PATH INTEGRAL FORMALISM IN QUANTUM FIELD THEORY

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- Green functions
- Path integral formalism
- Generating functionals
- Quantization of gauge theories

Assume some basic knowledge of QFT:
Field operators, *S*-matrix, cross sections, perturbation theory and Feynman rules, gauge theories

Literature: Weinberg: The Quantum Theory of Fields, I+II

Böhm, Denner, Joos: Gauge Theories of the Strong and Electroweak Interaction

Try to answer (at least these) two questions:

- How do I get Feynman rules from a Lagrangean ?
- Why and when do I need ghosts ?

S-matrix:

initial state $|\alpha\rangle_{in}$: observed at $t \to -\infty$
final state $|\beta\rangle_{out}$: observed at $t' \to +\infty$

$$S_{\beta\alpha} = {}_{out}\langle\beta|\alpha\rangle_{in}$$

match in/out states on N-particle states of a free theory

$$= {}_0\langle\beta|\hat{S}|\alpha\rangle_0$$
$$= \lim_{t\to+\infty, t'\to-\infty} \; {}_0\langle\beta|U(t,t')|\alpha\rangle_0$$

i.e. the S-matrix describes the time evolution of states in a full, interacting theory:

$$\hat{S} = T \exp\left(-i\int_{-\infty}^{\infty} d\tau \hat{H}_{int}(\tau)\right)$$

Lorentz invariance and causality is guaranteed, if $\hat{H}$ can be written as an integral over a scalar density: $\mathcal{H}(\vec{x}, t)$. Instead of $\mathcal{H}$, use the Lagrangean $\mathcal{L}$:

$$\hat{S} = T \exp\left(i\int d^4x \mathcal{L}_{int}(x)\right)$$

Perturbation theory: expand $\exp(\int \mathcal{L}_{int}) \to \sum(\int \mathcal{L}_{int})^n$ ➜ Feynman rules

${}_0\langle\beta|$ and $|\alpha\rangle_0$ from products of creation and annihilation operators
$\hat{S}$, from $\mathcal{L}_{int}$, is built from products of field operators:
i.e. F.Tr. of creation and annihilation operators
e.g.: $\ldots\hat{\phi}_l(x)|0\rangle = \ldots \frac{1}{(2\pi)^3}\int\frac{d^3q}{2q^0}u_l(q)e^{-iqx}a^\dagger(q)|0\rangle$

➜ Definition: n-point Green functions

$$G^{(n)}_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) = \langle\Omega|T\hat{\psi}_{\alpha_1}(x_1)\ldots\hat{\psi}_{\alpha_n}(x_n)|\Omega\rangle$$

where
$\hat{\psi}_\alpha(x)$ are Heisenberg operators with the time dependence of the full Hamiltonian $\hat{H}$;
$|\Omega\rangle$ is the ground state of $\hat{H}$ (the vacuum).
Index α to distinguish different fields and spin degrees of freedom: Lorentz index μ, Dirac index α, etc.

The set of all Green functions defines a theory

Side remark: assume $\hat{H} = \hat{H}_0 + \hat{H}_{int}$

$\hat{H}|n\rangle = E_n|n\rangle, \quad \hat{H}|\Omega\rangle = E_0|\Omega\rangle$ with $E_n > E_0$ for all n

$\hat{H}_0|0\rangle = 0|0\rangle$

Then: $|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)}\frac{1}{e^{-iE_0t}\langle\Omega|0\rangle}e^{-i\hat{H}t}|0\rangle$

Green functions are invariant with respect to translations:
$\hat{\psi}_\alpha(x) \to \hat{\psi}_\alpha(x+a) = U(a)\hat{\psi}_\alpha(x)U^\dagger(a)$, with $U^\dagger U = \mathbb{1}$
The vacuum is translation invariant:
$|\Omega\rangle \to U^\dagger(a)|\Omega\rangle = |\Omega\rangle$

Therefore:

$$\begin{aligned}
G^{(n)}(x_1,\ldots,x_n) &= \langle\Omega|T\hat{\psi}(x_1)\ldots\hat{\psi}(x_n)|\Omega\rangle \\
&= \langle\Omega|U(a)T\hat{\psi}(x_1)U^\dagger(a)\cdot U(a)\ldots U^\dagger(a)\cdot U(a)\hat{\psi}(x_n)U^\dagger(a)|\Omega\rangle \\
&= \langle\Omega|T\hat{\psi}(x_1+a)\ldots\hat{\psi}(x_n+a)|\Omega\rangle \\
&= G^{(n)}(x_1+a,\ldots,x_n+a) \qquad \text{(choose } a=-x_1\text{)} \\
&= G^{(n)}(0,x_2-x_1\ldots,x_n-x_1)
\end{aligned}$$

Green functions in momentum space, Fourier transformation:

$$G^{(n)}(p_1,\ldots,p_n) = \int d^4x_1\ldots\int d^4x_n e^{-i(p_1x_1+\ldots+p_nx_n)}G^{(n)}(x_1,\ldots,x_n)$$

set $\xi_i = x_i - x_1$, i.e. $x_i = \xi_i + x_1$:

$$= \int d^4x_1 e^{-ix_1\sum p_i}\int d^4\xi_2\ldots\int d^4\xi_n e^{-i(p_2\xi_2+\ldots+p_n\xi_n)}G^{(n)}(0,\xi_2\ldots,\xi_n)$$

$$= (2\pi)^4\delta^{(4)}(p_1+\ldots+p_n)\widetilde{G}^{(n)}(p_1,\ldots,p_n)$$

Connection to the S-matrix: Lehmann, Symanzik Zimmermann (without a proof)

consider the simplest case: $1 \to 1$: $|\vec{p}\,\rangle_0 \to |\vec{p}\,'\rangle_0$ in a free theory

S-matrix element: $\langle 0|a(\vec{p}\,')a^\dagger(\vec{p})|0\rangle = 2E_p\delta^{(3)}(\vec{p} - \vec{p}\,')$ with $E_p = \sqrt{\vec{p}^2 + m^2}$

Green function:
$$\begin{aligned}\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle &= G^{(2)}_{free}(x_1, x_2) \\ &= \text{propagator} \\ &= \frac{i}{(2\pi)^4}\int d^4p\, e^{ip(x_1-x_2)}\frac{1}{p^2-m^2+i\epsilon}\end{aligned}$$
(the Green function of the Klein-Gordon equation)

in momentum space: $\widetilde{G}^{(2)}_{free}(p_1, p_2) = \frac{i}{p_1^2-m^2+i\epsilon} = \widetilde{G}^{(2)}_{free}(p_1, -p_1)$

To obtain S-matrix elements from Green functions:

- remove a factor $\frac{i}{p^2-m^2}$ for each external particle;
- set momenta on-shell: $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$ ($\pm$ for in / out);
- adjust the normalization (the wave function renormalization $1/\sqrt{R}$):

$$R = (2\pi)^3 \left|\langle 0|a(\vec{p})\hat{\psi}(x)|\Omega\rangle\right|^2 = -i(2\pi)^3(p^2 - m^2)\,\widetilde{G}^{(2)}(p, -p)\Big|_{p^2=m^2}$$

LSZ-formula (for spin 0):

$$
{}_{out}\langle -\vec{q}_1, \ldots, -\vec{q}_M | \vec{p}_1, \ldots \vec{p}_N \rangle_{in} =
R^{-(N+M)/2} (-i)^{N+M} (p_1^2 - m^2) \ldots (q_M^2 - m^2) G^{(N+M)}(p_1, \ldots, q_M)|_{p_i^2 = m^2, p_i^0 > 0, q_i^2 = m^2, q_i^0 < 0}
$$

- all momenta are counted as incoming
- multiplication with $-i(p^2 - m^2)$, i.e. with the inverse propagator: truncation
  One defines truncated Green functions:

$$
\widetilde{G}^{(n)}_{trunc}(p_1, \ldots, p_n) = \left(\widetilde{G}^{(2)}(p_1, -p_1)\right)^{-1} \ldots \left(\widetilde{G}^{(2)}(p_n, -p_n)\right)^{-1} \widetilde{G}^{(n)}(p_1, \ldots, p_n)
$$

- S-matrix elements are the residues of the Green functions at their poles $p_i^2 = m^2$
- Green functions are the analytic continuation of S-matrix elements to off-shell momenta

For particles with spin: need additional factors:
Dirac spinors $u_\alpha(\vec{p}, \sigma)$ etc., polarization vectors $\epsilon_\mu(\vec{k}, \lambda)$.

From now on: talk about Green functions only

2-point functions:

spin 0: $G^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2; m)$

spin $\frac{1}{2}$: $\langle 0|T\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)|0\rangle = S_{F,\alpha\beta}(x_1 - x_2; m) = (i\not{\partial}_{x_1} + m)_{\alpha\beta} i\Delta_F(x_1 - x_2; m)$

spin 1, $m \neq 0$:

$\langle 0|T\,A_\mu(x_1)A_\nu(x_2)|0\rangle = D_{F,\mu\nu}(x_1 - x_2; m) = \left(-g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}\right) i\Delta_F(x_1 - x_2; m)$

(3-point functions are zero in a free field theory)

4-point function (spin 0):

$$
\begin{aligned}
G^{(4)}(x_1, x_2, x_3, x_4) &= \langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|0\rangle \\
&= i^2\,(\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) \\
&\qquad + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\
&\qquad + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3))
\end{aligned}
$$

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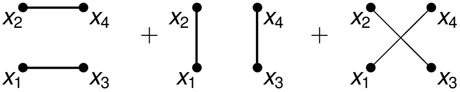


e.g. $\langle\Omega|T\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)|\Omega\rangle$

- relate $|\Omega\rangle$ to $|0\rangle$
- trace full time dependence of $\hat{\phi}_H$ from $\hat{H} = \hat{H}_0 + \hat{H}_{int}$

Without proof:

$$\langle\Omega|T\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)U(-t,t)|0\rangle}{\langle 0|U(-t,t)|0\rangle}$$

where $\hat{\phi}_I$: field operators in the interaction picture: $\hat{\phi}_I(x) = e^{i\hat{H}_0(t-t_0)}\hat{\phi}_S(\vec{x})e^{-i\hat{H}_0(t-t_0)}$
and $U(t_1,t_2) = T\exp\left\{-i\int_{t_1}^{t_2} d\tau \hat{H}_{int}(\tau)\right\}$

Generalization to n-point Green functions:

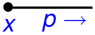
$$G^{(n)}(x_1,\ldots,x_n) = \langle\Omega|T\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)|\Omega\rangle$$

$$= \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n)\exp\left\{-i\int_{-t}^{+t} d\tau \hat{H}_{int}(\tau)\right\}|0\rangle}{\langle 0|\exp\left\{-i\int_{-t}^{+t} d\tau \hat{H}_{int}(\tau)\right\}|0\rangle}$$

From this: perturbation theory for Green functions like perturbation theory for S-matrix elements, in particular: vertex rules from $\hat{H}_{int}$

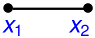
. . . as for S-matrix elements, but

- Field points x_i assigned to the arguments of $G^{(n)}$
- Incoming / outgoing lines end at a field point:



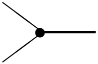
$$= \int \frac{d^4p}{(2\pi)^4} e^{-ipx}$$

- $G^{(n)}$ contains a δ-function for total 4-momentum conservation: $(2\pi)^4 \delta^{(4)}(\sum p_i)$
- Internal lines connect field points



$$= \langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle \quad = \quad i\Delta_F(x_1 - x_2; m)$$

- Interaction $\hat{H}_{int}$ ➜ vertex rules (as for the conventional S-matrix Feynman rules)
  e.g. from $\mathcal{L}_{int} = -g\phi^3(x) \longrightarrow$



$$= -ig$$

Path Integral Formalism and Generating Functionals

Reminder: the path integral in non-relativistic quantum mechanics

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t) e^{iS/\hbar}$$

with $S = \int_{t_i}^{t_f} dt\, L(x, \dot{x}) = S[x(t)]$, the classical action, $x(t_i) = x_i$, $x(t_f) = x_f$
and $\int \mathcal{D}x(t)$ from $\lim_{n\to\infty} C \int_{-\infty}^{+\infty} dx_n \ldots dx_1$: integration over paths $x(t)$

The analogue for eigen-states of field operators:

$$\langle \phi_f(\vec{x}); t_f | \phi_i(\vec{x}); t_i \rangle = \int \mathcal{D}\phi(x) \exp\left\{ i \int_{t_i}^{t_f} dt\, d^3x \mathcal{L}(x) \right\}$$

Without proof: Path integral representation of Green functions:

$$\langle \Omega | T \hat{\phi}_H(x_1) \ldots \hat{\phi}_H(x_n) | \Omega \rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) \phi(x_1) \ldots \phi(x_n) \exp\left\{ i \int_{-t}^{+t} d^4x \mathcal{L}(x) \right\}}{\int \mathcal{D}\phi(x) \exp\left\{ i \int_{-t}^{+t} d^4x \mathcal{L}(x) \right\}}$$

with:
- $\hat{\phi}_H(x)$: Heisenberg field operators
- $\phi(x)$ eigenvalues of $\hat{\phi}_H(x)$, i.e. classical, c-number fields

Calculate the probability to find a particle at time t_f at position x_f when it was at the position x_i at time t_i.

Use eigen-states $|x;t\rangle$ of the Heisenberg position operator $\hat{x}_H(t) = e^{i\hat{H}t/\hbar}\hat{x}_S e^{-i\hat{H}t/\hbar}$

$$\langle x_f; t_f | x_i; t_i\rangle = {}_S\langle x_f | e^{-i\hat{H}(t_f - t_i)/\hbar} | x_i\rangle_S$$

Split time interval $[t_i, t_f]$ in $n+1$ time steps: $t_i = t_0 < t_1 < \ldots < t_n < t_{n+1} = t_f$, insert complete set of states $\int dx |x;t_k\rangle\langle x;t_k| = \mathbb{1}$

$$\langle x_f; t_f | x_i; t_i\rangle = \int_{-\infty}^{+\infty} dx_n \ldots \int_{-\infty}^{+\infty} dx_1 \langle x_f; t_f | x_n; t_n\rangle\langle x_n; t_n | x_{n-1}; t_{n-1}\rangle \ldots \langle x_1; t_1 | x_i; t_i\rangle$$

Integration over all x_k at all times t_k: integration over paths $x_k = x(t_k)$

Calculate matrix elements for $\Delta t \to 0$: $\langle x_{k+1}; t_{k+1} | x_k; t_k\rangle = \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m\hbar}{i\Delta t}}\, e^{iL(x_k,\dot{x}_k)\Delta t/\hbar}$

$\Delta t \to 0$ implies $n \to \infty$: from a discrete set (x_k, t_k) to a continuous function $x(t)$

$$\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{n/2} \int_{-\infty}^{+\infty} dx_n \ldots \int_{-\infty}^{+\infty} dx_1 \to \int \mathcal{D}x(t)$$

and $\prod_{k=0}^{n} \exp\left\{\frac{i}{\hbar}(t_{k+1} - t_k)L(x_k, \dot{x}_k)\right\} \to \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t_f} L(x(t), \dot{x}(t))\right\} = \exp\left\{iS[x(t)]/\hbar\right\}$

All paths contribute, also those that are not allowed classically, but the dominating contribution is from paths where $S[x(t)]$ is extremal.

- one variable: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy\, e^{-\frac{1}{2} a y^2} = \frac{1}{\sqrt{a}} \quad (a \in \mathbb{C},\, a \neq 0)$
- n variables: $y = (y_1, \ldots, y_n)^T$, bilinear form $-\frac{1}{2} y^T A y = -\frac{1}{2} \sum_{i,j} A_{ij} y_i y_j$
  with A: real positive, non-singular, symmetric, with eigen-values a_i
  $\frac{1}{\sqrt{2\pi}^n} \int_{-\infty}^{+\infty} dy_1 \ldots \int_{-\infty}^{+\infty} dy_n e^{-y^T A y/2} = \frac{1}{\sqrt{\det A}}$
  Use $\ln \det A = \sum_i \ln a_i = Tr \ln A$, then: $\sqrt{\det A} = \exp\left(\frac{1}{2} Tr \ln A\right)$
- Linear terms in the exponent with vector b:
  $-\frac{1}{2} y^T A y + b^T y = -\frac{1}{2}(y - A^{-1} b)^T A (y - A^{-1} b) + \frac{1}{2} b^T A^{-1} b$
  and translation $y \to y - A^{-1} b$
  $\frac{1}{\sqrt{2\pi}^n} \int_{-\infty}^{+\infty} dy_1 \ldots \int_{-\infty}^{+\infty} dy_n e^{-y^T A y/2 + b^T y} = \frac{1}{\sqrt{\det A}} e^{(b^T A^{-1} b)/2}$

- Infinitely many variables: instead of a discrete (integer-valued) index $\rightarrow$ continuous variable x (usually real-valued, in $[-\infty, +\infty]$):

  replace $\quad y_i \rightarrow y_x = y(x)$

  and $\quad \prod_{i=1}^{n} \frac{dy_i}{2\pi} \rightarrow \mathcal{D}y(x),$

  Bilinear form: $\quad y^T A y \rightarrow \int dx \int dx' y(x) A(x,x') y(x')$

  $$\int \mathcal{D}y(x) \exp\left\{-\frac{1}{2}\int dx \int dx' y(x) A(x,x') y(x') + \int dx J(x) y(x)\right\}$$
  $$= \exp\left\{-\frac{1}{2} Tr \ln A\right\} \exp\left\{\frac{1}{2}\int dx \int dx' J(x) A^{-1}(x,x') J(x')\right\}$$

- $\int \mathcal{D}y(x)$ is called a functional integral

- Exercise: Calculate the Gaussian integral with the kernel

  $$A(x,x') = -\frac{d^2}{dx^2}\delta(x-x') + \omega^2 \delta(x-x'). \quad \text{Result: } 1 \Big/ \sqrt{\det\left(-\frac{d^2}{dx^2} + \omega^2\right)}$$

- In this example: the result depends on a function $J(x)$, i.e., it is a functional $F[J(x)]$

Definition:

$$\frac{\delta}{\delta J(y)} F[J(x)] = \lim_{\epsilon \to 0} \frac{F[J(x) + \epsilon \delta(x-y)] - F[J(x)]}{\epsilon}$$

Rules as for the usual derivative: linearity, Leibniz rule, chain rule,
in particular: $\frac{\delta}{\delta J(y)} J(x) = \delta(x-y)$

Consider:

$$\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \int \mathcal{D}y(x) \exp\left\{ -\frac{1}{2} \int dx \int dx' \, y(x) A(x,x') y(x') + \int dx J(x) y(x) \right\} \Bigg|_{J=0}$$

$$= \int \mathcal{D}y(x) \; y(x_1) \ldots y(x_n) \exp\left\{ -\tfrac{1}{2} \int \int yAy \right\}$$

$$= \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \exp\left\{ -\frac{1}{2} Tr \ln A \right\} \exp\left\{ \frac{1}{2} \int dx \int dx' \, J(x) A^{-1}(x,x') J(x') \right\}$$

From above:

$$\langle\Omega|T\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x)\,\phi(x_1)\ldots\phi(x_n)\exp\left\{i\int d^4x\mathcal{L}(x)\right\}}{\int \mathcal{D}\phi(x)\exp\left\{i\int d^4x\mathcal{L}(x)\right\}}$$

The generating functional for Green functions:

$$\mathcal{Z}[J(x)] = \int \mathcal{D}\phi(x)\exp\left\{i\int d^4x\,(\mathcal{L}(x)+J(x)\phi(x))\right\}$$

since

$$\frac{(-i)^n\delta^n}{\delta J(x_1)\ldots\delta J(x_n)}\mathcal{Z}[J(x)] = \int \mathcal{D}\phi(x)\,\phi(x_1)\ldots\phi(x_n)\exp\left\{i\int d^4x(\mathcal{L}(x)+J(x)\phi(x))\right\}$$

and

$$\langle\Omega|T\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)|\Omega\rangle = (-i)^n\frac{1}{\mathcal{Z}[0]}\frac{\delta^n}{\delta J(x_1)\ldots\delta J(x_n)}\mathcal{Z}[J(x)]\bigg|_{J=0}$$

$\mathcal{Z}[J]$ is the functional Fourier transform of $\exp(iS[\phi])$

Consider a power series of a function: $f(y) = \sum_{n=0}^{\infty} a_n y^n$

One can say, $f(y)$ *generates* the coefficients a_n since $a_n = \frac{1}{n!} \left. \frac{d^n f(y)}{dy^n} \right|_{y=0}$

Similarly, for k variables y_i, $i = 1, \ldots, k$, the Volterra series:

$$F(\vec{y}) = \sum_{n=0}^{\infty} \sum_{i_1=1}^{k} \cdots \sum_{i_n=1}^{k} \frac{1}{n!} a_{i_1 \ldots i_n} y_{i_1} \ldots y_{i_n}$$

generates the coefficients $a_{i_1 \ldots i_n}$

Transition to infinitely many variables:

$$i \to x, \quad y_i \to y(x), \quad \sum_i \to \int dx$$

$$F[y(x)] = \sum_{n=0}^{\infty} \int dx_1 \ldots dx_n \frac{1}{n!} a^{(n)}(x_1, \ldots, x_n) y(x_1) \ldots y(x_n)$$

is generating functional for the functions $a^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n}{\delta y(x_1) \ldots \delta y(x_n)} F[y(x)] \Big|_{y=0}$

For a free scalar field:

$$\mathcal{Z}_{free}[J(x)] = \int \mathcal{D}\phi(x) \exp\left\{ i \int d^4x \left(\mathcal{L}_{free}(x) + J(x)\phi(x) \right) \right\}$$

with

$$\mathcal{L}_{free} = \frac{1}{2} \left(\partial^\mu \phi(x) \partial_\mu \phi(x) - m^2 \phi^2(x) \right)$$

The exponent is $\int d^4x \left(-\frac{1}{2}\phi(x) \left[\Box + m^2 \right] \phi(x) + J(x)\phi(x) \right)$

i.e., we have a Gaussian integral with a linear term.

The Green function of the Klein-Gordon equation: $(\Box + m^2)\Delta_F(x - x') = -\delta(x - x')$

$$\mathcal{Z}_{free}[J] = \exp\left\{ -\frac{i}{2} \int d^4x \int d^4y \, J(x) \Delta_F(x-y) J(y) \right\} \cdot \mathcal{Z}_{free}[0]$$

with $\mathcal{Z}_{free}[0] = \int \mathcal{D}\phi(x) \exp\left\{ -(i/2) \int \int \phi(\Box + m^2)\phi \right\}$, independent of J

Exercise: check

$$G^{(2)}(x_1, x_2) = \frac{\delta^2}{i\delta J(x_1) i\delta J(x_2)} \mathcal{Z}_{free}[J]\Big|_{J=0} \Big/ \mathcal{Z}_{free}[0]$$

$$= \frac{\delta}{i\delta J(x_1)} \left[-\frac{1}{2} \int d^4y \, \Delta_F(x_2 - y) J(y) - \frac{1}{2} \int d^4x \, J(x) \Delta_F(x - x_2) \right]$$

$$\times \, e^{-(i/2) \int d^4x \int d^4y \, J(x) \Delta_F(x-y) J(y)} \Big|_{J=0}$$

$$= \left[\frac{i}{2} \Delta_F(x_2 - x_1) + \frac{i}{2} \Delta_F(x_1 - x_2) + \text{more terms with J} \right]$$

$$\times \, e^{-(i/2) \int d^4x \int d^4y \, J(x) \Delta_F(x-y) J(y)} \Big|_{J=0}$$

$$= i\Delta_F(x_1 - x_2)$$

Exercise: calculate $G^{(4)}(x_1, x_2, x_3, x_4)$

➜ The first element of Feynman rules: Connect field points by propagators in all possible ways

The idea: replace $\phi(x)$ in $\mathcal{L}_{int}(x) = \mathcal{L}_{int}(\phi(x))$ by a derivative $\delta/i\delta J(x)$

$$\mathcal{Z}[J(x)] = \int \mathcal{D}\phi(x) \exp\left\{i\int d^4x \mathcal{L}_{int}(\phi(x))\right\} \exp\left\{i\int d^4x(\mathcal{L}_{free}(x) + J(x)\phi(x))\right\}$$

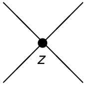
$$= \exp\left\{i\int d^4z \mathcal{L}_{int}\left[\frac{\delta}{i\delta J(z)}\right]\right\} \int \mathcal{D}\phi(x) \exp\left\{i\int d^4x(\mathcal{L}_{free}(x) + J(x)\phi(x))\right\}$$

$$\mathcal{Z}[J(x)] = \exp\left\{i\int d^4z \mathcal{L}_{int}\left[\frac{\delta}{i\delta J(z)}\right]\right\} \exp\left\{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\Delta_F(x-y)J(y)\right\} \cdot \mathcal{Z}[0]$$

Power series expansion of $e^{i\int \mathcal{L}_{int}}$ generates the perturbation series

Terms in $\mathcal{L}_{int}(z)$ are assigned to internal field points

Example: $\mathcal{L}_{int} = -g\phi^4(z) \to -g\frac{\delta^4}{\delta J^4(z)}$,



$= -ig$

In a free theory: all Green functions are obtained from $G^{(2)}$, i.e. the free propagator

Connected Green functions $G_c^{(n)}$: described by graphs where any two points are connected to each other by internal lines

E.g. $$G^{(4)}(x_1, x_2, x_3, x_4) = G_c^{(4)}(x_1, x_2, x_3, x_4) + G_c^{(2)}(x_1, x_2) G_c^{(2)}(x_3, x_4) + G_c^{(2)}(x_1, x_3) G_c^{(2)}(x_2, x_4) + G_c^{(2)}(x_1, x_4) G_c^{(2)}(x_2, x_3))$$

Can be inverted: $G_c^{(n)} = G^{(n)} - \sum G^{(n-i)} G^{(i)} + \sum G^{(n-i-j)} G^{(i)} G^{(j)} \mp \ldots$

Generalization to n-point Green functions: cluster decomposition

Generating functional of connected Green functions:

$$\mathcal{T}_c[J] = \log(\mathcal{T}[J]), \quad \text{with} \quad \mathcal{T}[J] = \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$$

1-p-i: One-particle irreducible Green functions, i.e. Green functions with graphs that can not be separated into two parts by cutting one line

Proper vertex functions, $i\Gamma(x_1, \ldots, x_n)$, are connected, completely truncated, 1-p-i Green functions

Example:

$G_c^{(3)}(x_1, x_2, x_3) = \int d^4z_1 \int d^4z_2 \int d^4z_3\, G_c^{(2)}(x_1, z_1) G_c^{(2)}(x_2, z_2) G_c^{(2)}(x_3, z_3) i\Gamma(z_1, z_2, z_3)$

Example: $G_c^{(4)}(x_1, x_2, x_3, x_4) = ?$

Generating functional of proper vertex functions:

$$\Gamma[a] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, \Gamma(x_1, \ldots, x_n)\, a(x_1) \ldots a(x_n)$$

is obtained from $\mathcal{T}_c[J]$ by a functional Legendre transformation:

$$i\Gamma[a] = -i \int d^4x\, J(x) a(x) + \mathcal{T}_c[J] \quad \text{with} \quad a(x) = \frac{\delta \mathcal{T}_c[J]}{i\delta J(x)}$$

For free scalar fields:

$a(x) = -\int d^4y\, \Delta_F(x-y)J(y)$

i.e. $J(x) = (\Box + m^2)a(x)$

and $\Gamma_{free}[a] = -\frac{1}{2}\int d^4x\, a(x)(\Box + m^2)a(x) = S_{free}[a]$, $\rightarrow \Gamma[a]$ is called effective action

To prove that $i\Gamma[a]$ generates proper vertex functions:

Determine the inverse Legendre transform, use $\delta\Gamma[a]/\delta a(x) = -J(x)$, use chain rule to calculate connected Green functions, i.e. derivatives of $\mathcal{T}_c[J]$

➜ Fynman rules for vertices are the proper vertex functions at tree level:
take functional derivatives of $\Gamma_{tree}[\phi] = S_{eff}[\phi] = \int d^4x \mathcal{L}_{int}(\phi(x))$

Anti-commuting field operators ➜ need anti-commuting classical variables

Grassmann algebra

Variables θ_i, $i = 1, \dots, n$ and conjugated variables θ_i^*, $i = 1, \dots, n$, with $(\theta_i^*)^* = \theta_i$

Addition and multiplication

Anti-commutation rules $\{\theta_i, \theta_j\} = \{\theta_i, \theta_j^*\} = \{\theta_i^*, \theta_j^*\} = 0$ for all i, j

$\Rightarrow \theta_i^2 = (\theta_i^*)^2 = 0$

Polynomials in θ_i, θ_i^*:

$F(\theta^*, \theta) = \sum_{r,s=1}^{n} \sum_{l_i,k_j=1}^{n} T(l_1, \dots, l_r | k_1, \dots, k_s) \theta_{k_s} \dots \theta_{k_1} \theta_{l_r}^* \dots \theta_{l_1}^*$

with anti-symmetric (in l_i and k_j) coefficients $T(.|.)$
$\Rightarrow$ Polynomials can have only a finite number of terms: products with more than n θ's are zero

Example: $n = 1$:
$F(\theta) = f_0 + f_1 \theta$
$F(\theta, \theta^*) = f_0 + f_1 \theta + f_2 \theta^* + f_3 \theta \theta^*$

Differentiation, to "generate" the coefficients $T(.|.)$ from polynomials $F(\theta,\theta^*)$

$$\frac{\partial}{\partial\theta_k}1 = 0, \quad \frac{\partial}{\partial\theta_k}\theta_i = \delta_{ik}, \quad \frac{\partial}{\partial\theta_k^*}\theta_i = 0$$

For the Leibniz rule: take care of the order: derivatives are left-derivatives

$$\frac{\partial}{\partial\theta_k}(F_1F_2) = \left(\frac{\partial}{\partial\theta_k}F_1\right)F_2 \pm F_1\left(\frac{\partial}{\partial\theta_k}F_2\right)$$

Integration defined as a linear, algebraic operation:

$$\int d\theta_k = 0, \quad \int d\theta_k\theta_i = \delta_{ik}$$

Note: (1) differentials $d\theta_i$ anti-commute; (2) Rules for integrals as usual, except signs

Exercise: substitution of variables $\eta_i = \sum_k A_{ik}\theta_k$

$$\int d\eta_n \ldots d\eta_1 F(\eta) = \frac{1}{\det A}\int d\theta_k \ldots d\theta_1 F(\eta(\theta))$$

Proof: write F as a power series: left-hand side: coefficients of $\eta_1,\ldots,\eta_n$ are $T_{1\ldots n}$;
right-hand side: $\frac{1}{\det A}\int d\theta_k \ldots d\theta_1 T_{1\ldots n}(A\theta)_1 \ldots (A\theta)_n$

Fubini formula: the Gaussian integral for $2n$ Grassmann variables

$$\int d\eta_1 \dots d\eta_n \, d\eta_n^* \dots d\eta_1^* \; \exp\left(\sum_{i,k} \eta_i^* A_{ik} \eta_k\right) = \det A$$

Proof: use substitution rule $A_{ik}\eta_k \to \theta_i$; or expand in a (finite !) power series: only terms $\propto (\eta^* A \eta)^n$ contributes; sort coefficients to recover the determinant

Functional Gaussian integral for fermion fields

$$\int \mathcal{D}\psi \mathcal{D}\psi^* \exp\left\{ \int dx' dx \, \psi^*(x') A(x, x') \psi(x) \right\} = \det A$$

The generating functional for free fermionic fields

$$\mathcal{Z}_{free}[\bar{\eta}, \eta] = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \int d^4x_n \dots d^4x_1\, d^4x_1' \dots d^4x_n'\ \eta(x_n') \dots \eta(x_1')\, \bar{\eta}(x_n) \dots \bar{\eta}(x_1)$$
$$\times \langle 0 | T\Psi(x_1) \dots \Psi(x_n) \bar{\Psi}(x_n') \dots \bar{\Psi}(x_1') | 0 \rangle$$

$$= \int \mathcal{D}\psi(x) \mathcal{D}\bar{\psi}(x) \exp\left(i \int d^4x (\mathcal{L}_{free} - \bar{\psi}\eta + \bar{\eta}\psi) \right)$$

$$= \exp\left(i \int d^4x d^4x' \bar{\eta}(x) S_F(x - x') \eta(x) \right) \mathcal{Z}_{free}[0, 0]$$

For interacting fields: add $\mathcal{L}_{int}(\psi, \bar{\psi})$ and replace $\psi \to \delta/\delta\bar{\eta}$ and $\bar{\psi} \to \delta/\delta\eta$

Path integral formalism for gauge theories

Consider a multiplet of fields $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$

with transformations under an n-dimensional representation of a symmetry group G (e.g. $SU(N)$, non-Abelian),

$\psi(x) \to \psi'(x) = U\psi(x)$, with $U = e^{i\theta^a T^a}$, $U^\dagger U = \mathbb{1}$

parameters θ^a, generators T^a ($n \times n$ matrices), $[T^a, T^b] = if^{abc}T^c$

Local transformations: make θ^a x-dependent: $\theta^a(x)$, $U(x) \to$ need covariant derivative:

$D_\mu = \partial_\mu - i\mathcal{A}_\mu(x)$

i.e. $D^{ij}_\mu = \partial_\mu \delta^{ij} - i\mathcal{A}^{ij}_\mu(x)$

Gauge fields: $\mathcal{A}_\mu(x) = gA^a_\mu(x)T^a$

Example: QCD, $G = SU(3)$, quark fields with 3 color, generators: $T^a = \frac{1}{2}\lambda^a$, the 3×3 Gell-Mann matrices, A^a_μ, $a = 1, \ldots, 8$: eight gluons, dim $SU(N) = N^2 - 1$

Then $\mathcal{L}_{Dirac} = \bar{\psi}(i\partial_\mu\gamma^\mu - m)\psi$ is locally invariant if $\partial_\mu \to D_\mu$: $\mathcal{L} = \bar{\psi}(iD_\mu\gamma^\mu - m)\psi$

Transformation of gauge fields:

$\mathcal{A}_\mu(x) \to \mathcal{A}'_\mu(x) = U(x)(\mathcal{A}(x) + i\partial_\mu)U^\dagger(x)$

Decompose $\mathcal{A}_\mu(x) = gA^a_\mu(x)T^a$, for infinitesimal transformations:

$\delta\mathcal{A}_\mu(x) = (i[T^a, \mathcal{A}_\mu(x)] + T^a\partial_\mu)\,\delta\theta^a(x)$

$\delta A^a_\mu(x) = \frac{1}{g}\partial_\mu\delta\theta^a(x) + f^{abc}A^b_\mu(x)\delta\theta^c(x)$

Field strength tensor, define: $\mathcal{F}_{\mu\nu} = i[D_\mu, D_\nu]$, (curvature)

transforms as $\mathcal{F}_{\mu\nu} \to \mathcal{F}'_{\mu\nu} = U(x)\mathcal{F}_{\mu\nu}U^\dagger(x)$

or, using its decomposition $\mathcal{F}_{\mu\nu} = gF^a_{\mu\nu}T^a$:

$F^a_{\mu\nu} = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^b_\mu(x)A^c_\nu(x)$ and $\delta F^a_{\mu\nu} = f^{abc}F^b_{\mu\nu}\delta\theta^c(x)$

$$\mathcal{L}_{SU(N)} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \bar{\psi}(iD_\mu\gamma^\mu - m)\psi$$

(for scalar fields: $(D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi$)

Gauge symmetries ➜ Construction principle for theories with interaction

Quantization with the path integral formalism?

Quantization? ➜ Find a definition of the set of all Green functions

Need the functional Gauss integral

$$\int \mathcal{D}\mathcal{A}_\mu(x) \exp\left\{i \int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + J^a_\mu A^{a,\mu}\right]\right\} =$$
$$\int \mathcal{D}\mathcal{A}_\mu(x) \exp\left\{i \int d^4x \left[-\frac{1}{2} A^{a,\mu} D^{-1}_{F,\mu\nu} A^{a,\nu} + \ldots\right]\right\}$$

But the inverse propagator $D_{F,\mu\nu}$ has eigenvalues 0

The reason: there are infinitely many different field configurations for $\mathcal{A}_\mu(x)$, related by a gauge transformation, which are physically equivalent

➜ Try to separate the integration over gauge-equivalent field configurations

A given $A_\mu^a(x)$ is connected by gauge transformations with parameters $\theta^a(x)$ to

$gA_\mu^a(x;\theta^a)T^a = U(x)(gA_\mu^a(x)T^a + i\partial_\mu)U^\dagger(x)$

Select a gauge by choosing the solution of

$G^c[A_\mu^a(x;\theta^b)] = B^c(x)$, where G^c are local functionals of the gauge fields

Examples:

- $G^c = n^\mu A_\mu^c$, axial gauge with a constant 4-vector n^μ
- $G^c = \partial^\mu A_\mu^c$, Lorenz gauge
- $G^c = \partial^\mu A_\mu^c + \xi^c\phi$, with a scalar field ϕ, R_ξ gauge

A problem (not relevant in perturbation theory):
Gribov copies, the solution of $G^c = B^c$ may be not unique!

Gauge fixing in the path integral: insert a functional δ-function, $\delta(G^c - B^c)$:

Consider n variables $y_1, \ldots, y_n$ *and functions* $g_i = g_i(y_1, \ldots, y_n)$, $i = 1, \ldots, n$, *unique, invertible*

Then: $\int dy_1 \ldots dy_n \prod_{i=1}^n \delta(g_i(y_1, \ldots, y_n)) \det\left(\frac{\partial g_i}{\partial y_j}\right) = 1$

To prove: substitution $y_i \to g_j$ *with Jacobian* $\partial g_i / \partial y_j$ *leads to*

$\int dg_1 \ldots dg_n \prod_{i=1}^n \delta(g_i) = 1$

$$\prod_c \int \mathcal{D}\theta^c(y) \prod_b \delta\left(G^b[A^a_\mu(x;\theta^c)] - B^b(x)\right) \det M_G = 1$$

$$\text{with} \quad (M_G)_{bc} = \frac{\delta G^b[A^a_\mu(x;\theta^c)]}{\delta\theta^c(y)}$$

Note: θ^c are the "coordinates" of the symmetry group, $\int \mathcal{D}\theta^c(y)$ is integration over the group

Assume $G^b[A_\mu(x;\theta)]$ is linear in θ, i.e. $\det M_G$ is θ-independent. Then

$$\det M_G \prod_c \int \mathcal{D}\theta^c(y) \prod_b \delta\left(G^b[A^a_\mu(x;\theta^c)] - B^b(x)\right) = 1$$

Now the generating functional for Green functions

$$\mathcal{Z}[0] = \int \mathcal{D}\mathcal{A}_\mu(x) e^{i\int d^4x \mathcal{L}(\mathcal{A}(x))} \det M_G \int \mathcal{D}\theta^c(x) \prod_b \delta\left(G^b[A^a_\mu(\theta^c)] - B^b\right)$$

$\mathcal{L}$ and $\int \mathcal{D}\mathcal{A}_\mu$ are invariant, replace $\mathcal{A}_\mu(x)$ by $\mathcal{A}_\mu(x;\theta)$

$$= \int \prod_c \mathcal{D}\theta^c(x) \int \mathcal{D}A^a_\mu(x;\theta^c(x)) \det M_G(A^a_\mu(x;\theta^c(x)))$$
$$\times \prod_b \delta(G^b[A^a_\mu(x;\theta^c(x))] - B^b(x)) \exp^{i\int d^4x \mathcal{L}(A^a_\mu(x;\theta^c(x)))}$$

$$= \left(\int \prod_c \mathcal{D}\theta^c(x)\right) \int \mathcal{D}A^a_\mu(x) \det M_G(A^a_\mu(x))$$
$$\times \prod_b \delta(G^b[A^a_\mu(x)] - B^b(x)) \exp^{i\int d^4x \mathcal{L}(A^a_\mu(x))}$$

$\int \mathcal{D}\theta^c(x)$ is the "group volume", an irrelevant factor that can be removed from the definition of $\mathcal{Z}[J]$. Define:

$$\mathcal{Z}[0] = \int \mathcal{D}A_\mu^a(x) \det M_G \, \delta(G - B) \exp^{i \int d^4x \mathcal{L}}$$

Since every choice of B is equally good, average over $B^c(x)$ with a weight

$$\int \mathcal{D}B^c(x) \;\; \exp\left(-\frac{i}{2\xi} \int d^4x \, B^c(x) B^c(x) \right)$$

Integration over B^c can be performed by virtue of the δ-functions,

$$\mathcal{Z}[J] = \int \mathcal{D}A_\mu^a(x) \det M_G \exp\left\{ i \int d^4x \left[\mathcal{L}(A_\mu^a) - \frac{1}{2\xi} G^c(A_\mu^a) G^c(A_\mu^a) + J^{a,\mu} A_\mu^a \right] \right\}$$

Remarks:

$G^c G^c$ is gauge-dependent, but its gauge dependence is compensated by the gauge dependence of $\det M_G$

Choose G^c such that the $\int \mathcal{D}\mathcal{A}_\mu$ stays Gaussian and can be evaluated

Write $\det M_G$ as a functional integral over anti-commuting fields

$$\det M_G = \int \mathcal{D}c^b(x)\mathcal{D}\bar{c}^a(x) \exp\left(i \int d^4x \int d^4y\, \bar{c}^a(y)(-gM_G^{ab})c^b(x) \right)$$

$c^a(x)$ and $\bar{c}^a(x)$ are different anti-commuting scalar (!) fields,
they are unphysical, there are no ”particle” states assigned to them

They are called Faddeev-Popov ghost fields or simply ghosts

$$\mathcal{Z}[J,\eta,\bar{\eta}] = \int \mathcal{D}A^c_\mu(x) \int \mathcal{D}c^b_\mu(x) \int \mathcal{D}\bar{c}^a_\mu(x)$$
$$\exp\left\{ i \int d^4x \, [\mathcal{L} + \mathcal{L}_{fix} + \mathcal{L}_{FP}\right.$$
$$\left. + J^{a,\mu}(x)A^a_\mu(x) + \bar{c}^{a,\mu}(x)\eta^a_\mu(x) + \bar{\eta}^{a,\mu}(x)c^a_\mu(x)] \right\}$$

with

$$\mathcal{L}_{fix} = -\frac{1}{2\xi} G^c[A^a_\mu(x)] G^c[A^a_\mu(x)]$$

$$\mathcal{L}_{FP} = -\bar{c}^a(x) g M^{ab}_G[A^a_\mu(x)] c^b(x)$$

$$M^{ab}_G = \frac{\delta G^a[A^a_\mu(x;\theta)]}{\delta\theta^b(x)}$$

Lorenz gauge: $G^a = \partial^\mu A^a_\mu$, i.e. $G^a[A^a_\mu(x;\theta)] = \partial^\mu A^a_\mu(x;\theta)$

To calculate M_G, need only infinitesimal gauge transformations:

$\delta A^a_\mu(x) = \frac{1}{g}\partial_\mu \delta\theta^a(x) + f^{abc} A^b_\mu(x)\delta\theta^c(x)$

Then

$$M^{ab}_G(x,y) = \frac{\delta}{\delta\theta^b(y)}\left[\partial^\mu\left(\frac{1}{g}\partial_\mu\delta\theta^a(x) + f^{acd} A^c_\mu(x)\delta\theta^d(x)\right)\right]$$

$$= \left(\frac{1}{g}\partial^\mu\partial_\mu\delta^{ab} - f^{abc}\partial^\mu A^c_\mu(x)\right)\delta^{(4)}(x-y)$$

$$= \frac{1}{g}\partial^\mu D^{ab}_\mu \delta^{(4)}(x-y)$$

$$\mathcal{L}_{FP} = -\bar{c}^a(x)\partial^\mu(\partial_\mu\delta^{ab} + g f^{abc} A^c_\mu(x))c^b(x)$$

- $\mathcal{L}_{FP}$ contains terms $-\bar{c}^a \Box c^a$: kinetic energy of massless scalar fields
- and terms $g\bar{c}^a f^{abc}\partial^\mu A^c_\mu c^b$: FP-ghosts couple to gauge fields
- in QED: gauge symmetry is Abelian, $f^{abc} = 0$: FP-ghosts decouple