Integrability in dynamical systems

Andrei Marin

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Abstract

This report concerns the activity of its author in the Departament of Theoretical Physics, National Institute for Physics and Nuclear Engineering, between July and September 2024, under the supervision of Dr. Stefan Carstea.
The report introduces nonlinear wave equations (such as Korteweg-de Vries equation) and the technique of multiple scales for obtaining asymptotic solutions of these equations in the long wavelength limit. This method is further applied to "envelope" equations (such as nonlinear Schrödinger). As an application for these techniques, the KdV hierarchy is obtained for water waves equations.
In addition to this, the report presents Hirota's bilinear formalism, starting from the definition of bilinear operators. They are used to obtain soliton solutions, which are further studied in terms of solitonic interaction and as test for integrability.
The last section of the report describes particular solitonic solutions and manner of interaction between solitons.

Contents

1 Asymptotic integrability and KdV hierarchy for fluid waves

1.1 Introduction to nonlinear waves

1.1.1 Overview of nonlinear equations in physics

Most of the physical theories rely on linear equations and the nonlinear terms are considered to be small and treated through perturbations. In the following, we will present some nonlinear systems. We will consider infinite-dimensional systems, such as:

$$\frac{\partial u}{\partial t} = \left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots \right) \tag{1}$$

One can denote $u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$ etc.
We will also encounter discrete-differential systems, such as:

$$\frac{\partial u}{\partial t} = F(u, \overline{u}, \overline{\overline{u}}, ..) \tag{2}$$

Here $\overline{u} = u(x+h,t), \overline{\overline{u}} = u(x+2h,t)$ etc. h is the discrete step of x.
Finally, we can encounter fully discrete systems, such as:

$$\tilde{u} - u = F(u, \overline{u}, \tilde{\overline{u}}, \dots) \tag{3}$$

Here $\tilde{u} = u(x, t+\delta), \tilde{\overline{u}} = u(x+h, t+\delta), \underline{\tilde{u}} = u(x-h, t+\delta)$ etc. In all examples above F denoted a nonlinear function.
There are two main branches of nonlinear physics: chaos and self-organization. Chaotic process are modeled by differential equations that are poor in symmetries [AL12]. Since these equations are unsolvable, chaotic systems can only be studied numerically. On the other hand, self-organized processes are described by fully (or partially) integrable equations, which possess a high amount of internal symmetry.
The simplest nonlinear self-organized field structure is the **soliton**. It appears as a solution of some fully integrable nonlinear equations and can interact in a perfectly elastic manner to any types of waves [DJ89]. There exist localized nonlinear wave solutions for non-integrable nonlinear equations, but these solutions do no interact in an elastic manner and therefore break apart. Such solutions are called solitary waves.
Solitons can, as we will investigate in this report, recover their original identity after colliding with another soliton [Hir04]. Such a behavior has been firstly proven through numerical simulations and is now investigated analytically. The following examples of nonlinear equations are fully integrable:

1. Korteweg - de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{4}$$

2. Modified KdV (mKdV) equation:

$$u_t \pm 6u^2 u_x + u_{xxx} = 0 \tag{5}$$

3. Sine-Gordon (SG) equation:

$$u_{tt} - u_{xx} = \sin u \tag{6}$$

4. Kadomtsev–Petviashvili (KP) equation:

$$(u_t + 6uu_x + u_{xxx})_x = \sigma u_{yy} \tag{7}$$

5. Țițeica equation:

$$u_{tt} - u_{xx} = e^u - e^{2u} \tag{8}$$

6. Toda or Toda lattice (TL) equation:

$$\frac{\partial^2 u_n}{\partial t^2} = e^{-(u_{n+1}-u_n)} - e^{-(u_n-u_{n-1})} \tag{9}$$

7. Volterra equation:

$$\frac{\partial u_n}{\partial} = (a + bu_n + cu_n^2)(u_{n+1} - u_{n-1}) \tag{10}$$

8. Self-dual Young-Mills (SDYM) equation:

$$F_{\mu\nu} = \varepsilon_{\mu\nu}^{\sigma\rho} F_{\sigma\rho} \tag{11}$$

9. Nonlinear Schrödinger equation:

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0 \tag{12}$$

All of the equations above possess a linear and a nonlinear part.

1.1.2 Brief reminder on linear equations

In the following we will review several techniques used in studying linear wave equations. Let us consider the following example:

$$u_t + u_{xxx} = 0 \tag{13}$$

We consider a solution $u(x,t) = A\exp(ikx - \omega t)$. We denote by oscillation mode a pair $(k, \omega(x))$. By replacing this solution in the equation, we arrive at the dispersion relation:

$$\omega(k) = -k^3 \tag{14}$$

Formally, the solution of the equation writes as:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} A(k)\exp(ikx + ik^3 t)\mathrm{d}x \tag{15}$$

Here we express $A(k)$ as function of the initial condition:

$$A(k) = \int_{\mathbb{R}} u(x,0)e^{-ikx}\mathrm{d}x \tag{16}$$

In the case when $\hat{u}_t + \hat{u}_{xxx} = 0$, with $\hat{u}$ a matrix, we can take $\hat{u} = \mathbb{A}\exp(ikx - i\omega t)$. By replacing in the equation we reach $\mathbb{M}\mathbb{A}\exp(ikx - i\omega t)$. By imposing $\det(\mathbb{M}) = 0$, we obtain the dispersion relation(s).
Discussion:

1. $\mathrm{Im}\omega(k) > 0$ for a certain k: that Fourier mode will increase in time and dominate the solution. Therefore, the solution is unstable.
2. $\mathrm{Im} \to \infty$ for a certain k: the problem is ill-posed (Hadamard sense).
3. $\mathrm{Im}\omega(k) < 0$ for all k: the problem is dissipative (for example, reaction-diffusion biological processes).
4. $\mathrm{Im}\omega(k) = 0$, for all k: we encounter wave propagation. From Parseval relation we have:

$$\int |u|^2\mathrm{d}x = \frac{1}{2\pi}\int |A(k)|^2\exp(2\mathrm{Im}(\omega(k))t\mathrm{d}k \tag{17}$$

   So in this case $\int |u|^2\mathrm{d}x$ is a conserved quantity. We can introduce $c_p = \frac{\omega}{k}$ as the phase velocity.

Observations:

- If $\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} \neq 0$ we have different phase velocities for the Fourier modes and a compact wave packet becomes dispersed. We call such an equation dispersive.
- If after a suficienty long time each wavenumber k dominates the solution in the region given by $x \sim c_g(k)t + \mathcal{O}(t^2)$, we have $c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ as the group velocity.

All the equations we will further study have a dispersive linear part ($\mathrm{Im}\omega(k) = 0$ and $\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} \neq 0$).

1.1.3 Travelling-wave reduction

Solutions of the type $u(x-vt,t)$, for an arbitrary v, are called "travelling-wave" (or solitary waves). If we substitute in a partial differential equation (PDE) in x and t by introducing $\xi = x - vt$, we obtain an ordinary differential equation (ODE) in ξ, which is called "travelling-wave reduction". Let us now calculate the solitary wave (or travelling-wave reduction) for KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{18}$$

By introducing $\xi = x - vt$, with v an arbitrary parameter, we arrive at:

$$-vu' + 6uu' + u''' = 0 \tag{19}$$

Here we replaced $u(x,t) \to u(\xi)$ and $u' = \frac{\mathrm{d}u}{\mathrm{d}\xi}$. We integrate once ($C$ is a constant):

$$-vu + C + 3u^2 + u'' = 0 \tag{20}$$

By multiplying with u' we obtain:

$$-\frac{v}{2}(u^2)' + Cu' + (u^3)' + \frac{1}{2}(u'^2)' = 0 \tag{21}$$

Which rewrites as(D is a constant):

$$-\frac{1}{2}u^2 + Cu + D + u^3 + \frac{1}{2}u'^2 = 0 \tag{22}$$

By imposing $u, u', u'' \to 0$ when $\xi \to \pm\infty$ we obtain $C = D = 0$ and the equation rewrites as:

$$u'^2 = -2u^3 + vu^2 \tag{23}$$

This integrates as:

$$\int \frac{\mathrm{d}u}{\sqrt{u^2(v-2u)}} = \pm \int \mathrm{d}\xi \tag{24}$$

We reach the solution:

$$u(\xi) = \frac{v}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{v}(x - vt)\right) \tag{25}$$

This is the solitary wave solution. Because KdV equation is fully integrable, the solution is 1-solitonic. We can obtain v from the initial condition $u(x,0) = F(x)$:

$$\frac{v}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{v}x\right) = F(x) \tag{26}$$

This equation is very restrictive for $F(x)$. One must observe the nonlinear character of the wave, as the velocity depends on the amplitude.

1.2 Multiple-scale method

1.2.1 Introduction

Multiple-scale method is an asymptotic method through which dispersion and nonlinearity are studied in as asymptotic regime where they balance each other. The result is a new equation with considerable chances of being integrable.

1.2.2 Chain of anharmonic oscillators

We start from the following physical model: an infinite linear chain of point masses of mass m are connected through anharmonic force of the type:

$$F(x) = -kx + \beta x^2 \tag{27}$$

We will label by x_n the displacement from the equilibrium position of point mass n. The Hamiltonian of this system is:

$$H = \sum_{n=-\infty}^{n=\infty} \left(\frac{p_n^2}{2} + V(x_{n+1} - x_n)\right) \tag{28}$$

Here $p_n = m\dot{x}_n$. The equation of motion writes as:

$$m\ddot{x}_n = -\frac{\partial H}{\partial x_n} = -\frac{\partial}{\partial x_n}\left(V(x_{n+1}-x_n) + V(x_n - x_{n-1})\right) \tag{29}$$

By inserting the expression of the force, we arrive at:

$$m\ddot{x} = k(x_{n+1} + x_{n-1} - 2x_n) + \beta(x_{n+1} + x_{n-1})(x_{n+1} + x_{n-1} - 2x_n) \tag{30}$$

One can try to find a direct solution by performing the travelling-wave reduction $x(n - vt)$, but this method is very hard. Instead, we will think about the behavior of the system for not too large oscillation which involve a large number of masses and considering large intervals of time.
We will simplify the equation by taking $\frac{k}{m} = 1$ and introduce $x_n \sim a\exp(ikn - i\omega t)$ in the linear part of the equation to obtain the dispersion relation:

$$\omega^2 = 2(1 - \cos k) = 4\sin^2\frac{k}{2} \tag{31}$$

We obtain two dispersion branches $\omega = \pm\sin\frac{k}{2}$, meaning that we encounter waves propagating in both directions. We consider the branch $\omega = \sin\frac{k}{2}$ and that we initially have a long wavelength wave (small wavenumber k). Therefore, we write $k = \varepsilon k_0$ with $\varepsilon \ll 1$. We can therefore perform a series expansion of the dispersion relation:

$$\omega(k) = k - \frac{k^3}{24} + \mathcal{O}(k^4) \tag{32}$$

The linear wave will be:

$$x_n \sim a\exp i(kn - kt + \frac{k^3}{24} + ...) \tag{33}$$

That suggests us a new scale for space and time, defined as follows (using $k \sim \varepsilon$):

$$\xi = \varepsilon(n - t)$$

$$\tau = \frac{\varepsilon^3}{24}$$

The new variables ξ and τ are called "stretched variables". We must also change the dependent variable $x_n(t) \to \varepsilon^p\Phi(\xi,\tau)$, were p is to be determined and Φ is the new function describing the process. The impact of nonlinearity is a change in amplitude that we do not know, but we measure through the factor ε^p. We will obtain p through the balance between dispersion and nonlinearity. By performing a series expansion on the terms in the equation of motion (30) we obtain:

$$x_{n+1} \to \varepsilon^p\Phi(\xi+\varepsilon,\tau) = \varepsilon^p\left(\Phi + \varepsilon\Phi_\xi + \frac{\varepsilon^2}{2}\Phi_{\xi\xi} + \frac{\varepsilon^3}{6}\Phi_{\xi\xi\xi} + \frac{\varepsilon^4}{24}\Phi_{\xi\xi\xi\xi} + ...\right) \tag{34}$$

$$x_{n-1} \to \varepsilon^p\Phi(\xi-\varepsilon,\tau) = \varepsilon^p\left(\Phi - \varepsilon\Phi_\xi + \frac{\varepsilon^2}{2}\Phi_{\xi\xi} - \frac{\varepsilon^3}{6}\Phi_{\xi\xi\xi} + \frac{\varepsilon^4}{24}\Phi_{\xi\xi\xi\xi} + ...\right) \tag{35}$$

$$\frac{\partial^2 x_n}{\partial t^2} \to \left(\frac{\varepsilon^3}{24}\partial_\tau - \varepsilon\partial_\xi\right)^2 \varepsilon^p\Phi(\xi,\tau) = \frac{\varepsilon^{p+6}}{576}\Phi_{\tau\tau} - \frac{\varepsilon^{p+4}}{12}\Phi_{\tau\xi} + \varepsilon^{p+2}\Phi_{\xi\xi} \tag{36}$$

Replacing in equation of motion (30) we arrive at:

$$\frac{\varepsilon^{p+6}}{576}\Phi_{\tau\tau} - \frac{\varepsilon^{p+4}}{12}\Phi_{\tau\xi} = \frac{\varepsilon^{p+4}}{12}\Phi_{\xi\xi\xi\xi} + 2\beta\varepsilon^{2p+3}\Phi_\xi\Phi_{\xi\xi} + \beta\frac{\varepsilon^{2p+5}}{3}(\Phi_{\xi\xi}\Phi_{\xi\xi\xi} + \frac{1}{2}\Phi_{xi}\Phi_{\xi\xi\xi\xi}) + ... \tag{37}$$

In order to find p, we use the maximum balance principle, that requires that at least two terms of the equation have the same power of ε and that all other terms have larger powers. The set of powers for this equation is: $\{p+4, p+6, 2p+3, 2p+5, 2p+7, ...\}$. The maximum balance is obtained for $2p+3 = p+4$, which leads us to $p = 1$. We can check that no power of ε smaller than 5 appears in the equation. On the other hand, an incorrect choice, such as $p+6 = 2p+3$ and $p = 3$, would have lead to the occurrence of smaller powers (in this case $p+4 = 7 < 9$).
For $p = 1$, equation (37) rewrites as:

$$-\frac{\varepsilon^5}{12}\Phi_{\xi\tau} = \frac{\varepsilon^5}{12}\Phi_{\xi\xi\xi\xi} + 2\beta\varepsilon^5\Phi_\xi\Phi_{\xi\xi} + ... \tag{38}$$

We have omitted writing the larger powers of ε, since we now divide by ε^5, take the limit $\varepsilon \to 0$ and replace $\Phi_\xi = u$, obtaining the KdV equation:

$$u_\tau + u_{\xi\xi\xi} + 24\beta u u_\xi = 0 \tag{39}$$

We remark that this limit is rigorously obtained, as we did not truncated any series, but only taken the limit $\varepsilon \to 0$. The solution of the initial problem writes as:

$$x_n(t) \to \varepsilon \int \mathrm{d}\xi u(\xi, \tau) \tag{40}$$

Here $u(\xi, \tau)$ is the solution of KdV equation in "streched variables" ξ and τ.

1.2.3 Obtaining KdV equation from Boussinesq equation

We will now apply the multiple scale method to obtain the KdV equation as the long wavelength limit for the Boussinesq equation. We start from the Boussinesq equation in the form presented below:

$$u_{tt} - u_{xx} - u_{xxxx} - 3(u^2)_{xx} = 0 \tag{41}$$

By considering a plane wave solution of the form $u(x,t) = a\exp(kx - \omega t)$ in the linear part of the equation, we arrive at the dispersion relation:

$$\omega^2 = k^2 - k^4 \tag{42}$$

By considering the positive solution for ω (forward propagating wave) and performing a Taylor expansion around $k = 0$ (the long wavelength limit), one arrives at:

$$\omega = k - \frac{k^3}{2} + \mathcal{O}(k^5) \tag{43}$$

By retaining only the first terms and returning to the plane wave solution, we find:

$$u = a\exp\left(k(x-t) + \frac{k^3}{2}t\right) \tag{44}$$

We will consider k proportional to some small parameter ε. This suggests the use of the following "streched variables":

$$\xi = \varepsilon(x - t)$$
$$\tau = \frac{\varepsilon^3}{2}t$$

The form of the solution in these variables will be $u = \varepsilon^p\Phi(\xi, \tau)$, with p to be determined from balancing the dispersion and the nonlinearity. To do this, we rewrite the equation in terms of the "stretched variables":

$$\left(\frac{\varepsilon^3}{2}\partial_\tau - \varepsilon\partial_\xi\right)^2 \varepsilon^p\Phi - \varepsilon^{p+2}\Phi_{\xi\xi} - \varepsilon^{p+4}\Phi_{\xi\xi\xi\xi} - 3\varepsilon^{2p+2}(\Phi^2)_{\xi\xi} = 0 \tag{45}$$

By expanding, one obtains:

$$\frac{\varepsilon^{p+6}}{4}\Phi_{\tau\tau} - \varepsilon^{p+4}\Phi_{\tau\xi} + -\varepsilon^{p+4}\Phi_{\xi\xi\xi\xi} - 3\varepsilon^{2p+2}(\Phi^2)_{\xi\xi} = 0 \tag{46}$$

The set of powers for ε is $p+6, p+4, 2p+2$. The balance is obviously obtained for $p = 2$, which leads to:

$$\frac{\varepsilon^8}{4}\Phi_{\tau\tau} - \varepsilon^6\Phi_{\tau\xi} - \varepsilon^6\Phi_{\xi\xi\xi\xi} - 3\varepsilon^6(\Phi^2)_{\xi\xi} = 0 \tag{47}$$

Now we can divide by ε^6 and take the limit $\varepsilon \to 0$:

$$\Phi_{\tau\xi} + \Phi_{\xi\xi\xi\xi} + 3(\Phi^2)_{\xi\xi} = 0 \tag{48}$$

Integrating the equation once with respect to ξ leads us to:

$$\Phi_\tau + \Phi_{\xi\xi\xi} + 6\Phi\Phi_\xi = 0 \tag{49}$$

This is precisely the KdV equation (in the "stretched varaibles" ξ and τ) that we attempted to obtain as a long wavelength asymptotic solution for the Boussinesq equation.

1.3 Multiple scale method for "envelope" equations

1.3.1 Nonlinear Schrödinger equation

One of the main examples of "envelope" equation is the Nonlinear Schrödinger equation:

$$i\psi_t + \psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{50}$$

We distinguish between two cases:

1. $\alpha > 0$: focusing NLS - it admits a large variety of solutions (solitons, breathers, rogues);
2. $\alpha < 0$: defocusing NLS - it usually does not admit solitons, unless it has non-zero boundary conditions ($\lim_{x\to\pm\infty} |\psi(x,t)|^2 \neq 0$), when it admits a dark-soliton (or hole-soliton) solution.

Since $\psi(x,t) \in \mathbb{C}$, what one can physically measure is the envelope $|\psi|^2$. The 1-soliton solution for NLS writes as:

$$\psi(x,t) = \exp i(k_0 x - \omega_0 t)\text{sech}(Kx - \Omega t) \tag{51}$$

There are two dispersions in this equation: $\omega_0 = \omega_0(k_0)$ and $\Omega = \Omega(K)$.
In order to understand how to reach such equations starting from physical problems, let us consider a real nonlinear equation, such as:

$$u_t + u_{xxx} + F(u, u_x) = 0 \tag{52}$$

Here $F(u, u_x)$ is a nonlinear function. The dispersive part of the equation ($u_t + u_{xxx}$) leads us to the dispersion relation $\omega = -k^3$. The periodic plane wave solution is the Fourier integral of terms of the form $u(x,t) = A \exp i(kx + k^3 t)$ + complex conjugate. The presence of a nonlinear term changes the amplitude and the frequency of the solution, so at larger time and space scales we will encounter a modulation in amplitude and frequency. The equation for this modulation is precisely NLS.
For the nonlinear problem we will seek solutions of the form:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x,t) \exp in(kx + k^3 t) \tag{53}$$

Here $u_n(x,t)$ is a set of complex functions obeying $u_{-n} = u_n^*$, because $u(x,t)$ needs to be real. Now, k is fixed and represents the initial dispersive wave that will be modulated. Therefore, the final equation we will obtain does not depend on this k.

1.3.2 Obtaining NLS from KdV

We consider the initial (real) equation to be KdV:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{54}$$

We plug solutions of the form:

$$u(x,t) = \sum_{n\in\mathbb{Z}} u_n e^{in\theta}, \ \theta \equiv kx + k^3 t \tag{55}$$

We equate to 0 the coefficients for every $e^{in\theta}$ and find an infinite set of equations for u_n, of the form:

$$\left(\frac{\partial}{\partial t} + ink^3\right) u_n + \left(\frac{\partial}{\partial x} + ink\right)^3 u_n + 3\left(\frac{\partial}{\partial x} + ink\right)\left(\sum_{j=-\infty}^{\infty} u_j u_{n-j}\right) = 0 \tag{56}$$

Since $u_{-n} = u_n^*$, it is enough to study the case $n \geq 0$. We introduce a small parameter ε and assume that the modulation is given by:

$$u_n = \varepsilon^{\alpha_n} v_n(\xi, \tau)$$

with:

$$\xi = \varepsilon(x + 3k^2 t)$$
$$\tau = -\varepsilon^2 6kt$$
$$\alpha_0 = 2; \ \alpha_n = \alpha_{-n} = n, n \geq 1$$

By substituting in equation (56) we arrive at:

- For $n = 0$ and order ε^3 we obtain an algebraic equation providing us with:

$$v_0 = -\frac{2}{k^2} v_1 v_{-1} = -\frac{2}{k^2} |v_1|^2 \tag{57}$$

- For $n = 2$ and order ε^2 we find another algebraic equation from which we obtain:

$$v_2 = \frac{1}{k^2} v_1^2 \tag{58}$$

- For $n = 1$ and order ε^3 we find the following differential equation:

$$-\frac{\partial v_1}{\partial \tau} + \frac{i}{2}\frac{\partial^2 v_1}{\partial \xi^2} + i(v_0 v_1 + v_2 v_1) = 0 \tag{59}$$

By substituting equations (57) and (58) in eq. (59), we reach:

$$-v_{1\tau} + \frac{i}{2} v_{1\xi\xi} - \frac{i}{k}|v_1|^2 v_1 = 0 \tag{60}$$

This is a defocusing NLS equation.
Observations:

- The other possible choices for n and order of ε do not provide anything supplementary.
- We need to perform an estimation of the Fourier integral in order to choose the powers α_n of ε. Let us start from the linear solution of the equation:

  $$u(x,t) = \int_{\mathbb{R}} A(k) e^{i(kt - \omega t)} \mathrm{d}k \tag{61}$$

  We will now take the case in which the profile of the solution becomes dominant for a value $k = k_0$. Let us define $k = k_0 + \varepsilon\mu$ and perform the series expansion $\omega(k) = \omega(k_0 + \varepsilon\mu) = \omega(k_0) + \varepsilon\mu\omega'(k_0) + \mathcal{O}(\varepsilon^2)$. Since this is the maximum contribution we can write $A(k_0 + \varepsilon\mu) = \frac{a(\mu)}{\varepsilon}$ and then the solution can be approximated as:

  $$u(x,t) \sim A(\xi, \tau) e^{i(k_0 x - \omega_0 t)} \tag{62}$$

  Here $\omega_0 = \omega(k_0)$, $\xi = \varepsilon(x - \omega'(k_0)t)$ and $\tau = \varepsilon^2 t$. This is how we obtain the stretched variables. So we can write $\xi = \varepsilon(x - v_g t)$, where $v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$ and deduce the scaling of the dependent variable:

  $$u(x,t) = \varepsilon v_1(\xi,\tau)e^{i\theta} + \varepsilon^2(v_0(\xi,\tau) + v_2(\xi,\tau)e^{2i\theta}) + \text{complex conjugates} \tag{63}$$

1.4 Introduction to KdV hierarchy

We will now be concerned with going to higher orders in ε in the type of equations presented before. The equations obtained will become more and more complicated, but if the first nonlinear equation obtained is integrable, then all the next equations will belong to the hierarchy of the completely integrable equation. The hierarchy of an integrable equation represents the set of all equations obtained from the first integrals (conservation laws) taken as "Hamiltonians". For example, if KdV equation has as conservation laws the quantities $I_1, I_2, ..., I_n, ...$ which depend on $u, u_x, u_x x$ etc., then the KdV hierarchy represents all the equations of the form $u_{t_1} = \{I_1, u\}, u_{t_2} = \{I_2, u\}, ..., u_{t_n} = \{I_n, u\}, ...$, where by $\{\cdot\}$ we denoted the Poisson brackets. KdV equation writes in Hamiltonian form as:

$$u_t = \frac{\partial}{\partial x}\frac{\delta}{\delta u} I_1 \tag{64}$$

The functional derivative $\frac{\delta}{\delta u}$ is applied on:

$$I_1 = \int_{-\infty}^{\infty} (u_y^2 - u^3) \mathrm{d}y \tag{65}$$

This quantity represents the "energy". The operator $\frac{\partial}{\partial x}$ in front of the functional derivative is called the "symplectic operator". We can deduce its analogous for the Hamiltonian equations of motion by writing them in matrix form as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \tag{66}$$

The Poisson bracket for the KdV equation is given by:

$$\{I_n, I_m\} = \int_{-\infty}^{\infty} \left(\frac{\delta I_n}{\delta u} \frac{\partial}{\partial x} \frac{\delta I_m}{\delta u} - \frac{\delta I_m}{\delta u} \frac{\partial}{\partial x} \frac{\delta I_n}{\delta u} \right) \mathrm{d}x = 0 \tag{67}$$

It is equal to 0 because the KdV equation is fully integrable and posses an infinity of conservation laws and they are in an involution: Poisson bracket between any two of them is 0.

1.5 Water waves and KdV equation

1.5.1 Statement of the problem

We consider a two-dimensional fluid in constant gravitational field. The space coordinates are x, z, as shown in 1. The velocity components are $\vec{v} = (u, w)$. The gravitational field acts on the negative z coordinate.

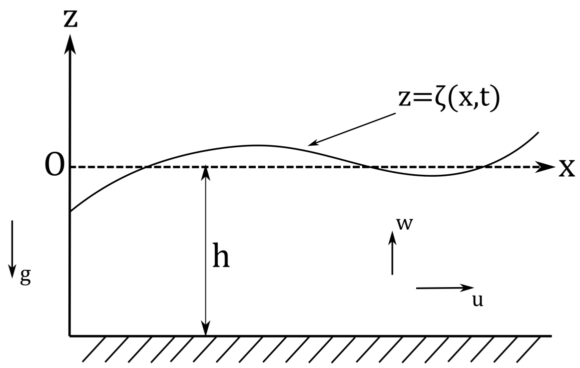

Figure 1: Configuration of the problem

The fundamental equations of the fluid are:

1. Incompressibility equation:
$$\nabla \vec{v} = 0 \tag{68}$$

2. Equation of motion (Euler):
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) v = -\frac{1}{\rho} \nabla P + \vec{g} \tag{69}$$

3. Condition that the fluid is irrotational (no vorticity):
$$\nabla \times \vec{v} = \vec{0} \tag{70}$$

We denoted by ρ the density of the fluid and by P the pressure. The nabla operator is defined as $\nabla = (\partial_x, \partial_z)$.
Because the fluid is irrotational ($\nabla \times \vec{v} = \vec{0}$), there exists a function ϕ such as $\vec{v} = \nabla \phi$. From the incompressibility equation ($\nabla \vec{v} = 0$), we have $\nabla^2 \phi = 0$, so the incompressibility condition can be reduced to the Laplace equation for velocity potential:

$$(\partial_x^2 + \partial_z^2)\phi = 0 \tag{71}$$

By replacing in the Euler equation, we have:

$$\nabla\left(\frac{\partial\vec{\phi}}{\partial t}+\frac{1}{2}(\nabla\phi)^2+gz+\frac{1}{\rho}(P-P_0)\right)=0 \tag{72}$$

By "integrating" this equation once, we arrive at the Bernoulli equation:

$$\phi_t+\frac{1}{2}(\nabla\phi)^2+gz+\frac{1}{\rho}(P-P_0)=0 \tag{73}$$

Here P_0 is an integration constant.
We will now impose the boundary conditions, keeping in mind that the velocity has the components $u=\phi_x$ and $w=\phi_z$. The height of the fluid is h and it is described by the equation $z=\zeta(x,t)$. Therefore, the vertical velocity of the fluid is given at the surface of the fluid by the following law:

$$w=\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{z=\zeta(x,t)} \tag{74}$$

On the other hand, we can express this velocity from the velocity potential as $\left.\frac{\partial\phi}{\partial z}\right|_{z=\zeta(x,t)}$. Consequently:

$$\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_{z=\zeta(x,t)}=\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\implies\zeta_t+\zeta_x\phi_x=\phi_z \tag{75}$$

Therefore, the boundary conditions writes as:

$$\zeta_t+\zeta_x\phi_x=\phi_z,\ z=\zeta(x,t) \tag{76}$$

Another condition is provided by the Euler equation written at the surface of the fluid, where $P=P_0=0$:

$$\phi_t+\frac{1}{2}(\phi_x^2+\phi_z^2)+g\zeta=0,\ z=\zeta(x,t) \tag{77}$$

Moreover, the vertical velocity of the fluid at its bottom ($z=-h$) needs to be 0:

$$\phi_z=0,\ z=-h \tag{78}$$

The system now writes as:

$$\phi_{xx}+\phi_{zz}=0,\ -h<z<\zeta(x,t) \tag{79}$$

$$\phi_t+\frac{1}{2}(\phi_x^2+\phi_z^2)+g\zeta=0,\ z=\zeta(x,t) \tag{80}$$

$$\zeta_t+\zeta_x\phi_x=\phi_z,\ z=\zeta(x,t) \tag{81}$$

$$\phi_z=0,\ z=-h \tag{82}$$

1.5.2 Linearized equations

This nonlinear system is very complicated. We begin its investigation by obtaining the dispersion relation in the linear limit of the system, which writes as:

$$\phi_{xx}+\phi_{zz}=0,\ -h<z<\zeta(x,t) \tag{83}$$

$$\phi_t+g\zeta=0,\ z=\zeta(x,t) \tag{84}$$

$$\zeta_t+=\phi_z,\ z=\zeta(x,t) \tag{85}$$

$$\phi_z=0,\ z=-h \tag{86}$$

We seek "oscillation mode" solutions of the form:

$$\zeta(x,t)=a\exp i(kt-\omega x) \tag{87}$$

$$\phi(x,z,t)=b(z)\exp i(kt-\omega x) \tag{88}$$

Here we performed a separation of variables and considered the Fourier mode only on x, since it is the only that interests us. By replacing in the Laplace equation, we have:

$$\mathcal{C}''(z) - k^2\mathcal{C}(z) = 0 \tag{89}$$

The solution is $\mathcal{C}(z) = Ae^{kz} + Be^{-kz}$, where A and B are integration constants. We now write the condition for the bottom of the fluid:

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=-h} = \mathcal{C}'(-h)e^{i(kx-\omega t)} = 0 \tag{90}$$

Or:

$$kAe^{-kh} - Bke^{kh} = 0 \tag{91}$$

By denoting $kAe^{-kh} = Bke^{kh} = \frac{1}{2}C$, we obtain $A = \frac{1}{2}Ce^{kh}$ and $B = \frac{1}{2}Ce^{-kh}$. The velocity potential rewrites as:

$$\phi = \left(\frac{1}{2}Ce^{kh} + \frac{1}{2}Ce^{-kh}\right)e^{i(kx-\omega t)} = C\cosh(k(h+z))e^{i(kx-\omega t)} \tag{92}$$

On the other hand, at the surface of the fluid ($z = \zeta(x,t)$), we have:

$$\phi_t + g\zeta = 0 \qquad\qquad \zeta_t + = \phi_z \tag{93}$$

Because the motion is linearized (thus assumed to have small amplitude) and horizontal, we can consider those conditions valid at $z \approx 0$, which is the rest position of the fluid. We arrive at the following equation:

$$\phi_{tt} + g\phi_z = 0,\ z = 0 \tag{94}$$

By replacing ϕ, we have:

$$-\omega^2 b(z)e^{i(kx-\omega t)} + g\mathcal{C}'(z)e^{i(kx-\omega t)} = 0 \tag{95}$$

We make use of eq. (92):

$$-\omega^2 C\cosh(k(h+z)) + gkC\sinh(k(h+z))\ |_{z=0} = 0 \tag{96}$$

Now we can solve for the linearized problem dispersion relation:

$$\omega^2 = gk\tanh(kh) \tag{97}$$

We obtain two solutions, corresponding to two propagation directions. We will limit ourselves to the positive solution:

$$\omega = \sqrt{gk\tanh(kh)} \tag{98}$$

1.5.3 Asymptotic solution

Returning to the nonlinear problem, we will now consider that the fluid is shallow (has a small depth) and the nonlinear waves have large wavelengths (small wavenumber), so that we can write $kh \ll 1$.
A second assumption we introduce is that the amplitude of the waves is small compared to the depth, which writes:

$$\frac{|\zeta|_{max}}{h} \ll 1 \tag{99}$$

We perform a series expansion for the dispersion relation:

$$\omega(k) = \sqrt{gh}k - \frac{1}{6}\sqrt{gh}k^3h^2 + \frac{19}{360}\sqrt{gh}k^5h^4 + ... \tag{100}$$

By introducing an asymptotic parameter ε and taking $k \sim \varepsilon^{1/2}$, we introduce the following stretched variables suggested by the dispersion relation:

$$\xi' = kx$$

$$\tau_1 = kt$$

$$\tau_3 = k^3 t$$
$$\tau_5 = k^5 t \text{ etc.}$$

We notice that $\frac{\omega}{k} = v_p = \sqrt{gh} + \mathcal{O}(k^2)$ is the phase velocity and it corresponds to the known result for the so-called "gravity waves".
In this problem we neglected the surface tension of the fluid. Including it implies additional terms in the Euler equation.
We return to the nonlinear equations:

$$\phi_{xx} + \phi_{zz} = 0, \ -h < z < \zeta(x,t) \tag{101}$$
$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\zeta = 0, \ z = \zeta(x,t) \tag{102}$$
$$\zeta_t + \zeta_x\phi_x = \phi_z, \ z = \zeta(x,t) \tag{103}$$
$$\phi_z = 0, \ z = -h \tag{104}$$

The stretched variables will be taken as follows:

$$\xi = \varepsilon^{1/2}x - \varepsilon^{1/2}v_p t \equiv \xi' - v_p\tau_1$$
$$\tau_1 = kt$$
$$\tau_3 = k^3 t$$
$$\tau_5 = k^5 t \text{ etc.}$$

We rewrite as function of the new variables, by computing the partial derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \varepsilon^{1/2}\frac{\partial}{\partial \xi} \tag{105}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \tau_1}\frac{\partial \tau_1}{\partial t} + \frac{\partial}{\partial \tau_3}\frac{\partial \tau_3}{\partial t} + ... = -\varepsilon^{1/2}\partial_\xi + \varepsilon^{3/2}\partial_{\tau_3} + \varepsilon^{5/2}\partial_{\tau_5} + ... \tag{106}$$

We introduced more time scales in order to investigate what happens at higher orders and deduce the KdV hierarchy. Now we consider solutions of the form:

$$\phi(x,z,t) \sim \varepsilon^q \hat{\phi}(\xi,\tau_3,z) + ... \tag{107}$$
$$\zeta(x,t) \sim \varepsilon^p \hat{\zeta}(\xi,\tau_3) + ... \tag{108}$$

We need to balance the terms in the two nonlinear equations to obtain the values of p and q. This will write as:

$$\varepsilon^{q+3/2}\hat{\phi}_{\tau_3} - \varepsilon^{q+1/2}v_p\hat{\phi}_\xi + \frac{1}{2}\varepsilon^{2q+1}\hat{\phi}_\xi^2 + \frac{1}{2}\varepsilon^{2q}\hat{\phi}_z^2 + g\varepsilon^p\hat{\zeta} = 0 \tag{109}$$
$$\varepsilon^{p+3/2}\hat{\zeta}_{\tau_3} - \varepsilon^{p+1/2}v_p\hat{\zeta}_\xi + \varepsilon^{p+q+1}\hat{\zeta}_\xi\hat{\phi}_\xi - \varepsilon^q\hat{\phi}_z = 0 \tag{110}$$

The set of powers to be balanced is:
$\{q+\frac{3}{2}, q+\frac{1}{2}, 2q+1, 2q, p\}$ for eq. (109);
$\{p+\frac{3}{2}, p+\frac{1}{2}, p+q+1, q\}$ for eq. (110).
The most convenient choice is $2q = p = 1$. Let us substitue in the two sets:
$\{2,1,2,1,1\}$: 3 dominant terms for eq. (109);
$\{\frac{5}{2}, \frac{3}{2}, \frac{5}{2}, \frac{1}{2}\}$ for eq. (110). We see that we obtained a balance, but not a maximum one.
However, no other balance can be obtained for eq. (110), because $q < p+q+1$ and $p+\frac{1}{2} < p+q+1$.
Consequently, we have:

$$\zeta = \varepsilon\hat{\zeta} = \varepsilon\zeta_0 + \varepsilon^2\zeta_1 + \varepsilon^2\zeta_2 + \varepsilon^3\zeta^3 + ... \tag{111}$$
$$\phi\varepsilon^{1/2}\hat{\phi} = \varepsilon^{1/2}\phi_0 + \varepsilon^{3/2}\phi_1 + \varepsilon^{5/2}\phi_2 + ... \tag{112}$$

We will now return to the variable $\xi' = \varepsilon^{1/2}x$ and use the expansions obtained in the initial set of equations:

$$\varepsilon\hat{\phi}_{\xi\xi} + \hat{\phi}_z = 0, \ -h < z < \varepsilon\hat{\zeta} \tag{113}$$
$$\hat{\phi}_z = 0, \ z = -h \tag{114}$$
$$\hat{\phi}_z = \varepsilon\hat{\zeta}_{\tau_1} + \varepsilon^2\hat{\zeta}_{\tau_3} + \varepsilon^3\hat{\zeta}_{\tau_5} + ... + \varepsilon^2\hat{\phi}_{\xi'}\hat{\zeta}_{\xi'}; \ z = \varepsilon\hat{\zeta} \tag{115}$$
$$2g\hat{\zeta} + 2\hat{\phi}_{\tau_1} + 2\varepsilon\hat{\phi}_{\tau_3} + 2\varepsilon^2\hat{\phi}_{\tau_5} + ... + \varepsilon\hat{\phi}_{\xi'}\hat{\phi}_{\xi'} + \hat{\phi}_z^2 = 0; \ z = \varepsilon\hat{\zeta} \tag{116}$$

We study separately for each order in ε:

- Order ε^0:

$$\varepsilon(\phi_0 + ...)_{\xi'\xi'} + (\phi_0 + ...)_z = 0 \tag{117}$$

$$(\phi_0 + ...)_z = 0;\ z = -h \tag{118}$$

We get $\phi_0 = \mathcal{F}(\xi', \tau_1, \tau_2, ...)$ and arbitrary function which does not depend on z.

- Order ε^1:

$$\phi_1 = -\left(\frac{z^2}{2} + hz\right)\mathcal{F}_{\xi\xi} + G(\xi', \tau_1, \tau_3, ...) \tag{119}$$

From here we obtain that $G(\xi', \tau_1, \tau_3, ...)$ is an arbitrary function that does not depend on z.

- Order ε^2:

$$\phi_2 = -\frac{1}{24}(z^4 + 4hz^2 - 8h^3 z)\mathcal{F}_{\xi'\xi'\xi'\xi'} - \frac{1}{2}(z^2 + 2hz)G_{\xi'\xi'} + H \tag{120}$$

Going further, one can calculate $\phi_3, \phi_4, \phi_5, ...$. We now study the two nonlinear equations:

$$\hat{\phi}_z = \varepsilon\hat{\zeta}_{\tau_1} + \varepsilon^2\hat{\zeta}_{\tau_3} + \varepsilon^3\hat{\zeta}_{\tau_5} + ... + \varepsilon^2\hat{\phi}_{\xi'}\hat{\zeta}_{\xi'};\ z = \varepsilon\hat{\zeta} \tag{121}$$

$$2g\hat{\zeta} + 2\hat{\phi}_{\tau_1} + 2\varepsilon\hat{\phi}_{\tau_3} + 2\varepsilon^2\hat{\phi}_{\tau_5} + ... + \varepsilon\hat{\phi}_{\xi'}\hat{\phi}_{\xi'} + \hat{\phi}_z^2 = 0;\ z = \varepsilon\hat{\zeta} \tag{122}$$

In order ε^0, eq. (122) does not provide us with anything, while from eq. (121) we get:

$$g\zeta_0 + \phi_{0\tau_1} = 0 \implies \zeta_0 = -\frac{1}{g}\mathcal{F}_{\tau_1} \tag{123}$$

In order ε^1, eq. (122) will provide us with:

$$\mathcal{F}_{\tau_1\tau_1} - gh\mathcal{F}_{\xi'\xi'} = 0 \tag{124}$$

This is a Klein-Gordon equation, accepting as solutions $\mathcal{F}(\xi' + \sqrt{gh}\tau_1)$ and $\mathcal{F}(\xi' - \sqrt{gh}\tau_1)$, which is precisely what the dispersion relation lead us to. We will limit ourselves to the case of a wave travelling to the right, so the independent variable will be $\xi = \xi' - v_p\tau_1$ from now on.
In order ε^1, eq. (121) provides us with:

$$2g\zeta_1 - 2v_p G_\xi + 2\mathcal{F}_{\tau_3} + \mathcal{F}_\xi^2 = 0 \tag{125}$$

We will take the derivative of this relation with respect to ξ and use that $\zeta_0 = -\frac{1}{g}\mathcal{F}_{\tau_1} \equiv \frac{h}{v_p}\mathcal{F}_\xi$:

$$v_p\zeta_{1\xi} - hG_{\xi\xi} = -\zeta_{0\tau_3} - \frac{v_p}{h}\zeta_0\zeta_{0\xi} \tag{126}$$

In order ε^2, eq. (122) gives us the same terms in the LHS:

$$v_p\zeta_{1\xi} - hG_{\xi\xi} = \zeta_{0\tau_3} + 2\frac{v_p}{h}\zeta_0\zeta_{0\xi} + \frac{v_p h^2}{3}\zeta_{0\xi\xi\xi} \tag{127}$$

By equating the terms from the RHS of the two equations, we obtain a KdV equation in stretched time τ_3:

$$\zeta_{0\tau_3} + \frac{3v_p}{2h}\zeta_0\zeta_{0\xi} + \frac{v_p h^2}{3}\zeta_{0\xi\xi\xi} = 0 \tag{128}$$

We remark that in the faster time τ_1 we have a plane wave evolution given by:

$$\zeta_{0\tau_1} - v_p\zeta_{0\xi} = 0 \tag{129}$$

Let us now investigate the higher orders. By taking eq. (121) in order ε^2 we obtain:

$$2g\zeta_2 + 2v_p^2\zeta_0\zeta_{0\xi\xi} - 2v_p H_\xi + 2G_{\tau_3} + 2\mathcal{F}_{\tau_5} + 2\mathcal{F}_\xi G_\xi + v_p^2\zeta_{0\xi}^2 = 0 \tag{130}$$

We take the derivative with respect to ξ and make use of $\zeta_0 = \frac{h}{v_p}\mathcal{F}_\xi$:

$$v_p\zeta_{2\xi} - hH_{\xi\xi} + \frac{h}{v_p}G_{\xi\tau_3} + (\zeta_{0\xi}G_\xi + \zeta_0 G_{\xi\xi}) = -\zeta_{0\tau_5} - 2v_p h\zeta_{0\xi}\zeta_{0\xi\xi} - v_p h\zeta_0\zeta_{0\xi\xi\xi} \tag{131}$$

By going in order ε^3 in eq. (122) we find:

$$v_p\zeta_{2\xi} - hH_\xi - (\zeta_{0\xi}G_\xi + \zeta_0 G_{\xi\xi}) - \frac{v_p}{h}(\zeta_{0\xi}\zeta_1 + \zeta_0\zeta 1\xi) - \frac{h^3}{3}G_{\xi\xi\xi\xi} = \zeta_{0\tau_5} + \frac{2}{15}v_p h^4 \zeta_{0\xi\xi\xi\xi\xi} \tag{132}$$

Combining eqs. (131 and 132), we obtain:

$$\begin{aligned} &2\zeta_{0\tau_5} + \frac{2}{15}v_p h^4\zeta_{0\xi\xi\xi\xi\xi} + 2v_p h\zeta_{0\xi}\zeta_{0\xi\xi} + v_p h\zeta_0\zeta_{0\xi\xi\xi} + \zeta_{1\tau_3} \\ &+ \frac{v_p}{h}(\zeta_0\zeta_1)_\xi + \frac{h}{v_p}G_{\xi\tau_3} + 2(\zeta_0 G_\xi)_\xi + \frac{h^3}{3}G_{\xi\xi\xi\xi} = 0 \end{aligned} \tag{133}$$

We will now use the KdV equation to describe $\mathcal{F}_{\tau_3}$ and rewrite eq. (125) as follows:

$$\zeta_1 - \frac{f}{v_p}G_\xi = \frac{1}{4h}\zeta_0^2 + \frac{h^2}{6}\zeta_{0\xi\xi} \tag{134}$$

We replace G_ξ and its derivatives from eq. (133) and find:

$$\zeta_{1\tau_3} + \frac{3v_p}{2h}(\zeta_0\zeta_1)_\xi + \frac{v_p h^2}{6}\zeta_{1\xi\xi\xi} = S(\zeta_0) \tag{135}$$

Here we have:

$$S(\zeta_0) = -\zeta_{0\tau_5} + \frac{19}{360}v_p h^4\zeta_{0\xi\xi\xi\xi\xi} - \frac{5}{12}v_p h\zeta_0\zeta_{0\xi\xi\xi} - \frac{23}{24}v_p h\zeta_{0\xi}\zeta_{0\xi\xi} + \frac{3v_p}{8h^2}\zeta_0^2\zeta_{0\xi} \tag{136}$$

In the LHS of eq. (135) we have the linearized KvD equation around ζ_0 (analogous to replacing in KdV $\zeta_0 \to \zeta_0 + \zeta_1$ and neglecting all nonlinear terms in ζ_1) and in the RHS we have $S(\zeta_0)$, which describes the yet unknown evolution of ζ_0 with respect to time τ_5. We can rewrite eq. (135) in operator form as:

$$L\zeta_1 = S(\zeta_0) \tag{137}$$

Here:

$$L = \frac{\partial}{\partial\tau_3} + \frac{3v_p}{2h}\zeta_0\frac{\partial}{\partial\xi} + \frac{3v_p}{2h}\zeta_{0\xi} + \frac{h^2 v_p}{6}\frac{\partial^3}{\partial\xi^3} \tag{138}$$

This is obviously a linear operator that depends on the solution $\zeta_0(\xi, \tau_3)$ of the KdV equation.

1.5.4 Discussion and KdV hierarchy

We will now state an important result. If $S(\zeta_0)$ contains a term belonging to $\ker L$, then ζ_1 will have components that increase algebraically in time (such as $\xi\tau_3 f(\xi)$ or $\xi f(\xi,\tau_3)$). These terms are known as secular terms and must be eliminated, in order for the series $\zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + ...$ to be uniformly valid.

We notice that $\zeta_{0\xi\xi\xi\xi\xi} \in S(\zeta_0)$ is such a term ($L\zeta_{0\xi\xi\xi\xi\xi} = 0$, because it reduces to the KdV eq.). It follows that $\zeta_{0\tau_5}$ must cancel this term in $S(\zeta_0)$.

The second important result is the scaling invariance. We noticed that the KdV equation that we obtained does not depend on ε. We start from the following equation:

$$\zeta_{0\tau_3} + \frac{3v_p}{2h}\zeta_0\zeta_{0\xi} + \frac{v_p h^2}{6}\zeta_{0\xi\xi\xi} = 0 \tag{139}$$

Here we replace:

$$\begin{aligned} x &= \varepsilon^{-1/2}\xi + v_p\varepsilon^{-3/2}\tau_3 \\ t &= \varepsilon^{-3/2}\tau_3 \\ \zeta_0 &= \varepsilon^{-1}\zeta \end{aligned} \tag{140}$$

By replacing in the KdV equation we have:

$$\zeta_t + v_p\zeta_x + \frac{3v_p}{2h}\zeta\zeta_x + \frac{v_p h^2}{6}\zeta_{xxx} = 0 \tag{141}$$

This is the same equation (up to a translation $x \to x + v_p t$). Moreover, the terms $\zeta_0 \zeta_{0\xi}$ and $\zeta_{0\xi\xi\xi}$ make up for all the possible terms of maximum order of 3 in the derivatives to obtain the scale invariance. We will apply the same procedure for the possible evolution $\zeta_{0\tau_5}$ with terms of maximum order of 5 in the derivatives. The scale invariant combination will be:

$$\alpha_2 \zeta_{0\xi\xi\xi\xi\xi} + \beta_2 \zeta_0 \zeta_{0\xi\xi\xi} + (\beta_2 + \gamma_2) \zeta_{0\xi} \zeta_{0\xi\xi} + \delta_2 \zeta_0^2 \zeta_{0\xi} = \zeta_{0\tau_5} \tag{142}$$

In order to obtain the coefficients $\alpha_2, \beta_2, \gamma_2, \delta_2$ we use the natural compatibility condition:

$$\frac{\partial}{\partial \tau_5} \left(\frac{\partial \zeta_0}{\partial \tau_3} \right) = \frac{\partial}{\partial \tau_3} \left(\frac{\partial \zeta_0}{\partial \tau_5} \right) \tag{143}$$

From this relation we find (α_2 will be chosen arbitrarily):

$$\zeta_{0\tau_5} + \alpha_2 \left(\zeta_{0\xi\xi\xi\xi\xi} + \frac{5}{3} \left(\frac{\beta_1}{\alpha_1} \right) \zeta_0 \zeta_{0\xi\xi\xi} + \frac{10}{3} \left(\frac{\beta_1}{\alpha_1} \right) \zeta_{0\xi} \zeta_{0\xi\xi} + \frac{5}{6} \left(\frac{\beta_1}{\alpha_1} \right)^2 \zeta_0^2 \zeta_{0\xi} \right) = 0 \tag{144}$$

Here $\beta_1 = \frac{3v_p}{2h}$ and $\alpha_1 = \frac{h^2 v_p}{6}$. Since α_2 is free, we take it to be $\alpha_2 = -\frac{19}{360} v_p h^4$, in order to eliminate the secular term from $S(\zeta_0)$. We have found $\zeta_{0\tau_5}$ (the evolution in τ_5) given by the equation (144), while the evolution of ζ_1 is given by the linear inhomogeneous equation $L\zeta_1 = S(\zeta_0)$.
Eq. (144) is the second equation from the KdV hierarchy (known as Lax 5 equation). Thus one can find the whole KdV hierarchy.

2 Bilinear methods for continuous/discrete integrable systems

2.1 Introduction

We observed that nonlinear equations contain both dispersion and nonlinearity. Through multiple scale method we balanced dispersion and nonlinearity in order to obtain an asymptotic equation that might be integrable, as it was the case of the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0;\ u = u(x,t) \in \mathbb{R} \tag{145}$$

In order to obtain the solutions of this equation, one must rigorously state the problem from a mathematical point of view. Since the equation is of the first order in time, we require one initial condition (Cauchy problem): $u(x,0) = f(x)$, where $f(x)$ is a given real function. We also need to specify the boundary conditions. The simplest case is cancelling the function at infinity (localized solution):

$$\lim_{x\to\pm\infty} u(x,t) = 0,\ \forall t \in \mathbb{R} \tag{146}$$

An equally important case is given by periodic boundary conditions in x:

$$u(x+a,t) = u(x,t), a \text{ fixed} \tag{147}$$

The Cauchy problem is difficult. It is studied using IST (Inverse Spectral Transform or Inverse Scattering Transform) [DEGM82]. We are not concerned here with IST, but we will rather try to check if such a nonlinear equation admits solutions of a certain type (in our case, solitonic solutions).
We return to the starting equation:

$$u_t + 6uu_x + u_{xxx} = 0;\ u = u(x,t) \in \mathbb{R} \tag{148}$$

Let us substitute $u = v_x$:

$$v_{xt} + 6v_xv_{xx} + v_{xxxx} = \frac{\partial}{\partial x}(v_t + 3v_x^2 + v_{xxx}) = 0 \tag{149}$$

We will take the simple case in which there is no time-dependent integration constant for the equation and write:

$$v_t + 3v_x^2 + v_{xxx} = 0 \tag{150}$$

Since these equations have both dispersion and nonlinearity, we will try to "remove" the nonlinearity and transform the equation into something "closer" to linearity. The simplest nonlinear substitution is the rational one. Therefore, we shall take:

$$v(x,t) = \frac{G(x,t)}{F(x,t)} \tag{151}$$

Here G and F are real functions. The substitution employed here is known as rational transformation. By replacing in the original equation, we arrive at:

$$\frac{G_tF - GF_t}{F^2} + 3\left(\frac{G_xF - GF_x}{F^2}\right)^2 + \frac{\partial^3}{\partial x^3}\left(\frac{G}{F}\right) = 0 \tag{152}$$

The last term in the LHS expands as:

$$\frac{\partial^3}{\partial x^3}\left(\frac{G}{F}\right) = \frac{G_{xxx}F - GF_{xxx} - 3G_{xx}F_x + 3G_xF_{xx}}{F^2} - 6\frac{G_xF - GF_x}{F^2}\frac{F_{xx}F - F_x^2}{F^2} \tag{153}$$

2.2 Bilinear operators and soliton solutions

2.2.1 Definition and properties

A linear expression in x_j is an expression of the form $\sum_{j=1}^{N} a_{ij}x_j$. By considering two variables, x_i and y_j, we can extend this definition and introduce bilinear forms through expressions of the

form $\sum_{i,j=1}^{N} a_{ij} x_i y_j$. These can be understood as linear expressions in x_i if we keep y_j fixed or, conversely, linear expressions in y_j by keeping x_i fixed. Not all the equations treated below will satisfy this definition and one could rigurously call the expression involved quadratic forms rather than bilinear forms. However, we will use the same term, bilinear, for all expressions.
We define the bilinear operators (also known as Hirota operators) as follows:

$$\mathbb{D}_x^n f(x) \bullet g(x) = (\partial_x - \partial_y)^n f(x)g(y) \mid_{x=y} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (\partial_x^{n-1} f)(\partial_x^i g) \tag{154}$$

We will now provide several examples for small n:

$$\begin{aligned} &\mathbb{D}_x f(x) \bullet g(x) = f_x g - f g_x \\ &\mathbb{D}_x^2 f(x) \bullet g(x) = f_{xx} g + f g_{xx} - 2 f_x g_x \\ &\mathbb{D}_x^3 G \bullet F = G_{xxx} F - G F_{xxx} - 3 G_{xx} F_x + 3 G_x F_{xx} \end{aligned} \tag{155}$$

One must notice that when writing $\mathbb{D}_x^n f(x) \bullet g(x) =$, the bullet ('•') does not denote a multiplication, but rather that $\mathbb{D}_x^n$ acts on a pair of functions f and g.
We will use two fundamental properties of bilinear operators:

$$\mathbb{D}_x^n e^{ab} \bullet b^{bx} = (a-b)^n e^{ax+bx} \tag{156}$$

The second property is called "gauge invariance":

$$\mathbb{D}_x^n (e^{ax} f) \bullet (e^{ax} g) = e^{2ax} \mathbb{D}_x^n f \bullet g \tag{157}$$

2.2.2 1-soliton solution for KdV

Let us now rewrite eq. (152) using the bilinear operators introduced before:

$$\frac{(\mathbb{D}_t + \mathbb{D}_x^3) G \bullet F}{F^2} + 3 \frac{\mathbb{D}_x G \bullet F}{F^2} \left(\frac{\mathbb{D}_x G \bullet F}{F^2} - \frac{\mathbb{D}_x^2 F \bullet F}{F^2} \right) = 0 \tag{158}$$

Or:

$$F^2 \left((\mathbb{D}_t + \mathbb{D}_x^3) G \bullet F \right) + 3 (\mathbb{D}_x G \bullet F)(\mathbb{D}_x G \bullet F - \mathbb{D}_x^2 F \bullet F) = 0 \tag{159}$$

We notice that our equation transformed into something more complicated, but with the property of multilinearity (in our case, quadri-linearity). On the other hand, in a formal sense, $(\mathbb{D}_t + \mathbb{D}_x^3)$ resembles $(\partial_t + \partial_x^3)$, which is precisely the linear part leading to dispersion. Therefore, by isolating the dispersion we obtain:

$$(\mathbb{D}_t + \mathbb{D}_x^3) G \bullet F = 0 \tag{160}$$

We are left with the non-trivial part, leading us to:

$$\mathbb{D}_x G \bullet F = \mathbb{D}_x^2 F \bullet F \tag{161}$$

This is the second equation needed to solve the problem, as we have two unknowns (G and F). Since these two equations are bilinear, we can search for solutions of the type:

$$G,\, F \sim \exp(kx + \omega t) \tag{162}$$

Let us further simplify the bilinear system by considering $G = 2F_x$. The second bilinear equation is identically satisified:

$$\mathbb{D}_x G \bullet F = 2 \mathbb{D}_x F_x \bullet F = 2 F_{xx} F - 2 F_x^2 = \mathbb{D}_x^2 F \bullet F \tag{163}$$

We are left with the first bilinear eq.:

$$(\mathbb{D}_t + \mathbb{D}_x^3) 2 F_x \bullet F = (\mathbb{D}_x \mathbb{D}_t + \mathbb{D}_x^4) F \bullet F \tag{164}$$

By taking $v = 2\partial_x \log F$ we have the following equivalence:

$$v_t + 3 v_x^2 + v_{xxx} = 0 \iff (\mathbb{D}_x \mathbb{D}_t + \mathbb{D}_x^4) F \bullet F \tag{165}$$

Therefore, if we find the solutions F of the bilinear equation, we have $u(x,t) = 2\frac{\partial^2}{\partial x^2}\log F$. The simplest solution is $F = 1 \implies u = 0$. Then, $F = 1+\exp(kx+\Omega t)$. Here, Ω and k are not related to the dispersion of the linear problem, but have to be determined. By replacing in the bilinear equation, we have:

$$(\mathbb{D}_x\mathbb{D}_t + \mathbb{D}_x^4)(1+e^\eta)\bullet(1+e^\eta),\ \eta = kx+\Omega t \tag{166}$$

For simplification, we denote $\mathbb{D}_x\mathbb{D}_t + \mathbb{D}_x^4 \equiv P(\vec{\mathbb{D}})$, which represents a polynomial in variables $\mathbb{D}_x$ and $\mathbb{D}_t$. By expanding the expression obtained, we have:

$$P(\vec{\mathbb{D}})1\bullet 1 + P(\vec{\mathbb{D}})1\bullet e^\eta + P(\vec{\mathbb{D}})e^\eta\bullet 1 + P(\vec{\mathbb{D}})e^\eta\bullet e^\eta = 0 \tag{167}$$

The constant solution is $P(\vec{\mathbb{D}})1\bullet 1 = 0$. Given that $\mathbb{D}_x^n 1\bullet e^\eta = \partial_x^n e^\eta, \forall x, n$, we have $P(\vec{\mathbb{D}})1\bullet e^\eta = P(-\vec{\partial})e^\eta$. Therefore:

$$P(\vec{\mathbb{D}}) = (\mathbb{D}_x\mathbb{D}_t + \mathbb{D}_x^4)1\bullet e^\eta = \left((-\partial_x)(-\partial_t) + (-\partial_x)^4\right)e^\eta = (\Omega k + k^4)e^\eta \tag{168}$$

Furthermore:

$$P(\vec{\mathbb{D}})e^\eta\bullet 1 = P(\vec{\partial})e^\eta = (\Omega k + k^4)e^\eta \tag{169}$$

By using the gauge-invariance:

$$P(\vec{\mathbb{D}})e^\eta\bullet e^\eta = P(\vec{\mathbb{D}})(e^\eta\cdot 1)\bullet(e^\eta\cdot 1) = e^{2\eta}P(\vec{\mathbb{D}})1\bullet 1 = 0 \tag{170}$$

By replacing in the initial expression we have:

$$2(k\Omega + k^4) = 0 \implies \Omega = -k^3 \tag{171}$$

We notice that for the linearized problem $u_t + u_{xxx} = 0$ and solutions of the form $\exp(kx+\omega t)$ we obtain $i\omega = ik^3 = 0 \implies \omega = k^3$, which is different from the bilinear relation.
Therefore, the solution of the nonlinear problem is:

$$F = 1+\exp(kx-k^3t) \implies u = 2\partial_x^2\log\left(1+\exp(kx-k^3t)\right) = 4k^2\text{sech}^2\left(k(x-k^2t)\right) \tag{172}$$

This solution is moving in the opposite direction compared to the linear waves with $\omega = k^3$. Moreover, unlike the linear waves (which have constant amplitude), for our solution the amplitude is dependent on the velocity (the term $4k^2$).
The solution $F = 1+\exp(kx-k^3t)$ is known as 1-soliton solution.

2.2.3 Multi-soliton solution

We will now attempt to generalize the previously obtained solution, by considering a solution of the form:

$$F = 1+e^{\eta_1}+e^{\eta_2},\ \eta_{1,2} = k_{1,2}x - k_{1,2}^3 t \tag{173}$$

Because $P(\vec{\mathbb{D}})F\bullet F \neq 0$, this is not a solution. Consequently, we need to consider a non-trivial interaction between the modes e^{η_1} and e^{η_2}. Let us consider the following expression of F:

$$F = 1+\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + ... \tag{174}$$

Here ε is a formal parameter, without physical meaning. By equating the various powers of ε we obtain:

- For ε^0:
$$P(\vec{\mathbb{D}})1\bullet 1 = 0 \tag{175}$$
- For ε^1:
$$2P(\vec{\mathbb{D}})1\bullet f_1 = 0 \implies (\partial_x\partial_t + \partial_x^4)f_1 = 0 \tag{176}$$
  We take $f_1 = \exp kx - k^3t$ and, by considering $f_2 = f_3 = ... = f_n = ... = 0$, we obtain $F = 1+f_1$ as solution.

- For ε^2:

$$2P(\vec{\mathbb{D}})1 \bullet f_2 = P(\vec{\mathbb{D}})f_1 \bullet f_1 \tag{177}$$

By taking $f_1 = e^{\eta_1} + e^{\eta_2}$ we have:

$$\begin{aligned} P(\vec{\mathbb{D}})1 \bullet f_2 &\equiv (\partial_x\partial_t + \partial_x^4)f_2 = P(\vec{\mathbb{D}})(e^{\eta_1} + e^{\eta_2}) \bullet (e^{\eta_1} + e^{\eta_2}) \\ &= P(\vec{\mathbb{D}})e^{\eta_1} \bullet e^{\eta_1} + P(\vec{\mathbb{D}})e^{\eta_1} \bullet e^{\eta_2} + P(\vec{\mathbb{D}})e^{\eta_2} \bullet e^{\eta_1} + P(\vec{\mathbb{D}})e^{\eta_2} \bullet e^{\eta_2} \end{aligned} \tag{178}$$

Because of gauge-invariance, $P(\vec{\mathbb{D}})e^{\eta_1} \bullet e^{\eta_1} = P(\vec{\mathbb{D}})e^{\eta_2} \bullet e^{\eta_2} = 0$. On the other hand:

$$P(\vec{\mathbb{D}})e^{\eta_1} \bullet e^{\eta_2} = \left((-k_1^3 + k_2^3)(k_1 - k_2) + (k_1 - k_2)^4\right) e^{\eta_1+\eta_2} \tag{179}$$

So $f_2 \sim e^{\eta_1+\eta_2}$ and we consider $f_2 = A_{12}e^{\eta_1+\eta_2}$. It follows that:

$$\begin{aligned} 2P(\vec{\mathbb{D}})1 \bullet f_2 &= 2A_{12}(\partial_x\partial_t + \partial_x^4)e^{\eta_1+\eta_2} \\ &= 2A_{12}\left((-k_1^3 - k_2^3)(k_1 - k_2) + (k_1 + k_2)^4\right) e^{\eta_1+\eta_2} \end{aligned} \tag{180}$$

It means that:

$$2P(\vec{\mathbb{D}})1 \bullet A_{12}e^{\eta_1+\eta_2} = 22P(\vec{\mathbb{D}})e^{\eta_1} \bullet e^{\eta_2} \tag{181}$$

This equation leads us to:

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \frac{(-k_1^3 + k_2^3)(k_1 - k_2) + (k_1 - k_2)^4}{(-k_1^3 - k_2^3)(k_1 - k_2) + (k_1 + k_2)^4} \tag{182}$$

We can now take $f_3 = f_4 = ... = f_n = ... = 0$ and the solution $F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}$ rigorously satisfies the equation $P(\vec{\mathbb{D}}F \bullet F$. This is known as 2-solition solution.

- For ε^3:

$$2(\partial_x\partial_t + \partial_x^4)f_3 = -2(\mathbb{D}_x\mathbb{D}_t + \mathbb{D}_x^4)f_1 \bullet f_2 \tag{183}$$

If $f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$ and $f_2 = A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3}$, one can obtain, through a longer calculation:

$$f_3 = A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3} \tag{184}$$

Therefore, the 3-soliton interaction is given by:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3} \tag{185}$$

The general formula for N-soliton interaction is:

$$F_N = \sum_{\mu_1,\ldots,\mu_N \in \{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i\eta_i + \sum_{1\le i<j} \mu_i\mu_j a_{ij}\right) \tag{186}$$

Here $e^{a_{ij}} = A_{ij} = \left(\frac{k_i = k_j}{k_i + k_j}\right)^2$ and $\eta_i = k_i x - k_i^3 t$.

2.3 Interaction between two solitons

We will now study the physical meaning of the interaction between two solitons. Let us take $k_1 < k_2$ and define:

$$\xi_i = x - k_i^2 t, \ i = 1, 2 \tag{187}$$

We can immediately reach the following identity:

$$\xi_2 = -(k_2^2 - k_1^2)t + \xi_1 \tag{188}$$

In the reference frame of the first soliton (fixed ξ_1) we take $t \to \infty$. Then, $\xi_2 \to -\infty$, $e^{\eta_2} \to 0$ and $F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \to 1 + e^{\eta_1}$. By taking $t \to -\infty$ we obtain $\xi_2 \to \infty$ and $e^{\eta_2} \to \infty$, so it becomes dominant:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} = e^{\eta_2}(e^{-\eta_2} + e^{\eta_1-\eta_2} + A_{12}e^{\eta_1} + 1) \sim e^{\eta_2}(A_{12}e^{\eta_1} + 1) \tag{189}$$

But:

$$2\partial_x^2 \log(e^{\eta_2}(A_{12}e^{\eta_1}+1)) = 2\partial_x^2 \log(A_{12}e^{\eta_1}+1) \tag{190}$$

Therefore, from a physical point of view, we have:

$$t \to \infty \implies u \sim k_1^2 \text{sech}^2 \eta_1 \tag{191}$$

$$t \to -\infty \implies u \sim k_1^2 \text{sech}^2 (\eta_1 + a_{12}) \tag{192}$$

We observe that the effect of the interaction is a phase shift given by $e^{a_{12}} = A_{12}$.
Let us generalize to a general bilinear equation:

$$P(\mathbb{D}_{x_1}, \mathbb{D}_{x_2}, ..., \mathbb{D}_t) F \bullet F = 0 \tag{193}$$

For any polynomial P the equation admits the 2-soliton solution $F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}$, with $\eta_1 = k_{11}x_1 + k_{12}x_2 + ... + \Omega_1 t$ and $\eta_2 = k_{21}x_1 + k_{22}x_2 + ... + \Omega_2 t$. The dispersion relation is given by $P(k_{i1}, k_{i2}, ..., \Omega_i) = 0$. The coefficient A_{12} is given by:

$$A_{12} = \frac{P(k_{11}-k_{21}, k_{12}-k_{22}, k_{13}-k_{23}, ..., \Omega_1 - \Omega_2}{P(k_{11}+k_{21}, k_{12}+k_{22}, k_{13}+k_{23}, ..., \Omega_1 + \Omega_2} \tag{194}$$

The condition for admitting three or more solitons (N-soliton solution) is not automatic. In fact, there are very few equations that admit 3-soliton solutions. If an equation admits at least a 3-soliton solution, then the equation is fully integrable. This means that the existence of a 3-soliton solution is an integrability detector.
One must be careful that the 3-soliton solution must depend solely on k_1, k_2, k_3, which must be arbitrary. Let us consider a 3-soliton solution of the type:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + Be^{\eta_1+\eta_2+\eta_3} \tag{195}$$

If $B \neq A_{12}A_{13}A_{23}$, then the equation is not integrable. Consequently, integrability implies that the interaction between an arbitrary number of solitons can be factorized as product of "two-body" interactions (described by A_{ij}).
The bilinear equations admitting at least a 3-soliton interaction and depending on a single function F are the following:

- KP equation:
$$(\mathbb{D}_x^4 + 4\mathbb{D}_x\mathbb{D}_t + a\mathbb{D}_y^2) F \bullet F = 0 \tag{196}$$
- Hunter-Saxton (HS) equation:
$$(\mathbb{D}_x^3\mathbb{D}_t + a\mathbb{D}_x^2 + \mathbb{D}_t\mathbb{D}_y) F \bullet F = 0 \tag{197}$$
- $$(\mathbb{D}_x^4 - \mathbb{D}_x^3\mathbb{D}_t + a\mathbb{D}_x^2 + b\mathbb{D}_x\mathbb{D}_t + c\mathbb{D}_t^2) F \bullet F = 0 \tag{198}$$
- Sawada-Kotera (SK) equation
$$(\mathbb{D}_x^6 + 5\mathbb{D}_x^3\mathbb{D}_t - 5\mathbb{D}_t^2 + \mathbb{D}_x\mathbb{D}_t) F \bullet F = 0 \tag{199}$$

2.4 Examples

2.4.1 Modified KdV equation

We will start from the modified KdV equation:

$$u_t \pm 6u^2 u_x + u_{xxx} = 0 \tag{200}$$

We perform the same substitution $u = \frac{G}{F}$ and compute the derivatives in bilinear formalism:

$$u_t = \partial_t \left(\frac{G}{F}\right) = \frac{\mathbb{D}_t G \bullet F}{F^2} \tag{201}$$

$$u_x = \partial_x \left(\frac{G}{F}\right) = \frac{\mathbb{D}_x G \bullet F}{F^2} \tag{202}$$

$$u_{xxx} = \partial_x^3 \left(\frac{G}{F}\right) = \frac{G_{xxx}F - GF_{xxx} - 3G_{xx}F_x + 3G_xF_{xx}}{F^2} - 6\frac{G_xF - GF_x}{F^2}\frac{F_{xx}F - F_x^2}{F^2} \tag{203}$$

$$u_{xxx} = \partial_x^3 \left(\frac{G}{F}\right) = \frac{\mathbb{D}_x^3 G \bullet F}{F^2} - 3\frac{\mathbb{D}_x G \bullet F}{F^2}\frac{\mathbb{D}_x^2 F \bullet F}{F^2} \tag{204}$$

The equation rewrites as:

$$(\mathbb{D}_t + \mathbb{D}_x^3)G \bullet F + \frac{6}{F^2}(\mathbb{D}_x G \bullet F)\left(\frac{1}{2}\mathbb{D}_x^2 F \bullet F - G^2\right) = 0 \tag{205}$$

By isolating the dispersion relation we obtain the following system:

$$\begin{aligned} &(\mathbb{D}_t + \mathbb{D}_x^3)G \bullet F = 0 \\ &\mathbb{D}_x^2 F \bullet F = 2G^2 \end{aligned} \tag{206}$$

In this case, we cannot find a relation between G and F that can simplify the problem. Therefore, we must use the expansions:

$$\begin{aligned} F &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + ... \\ G &= \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + ... \end{aligned} \tag{207}$$

We have considered even, respectively odd, powers of ε in order to simplify the calculations. We could have taken all powers into account and cancel the odd terms for F and the even terms for G.
From $(\partial_t + \partial_x^3)g_1 = 0$ we find $g_1 = e^{\eta_1}$ and $\eta_1 = k_1 x - k_1^3 t + \eta_1^{(0)}$, with $\eta_1^{(0)}$ an arbitrary constant. In the second order we find from the second bilinear equation $\partial_x^2 f_2 = g_1^2$, which implies $f_2 = \frac{1}{4k_1^2}e^{2\eta_1}$. Therefore, the 1-soliton solution is:

$$u = \frac{e^{\eta_1}}{1 + e^{2\eta_1/4k_1^2}} \equiv \frac{G}{F} \tag{208}$$

For the 2-soliton solution the result is more complicated:

$$\begin{aligned} G &= e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1 + 2\eta_2} + a_{21}e^{2\eta_1 + \eta_2} \\ F &= 1 + \frac{1}{4k_1^2}e^{2\eta_1} + \frac{1}{4k_2^2}e^{2\eta_2} + m_1 2 e^{\eta_1 + \eta_2} + n_1 2 e^{2\eta_1 + \eta_2} \end{aligned} \tag{209}$$

Here:

$$\begin{aligned} a_{12} &= \frac{1}{4k_2^2}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \\ a_{21} &= \frac{1}{4k_1^2}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \\ m_{12} &= \frac{2}{(k_1 + k_2)^2}e^{\eta_1 + \eta_2} \\ n_{12} &= \frac{(k_1 - k_2)^4}{16k_1^2 k_2^2 (k_1 + k_2)^4} \end{aligned} \tag{210}$$

This complicated form arises from the presence of both e^{η_i} and $e^{2\eta_i}$ modes and the requirement to consider all possible interactions between them.

2.4.2 Nonlinear Schrödinger equation

We start from the nonlinear Schrödinger equation in a complex function u:

$$iu_t + u_{xx} + |u|^2 u = 0 \tag{211}$$

We consider $u = \frac{G}{F}$ with F a real function and G a complex function. We will use the following formula:

$$\partial_x^2 \left(\frac{a}{b}\right) = \frac{\mathbb{D}_x^2 a \bullet b}{b^2} - \frac{a}{b}\frac{\mathbb{D}_x^2 b \bullet b}{b^2} \tag{212}$$

The equation writes in bilinear form:

$$\frac{1}{F^2}(i\mathbb{D}_t + \mathbb{D}_x^2)G \bullet F - \frac{G}{F^3}(\mathbb{D}_x^2 F \bullet F - GG^*) = 0 \tag{213}$$

By isolating the dispersion relation we find:

$$\begin{aligned} &(i\mathbb{D}_t + \mathbb{D}_x^2)G \bullet F = 0 \\ &\mathbb{D}_x^2 F \bullet F = |G|^2 \end{aligned} \tag{214}$$

The 1-soliton solution is:

$$\begin{aligned} &G = e^{\eta_1} \\ &F = 1 + e^{\eta_1 + \eta_1^* + \varphi_{11}^*} \end{aligned} \tag{215}$$

Here:

$$\begin{aligned} &\eta_1 = p_1 x - \Omega_1 t \\ &\Omega_1 = -ip_1^2 \\ &e^{\varphi_{11}^*} = \frac{1}{2(p_1 + p_1^*)^2} \end{aligned} \tag{216}$$

In these expressions p_1 and Ω_1 are complex.
For such complex equations, even the 2-soliton solution exists automatically only for fully integrable equations. The 2-soliton solution writes:

$$\begin{aligned} &G = e^{\eta_1} + e^{\eta_2} + a_{121} e^{\eta_1 + \eta_2 + \eta_1^*} + a_{212} e^{\eta_1 + \eta_2 + \eta_2^*} \\ &F = 1 + e^{\eta_1 + \eta_1^* + \varphi_{11}^*} + e^{\eta_2 + \eta_2^* + \varphi_{22}^*} + b_{12} e^{\eta_1 + \eta_2^*} + b_{21} e^{\eta_2 + \eta_1^*} + B_{1122} e^{\eta_1 + \eta_1^* + \eta_2 + \eta_2^*} \end{aligned} \tag{217}$$

3 Particular families of solutions for nonlinear equations

3.1 Introduction

We have previously introduced solitons as particular types of solutions for the equations we encountered. Following that, we discussed the simple case of an overtaking collision between two solitons in the case of KdV equation. In this section, we will concern about particular types of solitons, classified based on their behavior, as well as on other types of solutions resembling solitons in several important aspects. We will present numerous examples based mainly on Ref. [Hie02].

3.2 Fusion and Fission

3.2.1 The Burgers Equation

The Burgers equation is:

$$u_{xx} + 2uu_x + u_t = 0 \tag{218}$$

We will linearize the equation using the transformation (Cole-Hopf): $u = \partial_x \log F$. The equation rewrites as:

$$F_{xx} + F_t = 0 \tag{219}$$

The solution of this equation can be written as a sum of exponentials of the form e^{η_i} with $\eta_i = k_i x + \Omega_i t + \eta_i^{(0)}$. The dispersion relation is given by $\Omega_i = -k_i^2$. The 1-soliton solution of the equation is:

$$u = \frac{ke^{\eta}}{1 + e^{\eta}},\ \eta = kx - k^2 t + \eta_0 \tag{220}$$

The limits of these function are $\eta \to -\infty \implies u \to 0$ and $\eta \to \infty \implies u \to k$. A soliton-like shape is obtained for $w = \partial_x u = \partial_x^2 \log F$, which has the following expression:

$$w = \frac{k^2}{4\cosh^2\left(\frac{\eta}{2}\right)} \tag{221}$$

For the 2-soliton solution we take $F = 1 + e^{\eta_1} + e^{\eta_2}$ and obtain:

$$\begin{aligned} u &= \frac{k_1 e^{\eta_1} + k_2 e^{\eta_2}}{1 + e^{\eta_1} + e^{\eta_2}} \\ w &= \frac{k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} + (k_1 - k_2)^2 e^{\eta_1 + \eta_2}}{(1 + e^{\eta_1} + e^{\eta_2})^2} \end{aligned} \tag{222}$$

We will now introduce the velocities of individual solitons, as well as the relative velocity of soliton i with respect to soliton j. For soliton i, the velocity writes as $v_i = -\Omega_i / k_i$. The relative velocity is defined as:

$$v(i|j) = \text{sign}(k_i(v_j - v_i)) \tag{223}$$

By applying the definition above for our equation, we obtain:

$$v(i|j) = \text{sign}(k_i(-k_i + k_j)) \tag{224}$$

The asymptotic form of the 2-soliton solutions, we arrive at:

$$\begin{aligned} &\eta_2 \to -\infty:\ u \to \frac{k_1 e^{\eta_1}}{1 + e^{\eta_1}},\ w \to \frac{k_1^2 e^{\eta_1}}{(1 + e^{\eta_1})^2} \\ &\eta_1 \to -\infty:\ u \to \frac{k_2 e^{\eta_2}}{1 + e^{\eta_2}},\ w \to \frac{k_2^2 e^{\eta_2}}{(1 + e^{\eta_2})^2} \\ &\eta_2 \to \infty:\ u \to k_2,\ w \to 0 \\ &\eta_1 \to \infty:\ u \to k_1,\ w \to 0 \end{aligned} \tag{225}$$

Let us consider $0 < k_1 < k_2$ we get $v(2|1) = -1$ and $v(1|2) = 1$. This means that we have only this the first soliton only in the future and the second soliton only in the past.
Let us compute the limit for $\eta \to \infty$. We expand w and obtain in quadratic terms:

$$w = \frac{... + (k_1 - k_2)^2 e^{\eta_1 + \eta_2}}{... + 2e^{\eta_1 + \eta_2} + e^{2\eta_1} + e^{2\eta_2}} \tag{226}$$

In order to obtain a nonzero limit, we consider a comoving frame, where we denote $\eta_i = \tau_i + \tilde{\eta}$, where τ_i remains finite and $\tilde{\eta} \to \infty$. Then we obtain in the limit $\tilde{\eta} \to \infty$:

$$\begin{aligned} u &\to \frac{p_1 e^{\tau_1} + p_2 e^{\tau_2}}{e^{\tau_1} + e^{\tau_2}} \\ w &\to \frac{(p_1 - p_2)^2 e^{\tau_1+\tau_2}}{(e^{\tau_1} + e^{\tau_2})^2} = \frac{(p_1 - p_2)^2/4}{\cosh^2\left(\frac{1}{2}(\eta_1 - \eta_2)\right)} \end{aligned} \tag{227}$$

In the asymptotic regime we obtain a 1-soliton solution of the type $F = e^{\eta_1} + e^{\eta_2}$ which corresponds to a step size $k_1 - k_2$ and velocity $k_1 + k_2$.
Generally, one has to find a comoving frame where η_i go to infinity equally fast. Let us introduce $\xi = x - at$ as coordinate of the moving frame, $\tau = k_i\xi$ and $\eta_i = \tau_k + (\Omega_i + ak_i)t$. We need to choose a so as to obtain equal velocities, meaning $\tilde{\eta}_i = (\Omega_1 + ak_1)t = (\Omega_2 + ak_2)t$. The solution is:

$$a = -\frac{\Omega_1 - \Omega_2}{k_1 - k_2} \tag{228}$$

And:

$$\tilde{\eta} = t\frac{\Omega_2 k_1 - \Omega_1 k_2}{k_1 - k_2} \tag{229}$$

In our case, using the dispersion relation, we obtain $a = k_1 + k_2$ and $\eta = k_1 k_2 t$. Since we have chosen $0 < k_1 < k_2$ we obtain this soliton only in the future and obtain a fission of the type $u(\eta_2) \to u(\eta_1) + u(\eta_1 - \eta_2)$.

3.2.2 General aspects regarding fusion and fission

Let us consider a generic form of 2-soliton solutions given by $u = 2\partial_x^2 F$ where:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} \tag{230}$$

If one can choose k_1 and k_2 so as to have $A_{12} = 0$ and $k_1 \neq k_2$, we say that the solitons resonate. In such a scenario we obtain fussion and fision for the solitons.
We can apply this result on the shallow water-wave equation of Gibbon:

$$u_{xxt} - (4u^2 + 2w_x v_t)_x - u_x = u_t,\ v_x = u,\ w_t = u \tag{231}$$

This equation provides us with the dispersion relation:

$$k^2\omega = \omega + k \tag{232}$$

Although this equation is not integrable, we can obtain elastic two-soliton scattering given by the following term:

$$A_{12} = \frac{(k_1^2 - k_1k_2 + k_2^2 - 3)(k_1 - k_2)^2}{(k_1^2 + k_1k_2 + k_2^2 - 3)(k_1 + k_2)^2} \tag{233}$$

We notice that $A_{12} = 0$ for $k_1^2 - k_1k_2 + k_2^2 = 3$. This result provides us with a range of values for k that allow for fusion between a given soliton and another specific soliton.

3.3 Solitons of NLS

3.3.1 Bright solitons

We will consider the following focusing NLS:

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0 \tag{234}$$

A 1-soliton solution for this equation is given by:

$$\psi = \frac{e^{\eta}}{1 + \frac{e^{\eta+\eta^*}}{(k+k^*)^2}} \tag{235}$$

Here $\eta = kx + ik^2t + \eta_0$. Because η is complex, the soliton is characterized by two parameters instead of one: the imaginary part $k_I = \operatorname{Im} k$ determines the velocity, while $k_R = \operatorname{Re} k$ is responsible for its

amplitude. These parameters also determine the oscillations of the carrier wave. A consequence of this mathematical behavior is that one can create bound states by choosing two solitons with the same k_I and different k_R.
By expanding the expressions for η and u so as to include the real and imaginary components of k and writing $k = k_R + ik_I$, we have:

$$\eta = k_R(x - 2k_I t) + i(k_I x + (k_R^2 - k_I^2)t) + \eta_0 \tag{236}$$

$$u = \frac{p_R \exp i\left(k_I x + (k_R^2 - k_I^2)t + \eta_I^0\right)}{\cosh\left(k_R(x - 2k_I t) + \eta_R^0\right)} \tag{237}$$

In Figure 2 one can see the typical aspect of a bright soliton. Their name comes from the prominent aspect of these solitons, similar to a bright pulse in a dark background.

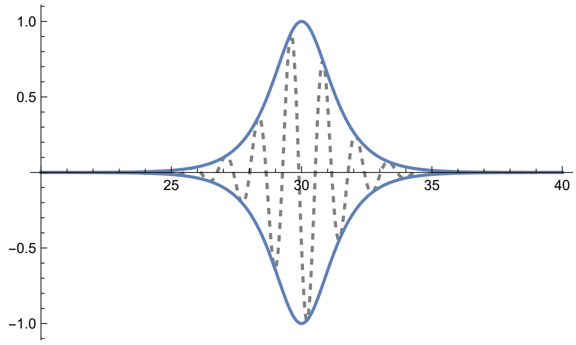

Figure 2: Example of bright soliton. The thick blue line shows the carrier wave, described by $|\psi|$, and the dashed gray line shows $\mathrm{Re}\,\psi$.

3.3.2 Dark solitons

Opposed to bright solitons, dark solitons appear as dark pulses in a bright background. These solitons are encountered for a defocusing NLS:

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0 \tag{238}$$

We use the form $\psi = g/f$ for the solution of this equation, where g is complex and f is real. The background carrier wave solution is determined by the following functions:

$$g \equiv g_0 = \rho e^{\theta}, \;\; f = 1 \tag{239}$$

Here $\theta = i(kx - \omega t)$ and $\omega = k^2 + 2\rho^2$. There are two free real parameters: ρ gives the background amplitude and k describes the oscillations of the carrier wave.
The envelope-hole solution is given by:

$$g = g_0(1 + Ze^{\eta}), \;\; f = 1 + e^{\eta} \tag{240}$$

Here $\eta = Kx - \Omega t$, $\Omega = K(2k - \sqrt{2\rho^2 - K^2})$, where K and Ω are real. The complex phase factor Z is given by:

$$Z = \frac{\sqrt{4\rho^2 - K^2} + iK}{\sqrt{4\rho^2 - K^2} - iK} \tag{241}$$

From this formula we obtain an inequality that needs to be satisfied: $|K| \leq 2\rho$.
One can derive the following relation:

$$\rho^2 - |\psi|^2 = \frac{K^2/4}{\cosh^2\left(\frac{\eta}{2}\right)} \tag{242}$$

From this expression it can be seen that the dark pulse has the same shape as an ordinary soliton. In Figure 3 one can see the typical aspect of a dark soliton.

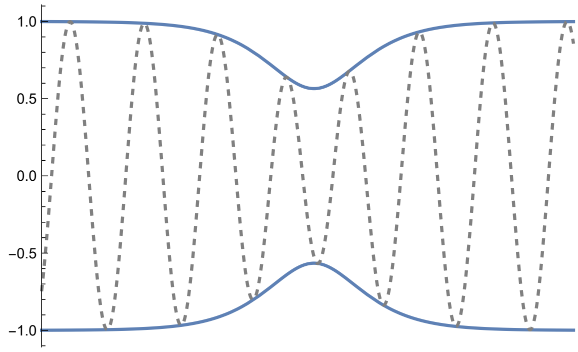

Figure 3: Example of dark soliton. The thick blue line shows the carrier wave, described by $|\psi|$, and the dashed gray line shows $\mathrm{Re}\,\psi$.

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