Path Integral Formalism in Quantum Field Theory

— Exercises —

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1:
Calculate the propagator for a scalar field.

Hints: The field operator for a scalar field is defined by

$$\hat{\phi}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2q^0} \left(e^{-iqx} a^\dagger(q) + e^{iqx} a(q) \right) . \tag{1}$$

The creation and annihilation operators are normalized as

$$\langle 0 | a(\vec{p}\,') a^\dagger(\vec{p}) | 0 \rangle = 2E_p \delta^{(3)}(\vec{p} - \vec{p}\,') \quad \text{with} \quad E_p = \sqrt{\vec{p}^{\,2} + m^2} \tag{2}$$

and the propagator is defined by

$$G^{(2)}_{free}(x_1, x_2) = \langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) | 0 \rangle . \tag{3}$$

The time-ordered product is defined by

$$T \hat{\phi}(x_1) \hat{\phi}(x_2) = \theta(x_1^0 - x_2^0) \hat{\phi}(x_1) \hat{\phi}(x_2) + \theta(x_2^0 - x_1^0) \hat{\phi}(x_2) \hat{\phi}(x_1) . \tag{4}$$

Use for the θ-functions the integral representation

$$\theta(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dt \frac{e^{itz}}{t - i\epsilon} \quad \text{and} \quad \theta(-z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dt \frac{e^{itz}}{t + i\epsilon} . \tag{5}$$

The propagator, as well as the θ-function, is a distribution. ϵ is a positive infinitesimal parameter and it is to be understood that after integration with a test function the limit $\epsilon \to 0$ has to be taken.

The result is

$$G^{(2)}_{free}(x_1, x_2) = \frac{i}{(2\pi)^4} \int d^4p \, e^{ip(x_1 - x_2)} \frac{1}{p^2 - m^2 + i\epsilon} . \tag{6}$$

2:
Show that the 4-point Green function in a free field theory can be decomposed in a sum of products of 2-point Green functions as

$$\begin{aligned} G^{(4)}(x_1, x_2, x_3, x_4) &= \langle 0 | T \hat{\phi}(x_1) \hat{\phi}(x_2) \hat{\phi}(x_3) \hat{\phi}(x_4) | 0 \rangle \\ &= i^2 (\Delta_F(x_1 - x_2) \Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3) \Delta_F(x_2 - x_4) \\ &\quad + \Delta_F(x_1 - x_4) \Delta_F(x_2 - x_3)) \end{aligned} \tag{7}$$

by performing contractions of the creation and annihilation operators, i.e., insert the expression for the field operators in terms of creation and annihilation operators and use (1): $a(p)|0\rangle = 0$ and (2): $\left[a(p), a^\dagger(q)\right] = 2q^0\delta(p-q)$.

Hints: Think about how to simplify the notation. Separate the calculation of the operator matrix element from the evaluation of factors related to the Fourier transforms.

3*:

Prove the equation on slide 9 of the lecture:

$$\langle\Omega|T\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|T\hat{\phi}_I(x_1)\hat{\phi}_I(x_2)U(-t,t)|0\rangle}{\langle 0|U(-t,t)|0\rangle}\,. \tag{8}$$

Hints: Let us remind ourselves of a few basic facts of quantum mechanics: Operators in the Heisenberg picture (ϕ_H, on which the definition of Green functions are based) are time-dependent. They are related to the time-independent operators in the Schrödinger picture, ϕ_S, by $\phi_H(x) = e^{iH(t-t_0)}\phi_S(\vec{x})e^{-iH(t-t_0)}$ where t_0 is some reference time. Therefore ϕ_H depends on the 4-vector coordinates including the time $t = x^0$, while ϕ_S depend only on 3-component coordinates $\vec{x}$. Operators in the interaction picture are obtained from Schrödinger operators by transforming away the time-dependence due to the free Hamiltonian ($H = H_0 + H_{\text{int}}$):

$$\phi_I(x) = e^{iH_0(t-t_0)}\phi_S(\vec{x})e^{-iH_0(t-t_0)}\,. \tag{9}$$

We define a time-evolution operator by

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{10}$$

such that it relates field operators in the Heisenberg and the interaction picture:

$$\phi_H(x) = U^\dagger(t,t_0)\phi_I(x)U(t,t_0)\,. \tag{11}$$

This operator obeys the following relations: $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ and $U^\dagger(t_1,t_2) = U(t_2,t_1)$.

In the following, consider the two cases $x_1^0 > x_2^0$ and $x_1^0 < x_2^0$ separately. For the first case, show that the Green function (left-hand side of Eq. (8) can be written as

$$\langle\Omega|U^\dagger(t,t_0)U(t,x_1^0)\phi_I(x_1)U(x_1^0,x_2^0)\phi_I(x_2)U(x_2^0,-t)U(-t,t_0)|\Omega\rangle\,. \tag{12}$$

Now use

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{1}{e^{-iE_0t}\langle\Omega|0\rangle} e^{-i\hat{H}t}|0\rangle \tag{13}$$

(see bottom of slide 4) for $t \to t + t_0$ and use the fact that the ground state of the free theory obeys $H_0|0\rangle = 0$ to show:

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{1}{e^{-iE_0(t+t_0)}\langle\Omega|0\rangle} U^\dagger(-t, t_0)|0\rangle \,. \tag{14}$$

Finally, use the fact that the vacuum state of the full theory, $|\Omega\rangle$, is normalized and show

$$\lim_{t\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0 t}|\langle\Omega|0\rangle|^2} = \lim_{t\to\infty(1-i\epsilon)} \frac{1}{\langle 0|U(t,-t)|0\rangle} \,. \tag{15}$$

It should be obvious now that the other part of the time-ordered product leads to a similar result so that one can combine the two parts to obtain

$$T\left(U(t, x_1^0)\phi_I(x_1)U(x_1^0, x_2^0)\phi_I(x_2)U(x_2^0, -t)\right) \,. \tag{16}$$

The final step is to re-order the factors in the time-ordered product.

4*:

Prove the equation on slide 12 of the lecture for the case of two field operators:

$$\langle\Omega|T\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2)\exp\left\{i\int_{-t}^{+t} d^4x\mathcal{L}(x)\right\}}{\int \mathcal{D}\phi(x)\exp\left\{i\int_{-t}^{+t} d^4x\mathcal{L}(x)\right\}} \,. \tag{17}$$

Hints: Start from

$$F = \int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2)\exp\left\{i\int_{-t}^{+t} dt d^3x \mathcal{L}(\phi(x))\right\} \,, \tag{18}$$

first for one specific choice of the time-ordering: $-t < x_1^0 < x_2^0 < t$. Assume boundary conditions at the times $-t$ and t: $\phi(-t,\vec{x}) = \phi_i(\vec{x})$ and $\phi(t,\vec{x}) = \phi_f(\vec{x})$. The functional integration over $\phi(x)$ can be performed in two steps: first integrate over functions $\phi(x)$ which obey additional boundary conditions at the intermediate times x_1^0 and x_2^0: $\phi(x_1^0,\vec{x}) = \phi_1(\vec{x})$ and $\phi(x_2^0,\vec{x}) = \phi_2(\vec{x})$ with fixed functions ϕ_1 and ϕ_2, then integrate over these functions:

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\substack{\phi(x_1^0,\vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^0,\vec{x}) = \phi_2(\vec{x})}} \mathcal{D}\phi(x) \,. \tag{19}$$

Now one can change the order of integrations and split up the integration over time contained in the exponential of Eq. (18) to obtain

$$\begin{aligned} F \;=\; & \int \mathcal{D}\phi_1(\vec{x})\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\phi_2(\vec{x}) \\ & \langle\phi_f(\vec{x};t)|\phi_2(\vec{x};x_2^0)\rangle\langle\phi_2(\vec{x};x_2^0)|\phi_1(\vec{x};x_1^0)\rangle\langle\phi_1(\vec{x};x_1^0)|\phi_i(\vec{x};-t)\rangle \,. \end{aligned} \tag{20}$$

Here we have used the relation given on slide 12:

$$\langle \phi_f(\vec{x}); t_f | \phi_i(\vec{x}); t_i \rangle = \int \mathcal{D}\phi(x) \exp\left\{ i \int_{t_i}^{t_f} dt\, d^3x \mathcal{L}(x) \right\} . \tag{21}$$

Since $|\phi_i(\vec{x}); x_i^0\rangle$ are eigen-states of the Heisenberg field operators, and using the completeness of these states, one finds

$$F = \langle \phi_f(\vec{x}); t | \phi_H(x_2) \phi_H(x_1) | \phi_i(\vec{x}); -t \rangle . \tag{22}$$

Now one can put the time-ordering back and perform the limit $t \to \infty(1 - i\epsilon)$. As shown in the previous problem, there will be a normalization factor which will disappear in the final result, Eq. (17).

5:
The Gaussian integral with the kernel (see slide 15):

$$A(x, x') = -\frac{d^2}{dx^2}\delta(x - x') + \omega^2 \delta(x - x') , \tag{23}$$

i.e.

$$\int \mathcal{D}y(x) \exp\left(-\frac{1}{2} \int dx \int dx' y(x) A(x, x') y(x') \right) \tag{24}$$

can be written as

$$\left\{ \det\left(-\frac{d^2}{dx^2} + \omega^2 \right) \right\}^{-1/2} . \tag{25}$$

Explain the meaning of this equation. In particular, what is the meaning of the determinant of an operator which contains derivatives? What are the eigenvalues and eigenfunctions of the operator A? Choose periodic boundary conditions for the eigenfunctions $y_n(x)$: $y_n(x_i) = y_n(x_f)$. Then the eigenvalues are given by $\lambda_n = \left(2\pi n/(x_i - x_f)\right)^2 + \omega^2$ where n is an integer.

6:
Once more: calculate the 4-point Green function for a free field theory from the generating functions

$$\mathcal{Z}_{free}[J] = \exp\left\{ -\frac{i}{2} \int d^4x \int d^4y\, J(x) \Delta_F(x - y) J(y) \right\} \cdot \mathcal{Z}_{free}[0] \tag{26}$$

by using

$$\langle \Omega | T \hat{\phi}_H(x_1) \dots \hat{\phi}_H(x_n) | \Omega \rangle = (-i)^n \frac{1}{\mathcal{Z}[0]} \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} \mathcal{Z}[J(x)] \Bigg|_{J=0} . \tag{27}$$

Note that the propagator $\Delta_F(x-y)$ is an even function of its argument. Compare with the result of problem 2.

7:
Now calculate the 4-point Green function for a scalar field theory with an interaction term

$$\mathcal{L}_{\text{int}} = -\frac{g}{4!}\phi^4 \tag{28}$$

to first order in the coupling constant.

Hints: Start from the generalization of Eq. (26) for an interacting theory, replacing the fields in $\mathcal{L}_{\text{int}}$ by derivatives with respect to the source $J(x)$. Treat $\Delta_F(0)$ as a constant even though it is divergent. Which Feynman diagrams can be drawn to visualize the result (three diagrams, up to trivial permutations!)?

8*:
Prove that $i\Gamma[a]$ is the generating functional of proper vertex functions by determining the inverse Legendre transform to calculate the connected Green functions (see slide 24).

9:
Prove the substitution rule for integrals over a function of Grassmann variables (slide 26),

$$\int d\eta_n \ldots d\eta_1 F(\eta) = \frac{1}{\det A}\int d\theta_k \ldots d\theta_1 F(\eta(\theta)) \tag{29}$$

where

$$\eta_i = \sum_k A_{ik}\theta_k \tag{30}$$

Hint: Write F as a (finite) power series and compare the coefficients of $\eta_1, \ldots, \eta_n$.

10:
Prove the Fubini formula (slide 27):

$$\int d\eta_1 \ldots d\eta_n \, d\eta_n^* \ldots d\eta_1^* \, \exp\left(\sum_{i,k} \eta_i^* A_{ik} \eta_k\right) = \det A \, . \tag{31}$$

Hint: As in the previous problem, expand the exponential in a finite power series of the η_i.