n$ and first integrals $f_1 = H, f_2, \ldots, f_n$. Let $c \in \mathbb{R}^n$ be a regular value of f. Then $f^{-1}(c)$ is Lagrangian, and:

(a) If the flows of $X_{f_1}, \ldots, X_{f_n}$ starting at $p \in f^{-1}(c)$ are complete, then the connected component of $f^{-1}(c)$ through p is of the form $\mathbb{R}^{n-k} \times \mathbb{T}^k$. On the torus factor there exist *angle coordinates*

$$\phi_1, \ldots, \phi_n$$

in which the Hamiltonian flows are linear.

(b) There exist *action coordinates* $\psi_1, \ldots, \psi_n$ such that each ψ_j is an integral of motion and

$$(\phi_1, \ldots, \phi_n,\ \psi_1, \ldots, \psi_n)$$

is a Darboux chart. Hence

$$\{\phi_j, \psi_k\} = \delta_{jk}.$$

However, in general $\psi_k \neq f_k$, because

$$(\phi_1, \ldots, \phi_n,\ f_1, \ldots, f_n)$$

need not form a Darboux chart.

3 Contact Geometry and Jet Bundles

Definition 24 (Distribution)**.** *Let M be a smooth n–manifold. A rank–k* distribution *on M is a rank–k subbundle $D \subset TM$; i.e. each $p \in M$ is assigned a k–dimensional subspace $D_p \subset T_pM$ varying smoothly with p.*

Definition 25 (Integrability, Foliation)**.** *A distribution D is* integrable *if through every $p \in M$ there exists an immersed k–dimensional submanifold $S \subset M$ with $T_qS = D_q$ for all $q \in S$. The maximal connected such submanifolds are the* leaves *of a (dimension k, codimension $q = n - k$)* foliation *$\mathcal{F}$.*

Frobenius, global form For a C^1 rank–k distribution $D \subset TM$, the following are equivalent:

1. D is integrable (i.e. is the tangent bundle of a foliation).
2. D is *involutive*: if X, Y are vector fields with values in D, then $[X, Y]$ also takes values in D.
3. (Local normal form) About each $p \in M$ there exist coordinates $(x^1, \ldots, x^n)$ with $D = \{\partial_{x^1}, \ldots, \partial_{x^k}\}$; leaves are given by fixing $x^{k+1}, \ldots, x^n$.

Annihilator / conormal bundle and normal bundle. Let $D \subset TM$ be a rank-k C^1 distribution on an n-manifold M. Its annihilator (conormal) bundle is

$$(D) \;:=\; \{\alpha \in T^*M : \;\alpha(v) = 0 \text{ for all } v \in D\}.$$

Fiberwise $(D)_p = \{\alpha_p \in T_p^*M : \alpha_p|_{D_p} = 0\}$, hence

$$(D) \;\cong\; (TM/D)^* \;=:\; N^*,$$

the dual of the normal bundle $N := TM/D$. Thus $\operatorname{rank} D = k$ and $\operatorname{rank}(D) = n - k$.

Transverse vector fields. A vector field ν is *transverse* to D if $\nu(p) \notin D_p$ for all $p \in M$. When $D = \ker \alpha$ with α nowhere zero, transversality is $\alpha(\nu) \neq 0$ everywhere.

Codimension 1: Coorientability For a hyperplane field $D \subset TM$ (i.e. $\operatorname{codim} D = 1$), the following are equivalent:

1. There exists a global nowhere-vanishing 1-form α with $D = \ker \alpha$.
2. The normal line bundle $N = TM/D$ is orientable (equivalently, N and $N^* = (D)$ are trivial real line bundles).
3. There exists a global nowhere-vanishing vector field ν everywhere transverse to D.

If these hold and D is integrable, the induced codimension-1 foliation is transversely orientable. Moreover, any two defining forms differ by a nowhere-zero function: if α defines D, then so does $f\,\alpha$ for any smooth $f : M \to \mathbb{R} \setminus \{0\}$; in this case $(D) = \operatorname{span}\{\alpha\} \subset T^*M$.

Submersion example. If $f : M \to \mathbb{R}$ is a submersion, then $D = \ker \mathrm{d}f$ has codimension 1 and $(D) = \operatorname{span}\{\mathrm{d}f\}$, a trivial line bundle generated by $\mathrm{d}f$.

Orientation vs. coorientation (when M is oriented). Let $D \subset TM$ be a hyperplane field with normal line bundle $N := TM/D$. Then the following are equivalent:

1. D is coorientable;
2. N is orientable (equivalently, N and $N^* = (D) \subset T^*M$ are trivial line bundles);
3. D is orientable and the orientations satisfy

$$\operatorname{or}(TM) = \operatorname{or}(D) \wedge \operatorname{or}(N).$$

If a global nowhere–vanishing 1–form α defines D (so $D = \ker \alpha$) and Ω_M orients M, then D is oriented by the unique Ω_D with

$$\alpha \wedge \Omega_D = \Omega_M.$$

Examples.

- **Coorientable:** On $S^1 \times S^2$, take $D = \ker(d\theta)$ (equivalently $D = T(S^2)$). Then $\alpha = d\theta$ is a global defining form, hence D is coorientable and oriented by $d\theta \wedge \Omega_D = \Omega_{S^1 \times S^2}$.
- **Not coorientable:** On the Möbius band, let D be the line field tangent to the circle fibers. Traversing once flips any chosen transverse normal, so there is no global nowhere–zero 1–form α with $\ker \alpha = D$; hence D is not coorientable.

3.1 Codimension–One Foliations

Remark 10 (Coorientable codimension–1 foliations)**.** *If a codimension–1 foliation $\mathcal{F}$ is coorientable, there exists a global nowhere–vanishing 1-form α with $T\mathcal{F} = \ker \alpha$. Frobenius then gives $\mathrm{d}\alpha = \alpha \wedge \beta$ for some 1-form β (not unique).*

Frobenius, codimension one Let α be a nowhere–vanishing 1-form on M. The following are equivalent:

$$\ker \alpha \text{ is integrable} \iff \alpha \wedge \mathrm{d}\alpha = 0 \iff \mathrm{d}\alpha = \alpha \wedge \beta \text{ for some 1-form } \beta.$$

If $\ker \alpha$ is integrable, then locally there exist a submersion $h : M \to \mathbb{R}$ and a nowhere–zero function μ with $\alpha = \mu \, \mathrm{d}h$; the leaves are the level sets $h = \text{const}$.

Remark 11 (First integrals)**.** *If $D = \ker \alpha$ is integrable and $\alpha = \mu \, \mathrm{d}h$ locally, then any vector field Y tangent to D satisfies $Y(h) = 0$. Thus h is a* first integral*: it is constant along leaves. Additionally, on a 3–manifold, if α is nowhere zero and a 2–form ω satisfies $\omega \wedge \alpha = 0$, then locally $\omega = \alpha \wedge \eta$ for some 1–form η.*

3.2 Contact Structures

Definition 26 (Contact form and structure)**.** *On an odd–dimensional manifold M^{2n+1}, a 1-form α is a* contact form *if the top–degree form*

$$\alpha \wedge (\mathrm{d}\alpha)^n$$

is nowhere vanishing. The associated contact structure *is the hyperplane field $\xi = \ker \alpha$. Multiplying α by a positive function does not change ξ. A contact structure is* coorientable *if such a global α exists.*

Proposition 13 (Maximal non–integrability)**.** *If α is contact, then the plane field* $\ker\alpha$ *is nowhere integrable (it violates the Frobenius condition as strongly as possible).*

Definition 27 (Reeb vector field)**.** *For a contact form α, the* Reeb vector field *R is uniquely defined by*

$$\alpha(R) = 1, \qquad \iota_R\, \mathrm{d}\alpha = 0.$$

In dimension 3, the kernel of $\mathrm{d}\alpha$ is the line field generated by the Reeb vector field:

$$\mathrm{Ker}(\mathrm{d}\alpha) = \mathrm{span}\{R\}.$$

Darboux Theorem *If (M^{2n+1}, α) is a contact manifold, then near every point there exist coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ such that*

$$\alpha = \mathrm{d}z + \sum_{i=1}^{n} x_i\, \mathrm{d}y_i.$$

Consequently, contact structures have no local invariants.

Standard forms *On $\mathbb{R}^{2n+1}$, the form $\alpha_0 = \mathrm{d}z + \sum_{i=1}^{n} x_i\, \mathrm{d}y_i$ is contact. On $S^{2n+1} \subset \mathbb{C}^{n+1}$, the restriction of*

$$\alpha = \tfrac{i}{2} \sum_{j=0}^{n} \left(\bar{z}_j\, \mathrm{d}z_j - z_j\, \mathrm{d}\bar{z}_j \right)$$

is a contact form.

Godbillon–Vey Class (Codimension One) *Let $\mathcal{F}$ be a C^2 codimension–one foliation on a 3–manifold M^3. Choose a nowhere–vanishing 1-form α with $T\mathcal{F} = \ker\alpha$. By Frobenius, $\mathrm{d}\alpha = \alpha \wedge \beta$ for some 1-form β.*

Proposition 14. *The 3–form $\beta \wedge \mathrm{d}\beta$ is closed, and the cohomology class*

$$\mathrm{GV}(\mathcal{F}) \;=\; [\,\beta \wedge \mathrm{d}\beta\,] \in H^3(M; \mathbb{R})$$

is independent of the choices; it is the Godbillon–Vey class *of $\mathcal{F}$.*

Definition 28 (k-jet of a section)**.** *Let $\pi : E \to M$ be a smooth fiber bundle and $k \in \mathbb{N}$. For local sections $s, t : M \to E$ and $x \in M$, we declare $s \sim_x^k t$ if, in any bundle chart, all partial derivatives of the component functions of s and t up to order k agree at x. The k-jet of s at x is the class $j_x^k s := [s]_x^k$.*

Definition 29 (Jet bundle). *The set* $J^k(E) := \bigsqcup_{x\in M}\{j^k_x s \mid s \text{ local section near } x\}$ *carries a unique smooth structure such that the maps*

$$\pi_{k,0} : J^k(E) \to M, \quad j^k_x s \mapsto x, \qquad \pi_k : J^k(E) \to E, \quad j^k_x s \mapsto s(x)$$

are smooth bundle projections. For $k \geq \ell$ *there are natural projections*

$$\pi_{k,\ell} : J^k(E) \to J^\ell(E), \qquad \pi_{k,\ell}(j^k_x s) = j^\ell_x s,$$

Definition 30 (Local coordinates and contact 1-forms). *Let* $\pi : E \to M$ *be a smooth bundle. In local coordinates* (x^i) *on* M *and fiber coordinates* (u^α) *on* E*, a point of* $J^k(E)$ *is written* $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{i_1 i_2}, \ldots, u^\alpha_I)$ *for* $|I| \leq k$*. The* contact 1-forms *on* $J^k(E)$ *are*

$$\omega^\alpha_I \;=\; du^\alpha_I \;-\; u^\alpha_{Ii}\, dx^i \qquad (|I| \leq k-1).$$

Definition 31 (Graded algebra). *A (unital)* graded algebra *over a ring* R *is an* R*-module* $\Omega^\bullet = \bigoplus_{p\in\mathbb{Z}} \Omega^p$ *together with an* R*-bilinear multiplication* $\cdot : \Omega^\bullet \times \Omega^\bullet \to \Omega^\bullet$ *such that*

$$\Omega^p \cdot \Omega^q \subset \Omega^{p+q} \quad \text{for all } p, q \in \mathbb{Z}.$$

An element $x \in \Omega^p$ *is said to have* degree $|x| = p$.

Definition 32 (Graded-commutative). *A graded algebra* $\Omega^\bullet$ *is* graded-commutative *if for all homogeneous* x, y *one has*

$$x \cdot y = (-1)^{|x|\,|y|}\, y \cdot x.$$

Definition 33 (Graded ideal). *Let* $\Omega^\bullet = \bigoplus_{p\geq 0} \Omega^p$ *be a graded algebra. A subset* $\mathcal{I} \subset \Omega^\bullet$ *is a* graded ideal *if*

$$\Omega^p \cdot \mathcal{I}^q \subset \mathcal{I}^{p+q} \quad \text{for all } p, q,$$

where $\mathcal{I}^q := \mathcal{I} \cap \Omega^q$*. Equivalently,* $\mathcal{I}$ *is an ideal that is closed under the grading, so that* $\mathcal{I} = \bigoplus_{p\geq 0} \mathcal{I}^p$ *with each* $\mathcal{I}^p \subset \Omega^p$.

Definition 34 (Differential ideal). *Let* M *be a smooth manifold and let* $(\Omega^\bullet(M), d)$ *be its graded-commutative algebra of differential forms with exterior derivative. A* differential ideal $\mathcal{J} \subset \Omega^\bullet(M)$ *is a graded ideal such that*

$$\omega \in \mathcal{J} \;\Rightarrow\; d\omega \in \mathcal{J}, \qquad \eta \in \Omega^\bullet(M),\ \omega \in \mathcal{J} \;\Rightarrow\; \eta \wedge \omega \in \mathcal{J}.$$

Given a subset $S \subset \Omega^\bullet(M)$*, the* differential ideal generated by S *is the smallest differential ideal containing* S*; denote it by* $\langle\!\langle S \rangle\!\rangle_d$.

Definition 35 (Contact ideal)**.** *Let $\Omega^\bullet(J^k(E))$ be the algebra of differential forms on $J^k(E)$. The* contact ideal *$\mathcal{J}^k \subset \Omega^\bullet(J^k(E))$ is the smallest differential ideal containing all contact 1-forms ω_I^α. Equivalently,*

$$\mathcal{C}^k := \langle \omega_I^\alpha \rangle \subset \Omega^1(J^k(E)), \qquad \mathcal{J}^k := \textit{the ideal generated by } \mathcal{C}^k \textit{ and closed under } d.$$

Definition 36 (Holonomy)**.** *A smooth map $\sigma : M \to J^k(E)$ is* holonomic *if there exists a section $s : M \to E$ with $\sigma = j^k s$. A point $\theta \in J^k(E)$ is* holonomic *if $\theta = j^k_x s$ for some $x \in M$ and local section s. A submanifold $S \subset J^k(E)$ is* holonomic *if $S = j^k s(M)$ for some section s.*

Proposition 15 (Contact ideal characterizes holonomy)**.** *For a smooth map $\sigma : M \to J^k(E)$ the following are equivalent:*

(i) σ is holonomic, i.e. $\sigma = j^k s$ for some section s;

(ii) $\sigma^*(\omega_I^\alpha) = 0$ for all $|I| \le k - 1$;

(iii) $\sigma^*(\eta) = 0$ for every $\eta \in \mathcal{J}^k$.

In particular, a submanifold $S \subset J^k(E)$ is holonomic iff $\iota^* \mathcal{J}^k = 0$, where $\iota : S \hookrightarrow J^k(E)$.

Definition 37 (Local coordinates and contact forms)**.** *Fix local coordinates (x^i) on M and fiber coordinates (u^α) on E. A point of $J^k(E)$ is written $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{i_1 i_2}, \dots, u^\alpha_I)$ for $|I| \le k$, where I is a multi-index. The* contact 1-forms *on $J^k(E)$ are*

$$\omega_I^\alpha = du_I^\alpha - u^\alpha_{Ii}\, dx^i \qquad (|I| \le k - 1),$$

and their ideal vanishes precisely on the image of holonomic *jets $j^k s(M) \subset J^k(E)$.*

Definition 38 (Prolongation of a section)**.** *For a section $s : M \to E$, its k-th prolongation is the smooth map*

$$j^k s : M \to J^k(E), \qquad x \mapsto j^k_x s.$$

Proposition 16 (Functoriality)**.** *A bundle map $F : E \to E'$ covering $f : M \to M'$ induces $J^k F : J^k(E) \to J^k(E')$ satisfying $\pi_{k,\ell} \circ J^k F = J^\ell F \circ \pi_{k,\ell}$ for all $k \ge \ell$, and $J^k F \circ j^k s = j^k(F \circ s)$.*

Trivial bundle *For $E = M \times \mathbb{R}^m$, sections are smooth maps $u : M \to \mathbb{R}^m$. In local coordinates, $J^k(E)$ has coordinates $(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_I)$ with $|I| \le k$, encoding all partials $\partial^{|I|} u^\alpha / \partial x^I$ at a point.*

$$\cdots \longrightarrow J^{k+1}(E) \xrightarrow{\pi_{k+1,k}} J^k(E) \xrightarrow{\pi_{k,k-1}} J^{k-1}(E) \longrightarrow \cdots \longrightarrow E \xrightarrow{\pi} M$$
$$J^{k+1}(E) \xrightarrow{\pi_{k+1,0}} M, \qquad J^k(E) \xrightarrow{\pi_{k,0}} M$$

Cartan's (canonical) contact structure on a 1-jet bundle

Definition 39 (Configuration manifold)**.** *For a mechanical system, the* configuration manifold Q *is a smooth manifold whose points parametrize all kinematically admissible configurations of the system. Holonomic constraints are incorporated by taking Q to be a submanifold of an ambient space (e.g. $\mathbb{R}^n$) defined by the constraint equations. The tangent bundle TQ represents velocities, and the cotangent bundle T^*Q (with its canonical symplectic form) is the phase space for Hamiltonian mechanics.*

Example *N labeled particles in $\mathbb{R}^3$: $Q = (\mathbb{R}^3)^N$. A single rigid body in $\mathbb{R}^3$: $Q = \mathrm{SE}(3)$. An n-link planar manipulator with revolute joints: $Q = (S^1)^n$.*

For a configuration manifold Q, the 1-jet bundle of real-valued functions is

$$J^1(Q) \;\cong\; T^*Q \times \mathbb{R}, \qquad (q^i, u, p_i),$$

with the Cartan contact form

$$\boxed{\alpha = \mathrm{d}u - p_i\,\mathrm{d}q^i},$$

and

$$\mathrm{d}\alpha = -\,\mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Here the Reeb vector field is $R_\alpha = \partial/\partial u$.

Definition 40 (Tangent submanifold to a distribution)**.** *Let $D \subset TM$ be a (smooth) distribution on a manifold M. A submanifold $L \subset M$ is* tangent to D *if*

$$T_pL \;\subset\; D_p \qquad \text{for every } p \in L.$$

If $D = \ker\alpha$ for a 1-form α, this is equivalent to $\alpha|_L = 0$.

Definition 41 (Tangent to a submanifold at a point)**.** *Let $L, S \subset M$ be submanifolds and $p \in L \cap S$. We say L* is tangent to S at p *if*

$$T_pL \;\subset\; T_pS.$$

If $T_pL = T_pS$, they have the same tangent space at p.

Definition 42 (Graph of a 1-jet)**.** *Let Q be an n-manifold and identify $J^1(Q) \cong Q \times \mathbb{R}_u \times T^*Q$ with contact form $\alpha = du - p_i\,dq^i$. For $S \in C^\infty(Q)$ the* graph of its 1-jet *is*

$$j^1S \;=\; \{\,(q,\; u = S(q),\; p = dS_q) \;:\; q \in Q\,\} \;\subset\; J^1(Q).$$

Then $\alpha|_{j^1S} = 0$, so j^1S is a Legendrian submanifold of $J^1(Q)$.

Legendrian submanifolds. *An n-dimensional submanifold is* Legendrian *if it is everywhere tangent to the contact distribution $\xi = \ker \alpha$ (i.e. $\alpha|_L = 0$). Graphs of 1-jets*

$$j^1 S = \{(q,\, u = S(q)),\, p = \mathrm{d}S_q\}$$

are Legendrian submanifolds.

Darboux (contact) theorem. *Locally, there exist coordinates (x^i, y_i, z) such that*

$$\alpha = \mathrm{d}z - \sum_i y_i \, \mathrm{d}x^i.$$

Consequently, all contact structures are locally equivalent.

Contact transformations. *A diffeomorphism Φ is a* contact transformation *if*

$$\Phi^* \alpha = f\, \alpha,$$

for some nowhere-vanishing function f (it is strict *if $f \equiv 1$).*

3.3 Affine–contact viewpoint

Definition 43 (Vertical bundle). *Let $\pi : E \to M$ be a smooth fiber bundle. The* vertical bundle *is the vector subbundle*

$$VE \;:=\; \ker(d\pi) \;\subset\; TE,$$

whose fiber at $e \in E$ is

$$V_e E \;=\; \{\, v \in T_e E \;\mid\; d\pi_e(v) = 0 \,\}.$$

Equivalently, $V_e E = T_e\big(\pi^{-1}(\pi(e))\big)$ is the tangent space to the fiber through e.

Proposition 17 (Vector-bundle case). *If $\pi : E \to M$ is a vector bundle, there is a canonical isomorphism*

$$VE \;\cong\; \pi^* E,$$

sending $(e, v) \in (\pi^* E)_e \simeq \{(e, v) \mid v \in E_{\pi(e)}\}$ to the vertical lift $v^{\uparrow}_e := \frac{d}{dt}\Big|_{t=0}(e + t\, v) \in V_e E$ *along the fiber through e.*

Definition 44 (Vertical lift of a section). *For a section $s \in \Gamma(E)$ of a vector bundle and $v \in \Gamma(E)$, the* vertical lift $v^{\uparrow} \in \mathfrak{X}(E)$ *is the vertical vector field defined by*

$$v^{\uparrow}_e \;=\; \frac{d}{dt}\bigg|_{t=0} \Big(e + t\, v(\pi(e))\Big) \;\in\; V_e E.$$

Definition 45 (Affine bundle modeled on a vector bundle). *Let be smooth and let $V \to M$ be a vector bundle. A surjective submersion $p : A \to M$ is an* affine bundle modeled on V *if each fiber A_x is an affine space modeled on V_x, with smooth fiberwise difference $(a, b) \mapsto a-b \in V_x$ and action $(v, a) \mapsto a+v \in A_x$ that is free and transitive. Equivalently, A is locally trivial with transition maps in the affine group bundle $V \rtimes \mathrm{GL}(V)$. If A admits a global section, then (non-canonically) $A \simeq V$ as bundles over M.*

Proposition 18. *The projection $\pi_{1,0} : J^1E \to E$ is an* affine bundle *modeled on $\pi^*T^*M \otimes_V VE$, where $VE \subset TE$ is the vertical bundle of π. For $j^1_x s$, $j^1_x t \in (J^1E)_{s(x)}$ over the same point $e = s(x)$, their difference is the linear map $T_xM \to V_eE$, i.e. an element of $T^*_xM \otimes V_eE$.*

Definition 46 (Cartan distribution on J^1E). *The* Cartan distribution *is the vector subbundle*

$$\mathcal{C} \;:=\; \ker(\theta) \;\subset\; T(J^1E),$$

equivalently the common annihilator of the θ^α:

$$X \in \mathcal{C} \iff \theta^\alpha(X) = 0 \;\; \textit{for all } \alpha.$$

Definition 47 (Projectable vector field). *Let $\pi : E \to M$ be a smooth bundle. A vector field $X \in \mathfrak{X}(E)$ is* projectable *if there exists $Y \in \mathfrak{X}(M)$ with $d\pi \circ X = Y \circ \pi$. We write $Y = \pi_* X$. X is* vertical *iff $\pi_* X = 0$.*

Proposition 19 (Coordinate criterion). *In coordinates (x^i, u^α), $X = \xi^i(x, u)\partial_{x^i} + \phi^\alpha(x, u)\partial_{u^\alpha}$ is projectable iff $\xi^i = \xi^i(x)$ depends only on base variables. Then $\pi_* X = \xi^i(x)\partial_{x^i}$.*

Definition 48 (First prolongation). *If X is projectable, its prolongation to J^1E is*

$$\mathrm{pr}^1 X = \xi^i \partial_{x^i} + \phi^\alpha \partial_{u^\alpha} + \Big(D_i\phi^\alpha - u^\alpha_j D_i \xi^j\Big)\partial_{u^\alpha_i}, \quad D_i = \partial_{x^i} + u^\beta_i \partial_{u^\beta}.$$

It preserves the contact forms $\theta^\alpha = du^\alpha - u^\alpha_i dx^i$ and the Cartan distribution.

(Remark: The Cartan 1-forms θ^a assemble to a vector–valued form $\theta : TJ^1E \to \pi^*_{1,0}VE$ whose kernel is precisely the Cartan distribution. Diffeomorphisms of E (and projectable vector fields) prolong to J^1E and act by transformations preserving θ and $\mathcal{C}$.)

3.4 Variational language

A kth-order Lagrangian is a horizontal n-form on J^kE. Using the variational bicomplex on jets(defined later if time permits), one recovers the Euler–Lagrange equations, Noether currents, and conservation laws; we only need the idea that the jet–space language cleanly packages "fields + derivatives + coordinate-free" structures.

4 Some Physical Applications

The geometry of Hamilton–Jacobi theory admits both symplectic and contact formulations, corresponding respectively to conservative and dissipative (or thermostatted) systems.

Symplectic Hamilton–Jacobi theory. *On the symplectic manifold* $(T^*Q, \omega = \mathrm{d}q^i \wedge \mathrm{d}p_i)$, *a function* $S \in C^\infty(Q)$ *determines the Lagrangian submanifold*

$$L_S = \{\, (q,p) \in T^*Q \mid p = \mathrm{d}S_q \,\}.$$

Such submanifolds are invariant under the Hamiltonian flow precisely when S *satisfies the classical Hamilton–Jacobi equation.*

Contact Hamilton–Jacobi theory. *For systems with dissipation or thermostats, one works on the contact manifold*

$$(J^1(Q), \alpha = \mathrm{d}u - p_i\,\mathrm{d}q^i),$$

where the dynamics is generated by a contact Hamiltonian. A smooth function $S \in C^\infty(Q)$ *defines the Legendrian submanifold*

$$j^1S = \{\, (q, u = S(q), p = \mathrm{d}S_q) \,\} \subset J^1(Q).$$

These Legendrian graphs encode solutions of the contact Hamilton–Jacobi equation*, and invariance of* j^1S *under the contact flow corresponds to the evolution of* S *along the dissipative dynamics.*

4.1 Example: the damped parametric oscillator

The contact Hamiltonian formalism is illustrated with the one-dimensional damped parametric oscillator of mass m *and time-dependent frequency* $\omega(t)$*, whose contact Hamiltonian is*

$$\mathcal{H}(q,p,S,t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)q^2 + \gamma S.$$

When $\omega(t) = \omega_0$ *we recover the damped harmonic oscillator, and when* $\omega(t) = 0$ *the damped free particle.*

Contact Hamiltonian equations. *The dynamics is given by*

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m},$$
$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} - p\,\frac{\partial \mathcal{H}}{\partial S} = -m\omega^2(t)q - \gamma p,$$
$$\dot{S} = p\frac{\partial \mathcal{H}}{\partial p} - \mathcal{H} = \frac{p^2}{2m} - \frac{1}{2}m\omega^2(t)q^2 - \gamma S.$$

4.1.1 First route: contact transformations

We perform the time-dependent contact transformation

$$(q_E, p_E, S_E, t) = \Big(q\, e^{\frac{\gamma t}{2}},\ [p + \tfrac{m\gamma}{2}q]\, e^{\frac{\gamma t}{2}},\ [S + \tfrac{m\gamma}{4}q^2]\, e^{\gamma t},\ t\Big),$$

which defines the so-called expanding coordinates. *In these variables the transformed Hamiltonian reads*

$$\mathcal{H}_E = e^{\gamma t}\mathcal{H} - \frac{\partial S_E}{\partial t} = \frac{p_E^2}{2m} + \frac{m}{2}\Big(\omega^2(t) - \frac{\gamma^2}{4}\Big)q_E^2.$$

This "expanding Hamiltonian" represents a parametric oscillator with shifted frequency $\omega_E^2(t) = \omega^2(t) - \gamma^2/4$.

4.1.2 Second route: invariants

The system admits the quadratic invariant

$$\mathcal{I}(q,p,t) = \frac{me^{\gamma t}}{2}\left[\Big(\frac{\alpha(t)p}{m} - [\dot{\alpha}(t) - \tfrac{\gamma}{2}\alpha(t)]q\Big)^2 + \Big(\frac{q}{\alpha(t)}\Big)^2\right],$$

where $\alpha(t)$ *satisfies the* Ermakov equation

$$\ddot{\alpha} + \Big(\omega^2(t) - \frac{\gamma^2}{4}\Big)\alpha = \frac{1}{\alpha^3}.$$

There is also an S*-dependent invariant*

$$\mathcal{G}(q,p,S,t) = e^{\gamma t}\Big(S - \frac{qp}{2}\Big).$$

Using $(\mathcal{I}, \mathcal{G})$ *one defines a time-dependent contact transformation*

$$\tilde{Q} = \arctan\Big(\alpha\,[\dot{\alpha} - \tfrac{\gamma}{2}\alpha] - \alpha^2\tfrac{p}{mq}\Big), \qquad \tilde{P} = \mathcal{I}, \qquad \tilde{S} = \mathcal{G}, \qquad t = t,$$

whose conformal factor is $f = e^{\gamma t}$. *In these variables the new contact Hamiltonian* *becomes*

$$\mathcal{K} = \frac{\mathcal{I}}{\alpha^2},$$

so that the equations of motion reduce to

$$\dot{\tilde{Q}} = \frac{1}{\alpha^2}, \qquad \dot{\tilde{P}} = 0, \qquad \dot{\tilde{S}} = 0.$$

The solutions are therefore

$$\tilde{Q}(t) = \int^t \frac{d\tau}{\alpha^2(\tau)}, \quad \tilde{P}(t) = \mathcal{I}, \quad \tilde{S}(t) = \mathcal{G}.$$

Inverting the transformation gives the physical solutions

$$\begin{aligned} q(t) &= \sqrt{\frac{2\mathcal{I}}{m}}\, e^{\frac{\gamma t}{2}} \alpha(t) \cos\phi(t), \\ p(t) &= \sqrt{2m\mathcal{I}}\, e^{\frac{\gamma t}{2}} \Big([\dot{\alpha} - \tfrac{\gamma}{2}\alpha] \cos\phi(t) - \frac{1}{\alpha} \sin\phi(t) \Big), \\ S(t) &= e^{-\gamma t}\mathcal{G} + \frac{1}{2} e^{-\gamma t} q(t) p(t), \end{aligned}$$

where $\phi(t) = \tilde{Q}(t)$.

4.1.3 Third route: the contact Hamilton–Jacobi equation

The same dynamics follows from the contact Hamilton–Jacobi (HJ) equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\omega^2(t) q^2 + \gamma S = -\frac{\partial S}{\partial t}.$$

Taking the ansatz

$$S(q,t) = mC(t)\left[\frac{q^2}{2} - \lambda(t) q\right] + mq\dot{\lambda}(t) + s(t),$$

one finds

$$p(q,t) = mC(t)\,[q - \lambda(t)] + m\dot{\lambda}(t),$$

and substitution into the HJ equation yields:

$$\begin{aligned} \dot{C} + C^2 + \gamma C + \omega^2(t) &= 0, && \textit{(Riccati equation)} \\ \ddot{\lambda} + \gamma\dot{\lambda} + \omega^2(t)\lambda &= 0, && \textit{(damped Newton eq.)} \\ \dot{s} = -\frac{m}{2}\Big[C^2\lambda^2 - 2C\lambda\dot{\lambda} + \dot{\lambda}^2\Big] - \gamma s. \end{aligned}$$

Integrating, one obtains

$$s(t) = \frac{m}{2}\Big[C\lambda^2 - \lambda\dot{\lambda}\Big],$$

and the full solution

$$S(q,t) = \frac{m}{2}C(t)[q-\lambda(t)]^2 + m\dot{\lambda}(t)[q-\lambda(t)] + \frac{m}{2}\lambda(t)\dot{\lambda}(t).$$

The functions $C(t)$ and $\lambda(t)$ are related by $C(t) = \dot{\lambda}(t)/\lambda(t)$, so solving either the Ricatti or the damped Newton equation suffices to determine $S(q,t)$.

Interpretation. *Each method provides an equivalent solution route:*

- *Section 3.6.1: solve the damped Newton equation for $\mathcal{H}_E$;*
- *Section 3.6.2: use the invariants $\mathcal{I}$ and $\mathcal{G}$ (Ermakov approach);*
- *Section 3.6.3: integrate the Riccati equation (HJ approach).*

Thus, the three geometric techniques in contact mechanics correspond to the three classical approaches (Newtonian, Ermakov, and Riccati) to the damped parametric oscillator.

4.2 Geometric Integration of Simple Thermodynamical Systems

In this section, geometric numerical methods for dissipative or thermodynamical systems formulated within the framework of contact geometry are discussed. Unlike purely symplectic systems, thermodynamical dynamics involve both conservative and dissipative aspects, which can be encoded by contact Hamiltonian or Lagrangian structures. The aim is to show how structure-preserving numerical schemes can be constructed by exploiting discrete geometric techniques—specifically, discrete gradients and the discrete Herglotz principle.

Integration Based on Discrete Gradients *For simplicity, let the configuration manifold be $Q = \mathbb{R}^n$, with phase-space coordinates $x = (q^i, p_i, S) \in T^*Q \times \mathbb{R}$. Let $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ be the contact Hamiltonian. The dynamics of the system are then governed by the contact Hamiltonian vector field*

$$\dot{x} = (\sharp_\Lambda)_x(\nabla H(x)),$$

where Λ denotes the contact bivector associated with the canonical contact form on $T^*Q \times \mathbb{R}$. The Euclidean gradient $\nabla H(x)$ is taken in the natural coordinates of $\mathbb{R}^{2n+1}$.

The central idea of discrete gradient methods is to approximate the continuous gradient by a discrete one that exactly preserves first integrals. This leads to energy-preserving geometric integrators that retain key structural properties of the underlying continuous system.

Definition 49. *Let $H : \mathbb{R}^N \to \mathbb{R}$ be differentiable. A* discrete gradient *of H is a continuous map $\bar{\nabla} H : \mathbb{R}^{2N} \to \mathbb{R}^N$ satisfying:*

$$\bar{\nabla} H(x, x')^\top (x' - x) = H(x') - H(x), \quad \forall\, x, x' \in \mathbb{R}^N, \tag{1}$$

$$\bar{\nabla} H(x, x) = \nabla H(x), \quad \forall\, x \in \mathbb{R}^N. \tag{2}$$

Several standard choices of discrete gradients appear in the literature:

Mean Value (Averaged) Discrete Gradient.

$$\bar{\nabla}_1 H(x, x') = \int_0^1 \nabla H((1 - \xi)x + \xi x')\, d\xi, \quad x' \neq x.$$

Midpoint (Gonzalez) Discrete Gradient.

$$\bar{\nabla}_2 H(x, x') = \nabla H\left(\frac{x + x'}{2}\right) + \frac{H(x') - H(x) - \nabla H\left(\frac{x+x'}{2}\right)^\top (x' - x)}{|x' - x|^2}(x' - x), \quad x' \neq x.$$

Coordinate Increment Discrete Gradient. *For each component i,*

$$[\bar{\nabla}_3 H(x, x')]_i = \frac{H(x'_1, \dots, x'_i, x_{i+1}, \dots, x_N) - H(x'_1, \dots, x'_{i-1}, x_i, \dots, x_N)}{x'_i - x_i}.$$

Once a discrete gradient has been chosen, one can define a structure-preserving contact integrator *using the midpoint rule:*

$$\frac{x_{k+1} - x_k}{h} = (\sharp_\Lambda)_{\frac{x_k + x_{k+1}}{2}}\, \bar{\nabla}_2 H(x_k, x_{k+1}),$$

where Λ is the canonical contact bivector. Due to the skew-symmetry of Λ, one has

$$H(x_{k+1}) - H(x_k) = \bar{\nabla}_2 H(x_k, x_{k+1})^\top (x_{k+1} - x_k) = h\Lambda(\bar{\nabla}_2 H(x_k, x_{k+1}), \bar{\nabla}_2 H(x_k, x_{k+1})) = 0,$$

ensuring that H is exactly preserved. However, the entropy S evolves according to

$$S_{k+1} - S_k = h\, \Lambda(\bar{\nabla}_2 H(x_k, x_{k+1}), dS).$$

Example 4.2: Damped Harmonic Oscillator. *Consider the contact Hamiltonian*

$$H(q, p, S) = \frac{p^2}{2} + \frac{q^2}{2} + \gamma S,$$

which describes a damped harmonic oscillator with damping coefficient γ. *Applying the midpoint discrete gradient scheme yields the explicit update*

$$q_1 = \frac{2\gamma h q_0 - h^2 q_0 + 4hp_0 + 4q_0}{2\gamma h + h^2 + 4}, \tag{3}$$

$$p_1 = \frac{2\gamma h p_0 + h^2 p_0 + 4hq_0 - 4p_0}{2\gamma h + h^2 + 4}, \tag{4}$$

$$S_1 = \frac{S_0 h^4 + (4S_0\gamma + 4q_0^2)h^3 + (4S_0\gamma^2 - 16p_0q_0 + 8S_0)h^2 + (16S_0\gamma + 16p_0^2)h + 16S_0}{(2\gamma h + h^2 + 4)^2}. \tag{5}$$

This integrator preserves the total energy exactly while exhibiting monotonic growth in the entropy S_k.

4.3 Integration Based on the Discrete Herglotz Principle

A complementary approach arises from discretizing the variational principle underlying the Herglotz equations, which govern contact Lagrangian systems. Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ *be a regular Lagrangian with coordinates* $(q^i, \dot{q}^i, S)$. *We define the canonical one-form*

$$\lambda_L = \frac{\partial L}{\partial \dot{q}^i}\, dq^i,$$

and the contact form

$$\eta_L = dS - \lambda_L = dS - \frac{\partial L}{\partial \dot{q}^i}\, dq^i.$$

If L *is regular,* $\eta_L \wedge (d\eta_L)^n \neq 0$, *and thus* $(TQ \times \mathbb{R}, \eta_L)$ *defines a contact manifold. The associated Reeb vector field is*

$$\mathcal{R}_L = \frac{\partial}{\partial S} - W^{ij} \frac{\partial^2 L}{\partial \dot{q}^j \partial S} \frac{\partial}{\partial \dot{q}^i},$$

where W^{ij} *is the inverse Hessian of* L *with respect to* $\dot{q}^i$. *The energy function is*

$$E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L.$$

The Herglotz variational principle leads to the generalized Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial S}, \qquad \dot{S} = L(q^i, \dot{q}^i, S).$$

These can be compactly expressed as the thermodynamical Herglotz equations

$$\mathcal{E}_L = \sharp_{\Lambda_L}(dE_L) \quad \Longleftrightarrow \quad b_L(\xi_L) = dE_L - (\mathcal{R}_L E_L + E_L)\eta_L,$$

where Λ_L is the contact bivector associated with η_L.

4.3.1 Discrete Herglotz Equations

Following this structure, we define a discrete Lagrangian $L_d : Q \times Q \times \mathbb{R} \to \mathbb{R}$ and obtain the discrete Herglotz evolution rule:

$$D_1 L_d(q_1, q_2, S_1) + (1 + D_S L_d(q_1, q_2, S_1)) D_2 L_d(q_0, q_1, S_0) = 0, \tag{6}$$

$$S_1 - S_0 = (q_1 - q_0) D_2 L_d(q_0, q_1, S_0). \tag{7}$$

These equations define an implicit geometric integrator for contact dynamics.

Example 4.3: Damped Harmonic Oscillator. *Consider again*

$$H(q, p, S) = \frac{p^2}{2} + \frac{q^2}{2} + \gamma S.$$

Its corresponding Lagrangian is

$$L(q, \dot{q}, S) = \frac{\dot{q}^2}{2} - \frac{q^2}{2} - \gamma S.$$

A standard discretization via midpoint quadrature gives

$$L_d(q_0, q_1, S_0) = \frac{(q_1 - q_0)^2}{2h} - h\frac{(q_1 + q_0)^2}{8} - h\gamma S_0.$$

Substituting into (6)–(7) *yields the explicit update rule*

$$q_2 = \frac{\gamma h^3 q_0 + \gamma h^3 q_1 + 4\gamma h q_0 - 4\gamma h q_1 - h^2 q_0 - 2h^2 q_1 - 4q_0 + 8q_1}{h^2 + 4}, \tag{8}$$

$$S_1 = S_0 + \frac{(q_1 - q_0)^2}{h} - h\frac{q_1^2 - q_0^2}{4}. \tag{9}$$

The discrete Herglotz integrator reproduces the exact motion of the damped oscillator with high fidelity, preserving the monotonic increase of entropy and maintaining a bounded oscillatory Hamiltonian.

Discussion. *The discrete gradient and discrete Herglotz integrators provide two consistent geometric approaches to numerically integrate contact Hamiltonian systems. Both preserve the underlying thermodynamic structure: the former preserves energy exactly and ensures entropy production, while the latter discretizes the Herglotz variational principle itself. In both schemes, the entropy S_k increases monotonically until stabilization, and the Hamiltonian oscillates around a nearly constant mean value, accurately reproducing the dissipative dynamics of the damped harmonic oscillator.*

5 Brief Application to Cosmology

5.1 Two-field cosmological models

Let $(\Sigma, \mathcal{G})$ *be a Riemannian target manifold of real dimension* 2 *and let* $\varphi : (M, g_{FRW}) \to (\Sigma, \mathcal{G})$ *be a spatially homogeneous map (the fields). We take the flat FRW metric*

$$g_{FRW} = -dt^2 + a(t)^2\, d\vec{x}^{\,2},$$

with Hubble parameter $H(t) = \dot{a}(t)/a(t)$ *and scale factor* $a(t) > 0$*. The action of a two-field model with potential* $V : \Sigma \to \mathbb{R}_{\geq 0}$ *is*

$$S[\varphi, a] = \int \left[\frac{M_{\mathrm{Pl}}^2}{2} R(g_{FRW}) - \frac{1}{2} \mathcal{G}_{ij}(\varphi)\, \dot{\varphi}^i \dot{\varphi}^j - V(\varphi) \right] a(t)^3\, dt\, d^3x\,. \tag{10}$$

Upon reduction to homogeneous profiles (comoving volume set to 1*), the Lagrangian reads*

$$L(\varphi, \dot{\varphi}, a, \dot{a}) = -3M_{\mathrm{Pl}}^2 a\dot{a}^{\,2} + \frac{a^3}{2}\, \mathcal{G}_{ij}(\varphi)\, \dot{\varphi}^i \dot{\varphi}^j - a^3 V(\varphi)\,. \tag{11}$$

The Friedmann constraint is

$$3M_{\mathrm{Pl}}^2 H^2 = \frac{1}{2} \mathcal{G}_{ij}(\varphi)\, \dot{\varphi}^i \dot{\varphi}^j + V(\varphi)\,. \tag{12}$$

The scalar equation is the covariant Euler–Lagrange equation:

$$\nabla_t \dot{\varphi}^i + 3H\dot{\varphi}^i + \mathcal{G}^{ij}(\varphi)\, \partial_j V(\varphi) = 0, \tag{13}$$

where ∇_t *is the pullback Levi-Civita covariant derivative along* $t \mapsto \varphi(t)$*. The second Friedmann equation is*

$$-2M_{\mathrm{Pl}}^2 \dot{H} = \mathcal{G}_{ij}(\varphi)\, \dot{\varphi}^i \dot{\varphi}^j\,. \tag{14}$$

5.2 Isothermal (complex) parameterization of the target

Since Σ *is a Riemann surface, there exist local complex coordinates* $z = x + iy$ *in which*

$$\mathcal{G} = \lambda(z, \bar{z})\, |dz|^2 = \lambda(x, y)\, (dx^2 + dy^2), \qquad \lambda > 0. \tag{15}$$

Writing $\varphi = (x(t), y(t))$ *and* $\dot{z} = \dot{x} + i\dot{y}$*, the kinetic term becomes*

$$\frac{a^3}{2}\, \mathcal{G}_{ij} \dot{\varphi}^i \dot{\varphi}^j = \frac{a^3}{2}\, \lambda(z, \bar{z})\, |\dot{z}|^2. \tag{16}$$

The Levi-Civita connection in isothermal charts satisfies

$$\Gamma^{z}_{zz} = \partial_z\Big(\log\lambda\Big), \qquad \Gamma^{\bar z}_{\bar z\bar z} = \partial_{\bar z}\Big(\log\lambda\Big), \qquad \Gamma^{z}_{z\bar z} = \Gamma^{z}_{\bar z z} = 0, \tag{17}$$

and the covariant equation (13) *becomes, in complex form,*

$$\ddot z + \partial_z(\log\lambda)\,\dot z^2 + 3H\dot z + 2\,\lambda^{-1}\partial_{\bar z}V = 0. \tag{18}$$

5.3 Slow-roll/Hamilton–Jacobi reformulation

Introduce the e-fold time N with $dN = H\,dt$ (when $H > 0$) and define the (reduced) HJ generating function $\mathcal{S} : \Sigma \to \mathbb{R}$ by the ansatz

$$\dot\varphi^i \equiv -\frac{1}{3H}\,\mathcal{G}^{ij}\partial_j V \quad\Longleftrightarrow\quad \dot\varphi^i = \mathcal{G}^{ij}\partial_j\mathcal{S}, \tag{19}$$

with H seen as a function of φ via the constraint (12)*. Eliminating velocities using* (14) *yields the Hamilton–Jacobi relation*

$$\frac{1}{2}\,\mathcal{G}^{ij}(\varphi)\,\partial_i\mathcal{S}\,\partial_j\mathcal{S} + V(\varphi) = 3M_{\rm Pl}^2\,H(\varphi)^2, \tag{20}$$

with H evolving consistently with (14)*. In isothermal coordinates,*

$$\frac{1}{2\lambda(z,\bar z)}\,|\partial\mathcal{S}|^2 + V(z,\bar z) = 3M_{\rm Pl}^2\,H(z,\bar z)^2, \qquad |\partial\mathcal{S}|^2 \equiv \partial_z\mathcal{S}\,\partial_{\bar z}\mathcal{S}. \tag{21}$$

5.4 Conformal class and Liouville form

Fix a representative $\mathcal{G}_0 = \lambda_0|dz|^2$ in the conformal class $[\mathcal{G}]$ and write

$$\mathcal{G} = e^{2u}\,\mathcal{G}_0, \qquad \lambda = e^{2u}\lambda_0. \tag{22}$$

Then (21) *becomes*

$$\frac{e^{-2u}}{2\lambda_0}\,|\partial\mathcal{S}|^2 + V = 3M_{\rm Pl}^2H^2. \tag{23}$$

The Gaussian curvatures satisfy the Liouville relation

$$K_{\mathcal{G}} = e^{-2u}\Big(K_{\mathcal{G}_0} - \Delta_{\mathcal{G}_0}u\Big), \qquad \Delta_{\mathcal{G}_0} = \frac{4}{\lambda_0}\,\partial_z\partial_{\bar z}. \tag{24}$$

5.5 The strong SRRT equation in a fixed conformal class

Let $\mathcal{R}$ denote a curvature-like scalar determined by the dynamics (e.g. target curvature, or a prescribed function along the flow). The strong SRRT problem asks for a conformal *factor u solving*

$$-\Delta_{\mathcal{G}_0} u + K_{\mathcal{G}_0} = \mathcal{R}\, e^{2u}, \tag{25}$$

possibly coupled to the HJ relation (23) *through* H *and/or* V*. Given* V *and* $[\mathcal{G}]$*, one studies existence/uniqueness and regularity of* u *under appropriate boundary conditions.*

5.6 Hamilton–Jacobi form as a first-order PDE for u

In settings where $\mathcal{S}$ *can be parameterized in terms of* u *and auxiliary data (e.g. through momentum variables), the HJ constraint can be recast as a first-order nonlinear PDE for* u *of the form*

$$F(x_1, x_2, u, \partial_1 u, \partial_2 u) = 0, \tag{26}$$

whose key feature is properness *in* u *(monotone dependence), characteristic of contact-type Hamilton–Jacobi equations. This reformulation admits a method-of-characteristics analysis and viscosity solution theory.*

5.7 Boundary value formulations

Let $\Omega \subset \Sigma$ *be a simply connected domain endowed with an isothermal chart* z*. Typical boundary problems include Dirichlet data on a closed curve* $\gamma \subset \partial\Omega$*,*

$$u|_{\gamma} = u_0, \tag{27}$$

or mixed conditions coupling u *and its normal derivative in* (25)*. In the HJ reformulation* (26)*, admissible boundary momenta must satisfy tangency and on-curve constraints.*

5.8 The Strong SRRT Equation

On $(\Omega, \mathcal{G}_0)$ *with smooth boundary, the strong SRRT equation* (25) *reads*

$$-\Delta_{\mathcal{G}_0} u + K_{\mathcal{G}_0} = \mathcal{R}\, e^{2u} \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = u_0. \tag{28}$$

When $\mathcal{R} > 0$ *is smooth and* Ω *is topologically a disk,* (28) *is a Liouville-type elliptic PDE with standard variational structure. Maximum principles and Moser–Trudinger inequalities provide a priori bounds in appropriate regimes.*

5.9 Fixed potential V and fixed conformal class

When V is fixed and the metric is sought in the conformal class $[\mathcal{G}_0]$, the coupled system (23)–(28) *becomes*

$$\frac{e^{-2u}}{2\lambda_0}\,|\partial\mathcal{S}|^2 + V = 3M_{\rm Pl}^2 H^2, \tag{29}$$

$$-\Delta_{\mathcal{G}_0} u + K_{\mathcal{G}_0} = \mathcal{R}\, e^{2u}. \tag{30}$$

One may eliminate H via (12) *or treat H as a derived quantity along solutions. In models where $\partial\mathcal{S}$ admits an algebraic parameterization in terms of u and V,* (29) *becomes a first-order constraint for u.*

5.10 Characteristic formulation and admissible data

Writing (26) *in local coordinates $x = (x_1, x_2)$, its characteristics $(x(t), u(t), p(t))$ with $p = \nabla u$ satisfy*

$$\dot{x}_i = F_{p_i}, \qquad \dot{u} = p_i F_{p_i}, \qquad \dot{p}_i = -F_{x_i} - p_i F_u, \qquad i = 1, 2, \tag{31}$$

restricted to the hypersurface $F = 0$. Boundary data (γ, ϕ_0) along a smooth curve γ are admissible *if there exists p_0 on γ such that*

$$F(\gamma, \phi_0, p_0) = 0, \qquad \gamma' \cdot p_0 = \phi_0',$$

and a noncharacteristic condition holds ($F_p \wedge \gamma' \neq 0$). Local solvability then follows by the method of characteristics.

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