Trans-Carpathian Seminar on Geometry & Physics

Some results in the spectral analysis of quantum Hamiltonians with magnetic fields

Radu Purice

INSTITUTUL DE MATEMATICA "SIMION STOILOW" AL ACADEMIEI ROMANE
IMAR

March 13, 2024

These results and the covariant 'magnetic' pseudo-differential calculus used in proving them, are the fruits of more than 20 years of research in cooperation mainly with:

- Horia Cornean (Aalborg University),
- Bernard Helffer (Nantes University),
- Viorel Iftimie (Bucharest University),
- Marius Mantoiu† (IMAR and University of Chile at Santiago),
- Serge Richard (Nagoya University and former at Lyon University).

The class of Hamiltonians

We work on $\mathcal{X} \cong \mathbb{R}^d$ for some $d \geq 2$ with the *phase space* $\Xi := \mathcal{X} \times \mathcal{X}^*$. We consider *classical Hamiltonians* defined by positive, elliptic Hörmander type symbols of type $S^p_1(\mathcal{X})$ with $p > 0$, i.e.:

$h \in C^\infty_{\mathtt{pol}}(\Xi; \mathbb{R}_+)$:

- $\sup_{(x,\xi)\in\Xi} < \xi >^{-p+|\beta|} \max_{|\alpha|\leq n} \max_{|\beta|\leq m} \left|\left(\partial^\alpha_x \partial^\beta_\xi h\right)(x,\xi)\right| =: \nu_{n,m}(h) < \infty,$
- $\exists (R, C) \in \mathbb{R}^2_+,\ < \xi >^p \leq\ h(x,\xi),\ \forall (x,\xi) \in \big\{|x| \geq R\big\} \times \mathcal{X}^*\}.$

By the Weyl quantization this symbol defines
an essentially self-adjoint operator $\mathfrak{Op}(h) : C^\infty_0(\mathcal{X}) \to L^2(\mathcal{X})$,
its closure having as domain the Sobolev space of order p on $\mathcal{X}$:

$$\mathscr{H}^p(\mathcal{X}) := \big\{f \in L^2(\mathcal{X}),\, (1 - \Delta)^{p/2} f \in L^2(\mathcal{X})\big\}.$$

The magnetic field

- The magnetic field is described by a closed 2-form on $\mathcal{X}$:

$$B = \sum_{j,k=1}^{n} B_{jk}(x) dx_j \wedge dx_k,$$
$$B_{jk}(x) = -B_{kj}(x), \quad B_{jk} \in BC^\infty(\mathcal{X}), \quad dB = 0.$$

- On $\mathbb{R}^d$ there exists a 1-form, the vector potential A such that $B = dA$.
- The association of a vector potential to a magnetic field B is highly non unique and we have that

$$A - A' = \nabla\Phi \quad \Leftrightarrow \quad dA = dA' = B.$$

- The formula $A_j(x) := -\sum_{k=1}^{n} \int_0^1 ds\, B_{jk}(sx) s x_k$ provides a vector potential with components of class $C^\infty_{\text{pol}}(\mathcal{X})$:

The magnetic quantization 1

We start from the 'minimal coupling' paradigm:

Replace:

- the *'momentum variables'* $\xi \in \mathcal{X}^*$ $\qquad \Pi := -i\nabla$,
- by *'magnetic momenta'* $\xi - (e/c)A(x)$ $\qquad \Pi^A := -i\nabla - (e/c)A(Q)$.

Proposition

For any $v \in \mathcal{X}$ the operator $v \cdot \Pi^A$ defines an essentially self-adjoint operator on $C_0^\infty(\mathcal{X})$.
The 1-parameter C_0 group generated by the closure of $v \cdot \Pi^A$ is given by:

$$\mathbb{R} \ni t \mapsto U_v^A(t) := e^{-i\int_{[Q,Q+tv]} A} U(tv),$$

where $\big(U(v)f\big)(x) := f(x+v), \ \forall f \in L^2(\mathcal{X})$.

The magnetic quantization 2

We can define now *the 'magnetic' Weyl system*:

$$W^A(z,\zeta) \ := \ e^{-(i/2)<\zeta,z>}\, e^{-i<\zeta,Q>}\, U^A_z(1)$$

and the associated *'magnetic' Weyl quantization*:

$$\mathfrak{Op}^A(F) := (2\pi)^{-d} \int_{\mathcal{X}} dz \int_{\mathcal{X}^*} d\zeta \,\big(\mathcal{F}_{\Xi} F\big)(z,\zeta) W^A(z,\zeta), \quad \forall F \in \mathscr{S}'(\Xi)$$

with $\mathcal{F}_{\Xi} : \mathscr{S}'(\Xi) \xrightarrow{\sim} \mathscr{S}'(\Xi)$ is the Fourier transform bijection.
Explicitly we have the formula:

$$\big\langle \big(\mathfrak{Op}^A(F)\big)\phi\,,\,\psi\big\rangle_{\mathscr{S}(\mathcal{X})} = (2\pi)^{-d} \int_{\mathcal{X}} dx \int_{\mathcal{X}} dy\, \phi(x)\, \psi(y) \times$$

$$\times\ \exp\Big(-i \int_{[x,y]} A\Big) \int_{\mathcal{X}^*} d\zeta\, e^{i<\zeta,x-y>} F\big((x+y)/2,\zeta\big),$$

$$\forall F \in \mathscr{S}'(\Xi),\ \forall (\phi,\psi) \in \big[\mathscr{S}(\mathcal{X})\big]^2.$$

The magnetic quantization 3

We have the topological and linear isomorphisms

$$\mathfrak{Op}^A : \mathscr{S}(\Xi) \xrightarrow{\sim} \mathcal{L}\big(\mathscr{S}'(\mathcal{X}); \mathscr{S}(\mathcal{X})\big),$$
$$\mathfrak{Op}^A : \mathscr{S}'(\Xi) \xrightarrow{\sim} \mathcal{L}\big(\mathscr{S}(\mathcal{X}); \mathscr{S}'(\mathcal{X})\big).$$

Definition

The 'magnetic' Moyal product:

$$\mathfrak{Op}^A\big(F \sharp^B G\big) := \mathfrak{Op}^A(F)\,\mathfrak{Op}^A(G).$$

Notation

$[F]_B^-$ such that $F \sharp^B [F]_B^- = 1$ (when it exists).

The magnetic quantization 4

Main results concerning the 'magnetic' pseudo-differential calculus:

- The map $S^{p_1}_1(\mathcal{X}) \times S^{p_2}_1(\mathcal{X}) \ni (F, G) \mapsto F\sharp^B G \in S^{p_1+p_2}_1(\mathcal{X})$ is continuous.
- $\mathfrak{Op}^A(F)$ belongs to $\mathcal{L}(\mathscr{S}(\mathcal{X}); \mathscr{S}(\mathcal{X}))$ for any $F \in S^p_1(\mathcal{X})$.
- $\mathfrak{Op}^A(F)$ belongs to $\mathbb{B}(L^2(\mathcal{X}))$ for any $F \in S^0_1(\mathcal{X})$.
- If $F \in S^p_1(\mathcal{X})$ and is invertible for the magnetic Moyal product, then $[F]^-_B \in S^{-p}_1(\mathcal{X})$.
- If $F \in S^p_1(\mathcal{X})$ is a positive elliptic symbol, with $p > 0$, then $\mathfrak{Op}^A(F)$ is lower bounded and essentially self-adjoint on $C^\infty_0(\mathcal{X})$ and its closure has as domain *the 'magnetic' Sobolev space of order* p on $\mathcal{X}$ associated to the vector potential A:

$$\mathscr{H}^p_A(\mathcal{X}) := \{f \in L^2(\mathcal{X}) \,,\, \mathfrak{Op}^A(\mathfrak{s}_p)f \in L^2(\mathcal{X})\}, \quad \mathfrak{s}_p(x, \xi) :=< \xi >^p .$$

The minimal C^*-algebra associated to the Hamiltonian 1

Suppose given an abelian C^*-algebra $\mathcal{A}(\mathcal{X}) \subset BC_u(\mathcal{X})$ such that:

- $\mathcal{A}(\mathcal{X})$ is a unital C^*-subalgebra of $BC_u(\mathcal{X})$
- $\mathcal{A}(\mathcal{X})$ is left invariant by all the translations τ_x (with $x \in \mathcal{X}$).
- $C_\infty(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$.

We shall call it *an interaction algebra*.

Given a magnetic field B with components of class $\mathcal{A}(\mathcal{X}) \bigcap C^\infty(\mathcal{X})$ we may define the 2-cocycle:

$$\omega^B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{U}\big(\mathcal{A}(\mathcal{X})\big), \quad [\omega^B(v,w)](x) := \exp\Big(-i\int_{<x,x+v,x+v+w>} B\Big).$$

Let us endow the space $L^1\big(\mathbb{R}^d; \mathcal{A}(\mathcal{X})\big)$ with the twisted cross product:

$$\big(F \rtimes_\tau^{\omega^B} G\big)(w) := \int_{\mathbb{R}^d} dv\, \tau_{\frac{v-w}{2}}[F(v)] \cdot \tau_{\frac{v}{2}}[G(w-v)] \cdot \tau_{-\frac{w}{2}}[\omega^B(v, w-v)].$$

The minimal C^*-algebra associated to the Hamiltonian 2

Definition

Let $\mathcal{A}(\mathcal{X}) \rtimes_{\tau}^{\omega^B} \mathbb{R}^d$ be the completion of the Banach *-algebra $L^1(\mathbb{R}^d; \mathcal{A}(\mathcal{X}))$ with the twisted cross product defined above and the involution $f^{\rtimes}(v) := \overline{f(-v)}$, with respect to the maximal C^*-norm associated to covariant representations.

Proposition

Given $h \in S^p_1(\mathcal{X})$ with $p > 0$ a positive, elliptic symbol such that $h(\cdot, \xi) \in \mathcal{A}(\mathcal{X})$ for any $\xi \in \mathcal{X}^*$ and a magnetic field B with components in $\mathcal{A}(\mathcal{X}) \bigcap C^\infty(\mathcal{X})$, choosing A with components in $C^\infty_{\mathtt{pol}}(\mathcal{X})$, the resolvent $R_{\mathfrak{z}}(h) := \big(\overline{\mathfrak{Op}}^A(h) - \mathfrak{z}\mathbb{1}\big)^{-1}$ for any $\mathfrak{z} \in \mathbb{C} \setminus \sigma\big(\overline{\mathfrak{Op}}^A(h)\big)$ satisfies:

$$R_{\mathfrak{z}}(h) = \mathfrak{Op}^A(r_{\mathfrak{z}}[h]), \quad (\mathbb{1}_{\mathcal{X}} \otimes \mathcal{F}_{\mathcal{X}^*}) r_{\mathfrak{z}}[h] \in \mathcal{A}(\mathcal{X}) \rtimes_{\tau}^{\omega^B} \mathbb{R}^d.$$

We say that $\mathfrak{Op}^A(h)$ is *affiliated to* the C^*-algebra $\mathcal{A}(\mathcal{X}) \rtimes_{\tau}^{\omega^B} \mathbb{R}^d$.

Spectral regularity

Continuity of the spectra 1

Hypothesis

Suppose given:

- A map $I \ni \epsilon \mapsto h^\epsilon \in S^p_1(\mathcal{X})$ with $h^\epsilon \geq (-C)$ elliptic $\forall \epsilon \in I$ continuous for the Fréchet topology, (with some $C > 0$),
- a map $I \ni \epsilon \mapsto \{B^\epsilon_{jk}\}_{1 \leq j,k \leq d} \in \big[BC^\infty(\mathcal{X})\big]^{d(d-1)/2}$ continuous for the Fréchet topology on $BC^\infty(\mathcal{X})$.

For any $\mathfrak{z} \in \mathbb{C} \setminus \mathbb{R}$ the family of resolvents $R^\epsilon_{\mathfrak{z}} := \big(\overline{\mathfrak{Op}}^{A^\epsilon}(h^\epsilon) - \mathfrak{z}\mathbf{1}\big)^{-1}$ define an element of the C^*-algebra $C\big(I; BC_u(\mathcal{X})\big) \rtimes^{\omega^B}_{\mathrm{id} \times \tau} \mathbb{R}^d$ with $[\omega^B(u,v)](\epsilon, x) := [\omega^{B^\epsilon}(u,v)](x)$.

We can then view $C\big(I; BC_u(\mathcal{X})\big) \rtimes^{\omega^B}_{\tau} \mathbb{R}^d$ as a C^*-algebra of continuous sections of the continuous field of C^*-algebras $\Big\{BC_u(\mathcal{X}) \rtimes^{\omega^{B^\epsilon}}_{\tau} \mathbb{R}^d\Big\}_{\epsilon \in I}$.

Continuity of the spectra 2

- Abstract results on continuous fields of C^*-algebras imply the upper semi-continuity of the norm.
- Writing the norm as an infimum of meanvalues in a representation, the weak continuity of the quantization implies lower semi-continuity of the norm.
- Then, abstract results imply that for a continuous choice for $\{A^\epsilon\}_{\epsilon\in I}$ with $C^\infty_{\mathtt{pol}}(\mathcal{X})$ components, if we denote by $\sigma^\epsilon := \sigma\big(\overline{\mathfrak{Op}^{A^\epsilon}(h^\epsilon)}\big)$,

we have that:

- $\forall\epsilon_0 \in I$ and for any compact $K \subset \mathbb{R}$ with $K \cap \sigma^{\epsilon_0} = \emptyset$, $\exists V^{\epsilon_0}_K$ neighbourhood of ϵ_0 with $K \cap \sigma^\epsilon = \emptyset$, $\forall\epsilon \in V^{\epsilon_0}_K$.
- $\forall\epsilon_0 \in I$ and for any open $\mathcal{O} \subset \mathbb{R}$ such that $\mathcal{O} \cap \sigma^{\epsilon_0} \neq \emptyset$, $\exists V^{\epsilon_0}_{\mathcal{O}} \subset I$ neighbourhood of ϵ_0 with $\mathcal{O} \cap \sigma^\epsilon \neq \emptyset$, $\forall\epsilon \in V^{\epsilon_0}_{\mathcal{O}}$.

Magnetic Weyl integral kernel

As before, let:

- $h \in S^p_1(\mathcal{X})$ positive and elliptic, with $p > 0$,
- a magnetic field ϵB with components $BC^\infty(\mathcal{X})$ and $\epsilon > 0$,
- a vector potential ϵA for ϵB with components $C^\infty_{\mathtt{pol}}(\mathcal{X})$,
- $H^\epsilon := \overline{\mathfrak{Op}}^{\epsilon A}(h) : \mathscr{H}^p_{\epsilon A}(\mathcal{X}) \to L^2(\mathcal{X})$, and $R^{\epsilon A}_{\mathfrak{z}}[h] := \big(H - \mathfrak{z}\mathbb{1}\big)^{-1}$, $\forall \mathfrak{z} \in \mathbb{C} \setminus \sigma(H)$.

We consider the distribution kernel of $R^{\epsilon A}_{\mathfrak{z}}[h] =: \mathfrak{Op}^{\epsilon A}(r^{\epsilon A}_{\mathfrak{z}}[h])$:

$$\mathfrak{K}^{\epsilon A}[r^{\epsilon A}_{\mathfrak{z}} h](x,y) = (2\pi)^{-d/2} \Lambda^{\epsilon A}(x,y) \big[\big((\mathbb{1} \otimes \mathcal{F}_{\mathcal{X}^*}) r^{\epsilon A}_{\mathfrak{z}}[h] \big) \circ \Upsilon \big](x,y)$$

where: $\Upsilon(x,y) := \big((x+y)/2, x-y\big)$ and $\Lambda^{\epsilon A}(x,y) := \exp\big(- i\epsilon \int_{[x,y]} A\big)$.

Proposition

$\mathfrak{K}^{\epsilon A}[r^{\epsilon A}_{\mathfrak{z}}[h]] \in L^1(\mathcal{X} \times \mathcal{X}) \bigcap C^\infty(\mathcal{X} \times \mathcal{X} \setminus \{(x,x), x \in \mathcal{X}\})$ with rapid decay outside $\{(x,x), x \in \mathcal{X}\}$ uniformly for $\epsilon \in [0,1]$.

Regularity of the Hausdorff distance between spectra

- Let $\Gamma \subset \mathcal{X}$ be the image of $\mathbb{Z}^d \subset \mathbb{R}^d$.
- We define a locally finite partition of unity:
  $g \in C_0^\infty\big(\mathcal{X};[0,1]\big),\ g(x)=1 \text{ if } |x|\leq 1/2,\ g(x)=0 \text{ if } |x|\geq 2,,$
  $\sum_{\gamma\in\Gamma} g(x-\gamma)^2 = 1.$
- We use the scaling $\forall\gamma\in\Gamma,\ g_\gamma^\epsilon(x):=g\big(\epsilon^{1/2}x-\gamma\big)$.
- We make local changes of gauge around each $\gamma\in\Gamma$:
  $\Theta_\gamma^\epsilon(x):=\exp\left(i\int_{[x,\epsilon^{-1/2}\gamma]}A\right)$.
- We approximate the kernel $\mathfrak{K}^{\epsilon A}[r_{\mathfrak{z}}^{\epsilon A}[h]]$, by $\sum_{\gamma\in\Gamma}\Theta_\gamma^\epsilon g_\gamma^\epsilon \mathfrak{K}^0[r_{\mathfrak{z}}^0[h]]g_\gamma^\epsilon\overline{\Theta_\gamma^\epsilon}$.

Theorem

Under the above hypothesis, if $\mathrm{d_H}(M_1,M_2)$ denotes the Hausdorff distance:
$\exists C>0,\exists\delta_0>0,\exists\epsilon_0>0:\ \mathrm{d_H}\Big(\sigma\big(H^{\epsilon_0+\delta}\big),\sigma\big(H^{\epsilon_0}\big)\Big)\leq C\delta^{1/2},\ \forall\delta\in[0,\delta_0]$.

Quasi-Lipshitz regularity of spectral edges

Remark

For $f \in C_0^\infty(\mathcal{X})$ let $\tilde{f} := f * f$ and $\tilde{f}_\delta$ with $f_\delta(x) := f(\delta x)$, $\forall \delta \in (0,1]$.
We consider the kernels $F_\delta(x,y) := f_\delta(x-y)$, $\tilde{F}_\delta(x,y) := \tilde{f}_\delta(x-y)$.
Then we have the identities:

a) $\int \| F_\delta(x,y)|^2 dy = \| f\delta\|^{-2} \tilde{F}_\delta(x,x)^2 = \tilde{f}_\delta(0) = \|f_\delta\|^2$.

b) $$\begin{aligned}(e^{i\epsilon\phi}\tilde{F}_\delta)(x,x') &= \int_{\mathcal{X}} dy \left[\overline{(e^{i\epsilon\phi}F_\delta)(y,x)}\right] \left[(e^{i\epsilon\phi}F_\delta)(y,x')\right] + \\ &+ e^{i\epsilon\phi}(x,x') \int_{\mathcal{X}} dy \left[e^{i\epsilon\Phi 1(x,y,x')} - 1\right] F_\delta(x,y)\, F_\delta(y,x') + \\ &+ e^{i\epsilon\phi}(x,x') \int_{\mathcal{X}} dy \left[e^{i\epsilon\Phi 1(x,y,x')} - 1\right]^2 F_\delta(x,y)\, F_\delta(y,x').\end{aligned}$$

Theorem

Suppose we have a real function $f \in S^{-p}(\mathcal{X})$ with $p > 0$.
Then $\mathfrak{Op}^{\epsilon A}(f)$ is a bounded self-adjoint operator and $\exists C > 0$:

$$\left| \|\mathfrak{Op}^{\epsilon A}(f)\|_{\mathbb{B}(L^2(\mathcal{X}))} - \|\mathfrak{Op}^0(f)\|_{\mathbb{B}(L^2(\mathcal{X}))} \right| \leq C|\epsilon| \ln(1/|\epsilon|), \quad |\epsilon| \leq 1/2.$$

Regularity and decay of eigenfunctions

Eigenfunction decay

Theorem

Let us suppose that

- $h \in S^p_1(\mathbb{R}^d)$ (with $p > 0$) is positive and elliptic;
- the magnetic field B has components of class $BC^\infty(\mathcal{X})$ and A has been choosen with components of class $C^\infty_{\mathtt{pol}}(\mathcal{X})$;
- $\lambda \in \sigma_{\text{disc}}(\mathfrak{Op}^A(h))$ and $u \in \mathrm{Ker}\big(\mathfrak{Op}^A(h) - \lambda\big)$.

Then $u \in \mathcal{S}(\mathcal{X})$.

If $a := \widetilde{a}\big|_{\mathbb{R}^d \times \mathbb{R}^d}$ for some $\widetilde{a} : \{\zeta \in \mathbb{C}^d \,,\, |\Im \zeta_j| \leq \delta\} \to \mathbb{C}$ analytic, $\delta > 0$,

and $(x, \xi) \mapsto \widetilde{(}x, \xi + i\zeta)$ of class $S^p_1(\mathcal{X})$ uniformly for $|\zeta_j| \leq \delta$,

then $\exists \epsilon_0 > 0$ such that $e^{\epsilon <\cdot>} u \in \mathscr{S}(\mathcal{X})$.

Structure of the essential spectrum

The asymptotic structure 1

Suppose given an interaction algebra $\mathcal{A}(\mathcal{X})$ as above.

The spectrum of $\mathcal{A}(\mathcal{X})$: By Gelfand theory we have that

- $\mathcal{A}(\mathcal{X}) \cong C(\mathfrak{S}_{\mathcal{A}})$, with $\mathfrak{S}_{\mathcal{A}}$ a compactification of $\mathcal{X}$.
- we can therefore identify $\mathcal{X}$ with a dense open subset of $\mathfrak{S}_{\mathcal{A}}$.
- The group law $\theta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ extends to a continuous map:

$$\tilde{\theta} : \mathcal{X} \times \mathfrak{S}_{\mathcal{A}} \to \mathfrak{S}_{\mathcal{A}};$$

- the complement $\mathfrak{F}_{\mathcal{A}}$ of $\mathcal{X}$ in $\mathfrak{S}_{\mathcal{A}}$ is closed and invariant under $\tilde{\theta}$;
- For any $\mathfrak{z} \in \mathfrak{F}_{\mathcal{A}}$, let $\mathfrak{K}_{\mathfrak{z}} := \overline{\{\tilde{\theta}(x, \mathfrak{z}), x \in \mathcal{X}\}}$ be *the quasi-orbit of* $\mathfrak{z}$.
- We have a canonical surjection: $\mathfrak{P}_{\mathcal{A}} : \mathcal{A}(\mathcal{X}) \ni F \mapsto F\big|_{\mathfrak{F}_{\mathcal{A}}} \in C\big(\mathfrak{F}_{\mathcal{A}}\big)$;
- For $\mathfrak{K}$ a quasi-orbit let $\mathcal{A}^{\mathfrak{K}} := \{f \in \mathcal{A}(\mathcal{X}) \mid f|_{\mathfrak{K}} = 0\}$.

The Asymptotic structure 2

Local sections for $\mathfrak{P}_{\mathcal{A}}$

For $\mathfrak{K}$ a quasi-orbit let $\mathfrak{z} \in \mathfrak{F}_{\mathcal{A}}$ be an element generating it.
For any $f \in C(\mathfrak{K})$ and for $x \in \mathcal{X}$, set

$$\mathfrak{J}_{\mathfrak{z}}[f](x) := f[\tilde{\theta}(x, \mathfrak{z})].$$

One has thus obtained an embedding homomorphism:

$$\mathfrak{J}_{\mathfrak{z}} : C(\mathfrak{K}) \to \mathcal{A}(\mathcal{X}) \subset BC_u(\mathcal{X}).$$

Suppose B has components $B_{jk} \in \mathcal{A}(\mathcal{X}) \cap BC^{\infty}(\mathcal{X})$.

Given $\mathfrak{K}$ a quasi-orbit at infinity and $\mathfrak{z} \in \mathfrak{K}$ an element generating it, we define the associated

asymptotic magnetic field

$$B_{jk}^{(\mathfrak{K})} := \mathfrak{J}_{\mathfrak{z}}[\mathfrak{P}_{\mathcal{A}}(B_{jk})\big|_{\mathfrak{K}}] \in \mathcal{A} \subset BC_u(\mathcal{X}).$$

The Asymptotic structure 3

Let $\mathscr{A}(\mathcal{X}) \subset BC_u(\mathcal{X})$ be an interaction algebra.
Suppose $\{\mathfrak{K}_\kappa\}_{\kappa \in \mathscr{K}}$ gives a covering of $\mathfrak{F}_{\mathscr{A}}$ with quasi-orbits and let us choose for each $\kappa \in \mathscr{K}$ an element $\mathfrak{z}_\kappa \in \mathfrak{K}_\kappa$ generating it.

- Then $\bigcap\limits_{\kappa \in \mathscr{K}} \mathscr{A}(\mathcal{X})^{\mathfrak{K}_\kappa} = C_\infty(\mathcal{X})$

Suppose B has components $B_{jk} \in \mathscr{A}(\mathcal{X}) \cap BC^\infty(\mathcal{X})$
and let $B^\kappa =: dA^\kappa$ be the asymptotic magnetic field associated to $\mathfrak{K}_\kappa$.

- Then for any $\kappa \in \mathscr{K}$ we can define the corresponding *asymptotic twisted cross-product algebra* $C(\mathfrak{K}_\kappa) \rtimes_\tau^{\omega^{B^\kappa}} \mathcal{X}$.

Proposition

Given the above structure, there exists a canonical injective morphism (thus isometric):

$$\prod_{\kappa \in \mathscr{K}} \pi_\kappa : \left(\mathscr{A}(\mathcal{X}) \rtimes_\tau^{\omega^B} \mathcal{X} / C_\infty(\mathcal{X}) \rtimes_\tau^{\omega^B} \mathcal{X}\right) \hookrightarrow \prod_{\kappa \in \mathscr{K}} \mathfrak{I}_{\mathfrak{z}}[C(\kappa)] \rtimes_\tau^{\omega^{B^\kappa}} \mathcal{X}$$

The structure of the essential spectrum

Proposition

Given a magnetic field $B = dA$ with components of class $BC^\infty(\mathcal{X})$ we have the following C^*-algebra isomorphism:

$$\left\{\mathfrak{Op}^A(h)\,,\ h \in\ C_\infty(\mathcal{X}) \rtimes_\tau^{\omega^B} \mathbb{R}^d\right\} \cong \mathbb{B}_\infty\big(L^2(\mathcal{X})\big),$$

(*the ideal of compact operators*).

Suppose given a positive, elliptic symbol $h \in S^p_1(\mathcal{X})$ with $p > 0$ and a magnetic field B with components of class $BC^\infty(\mathcal{X}) \bigcap \mathcal{A}$, with a vector potential A of class $C^\infty_{\mathtt{pol}}(\mathcal{X})$

and suppose that h is affiliated to the C^*-algebra $\mathcal{A} \rtimes_\tau^{\omega^B} \mathbb{R}^d$.

Suppose that $\mathcal{A}$ has the above asymptotic structure.

Theorem

$$\sigma_{\mathsf{ess}}\big(\mathfrak{Op}^A(h)\big) \;=\; \overline{\bigcup_{\kappa\in\mathcal{K}} \sigma\big(\mathfrak{Op}^{A^\kappa}(\pi_\kappa h)\big)}.$$

2-dimensional periodic Schrödinger operators in magnetic fields.

The Problem:

Suppose we consider a periodic Schrödinger operator with magnetic field.

How does the spectrum depend on the intensity of the magnetic field?

The periodic Hamiltonian:

The real affine space: $\mathcal{X} \cong \mathbb{R}^d; d \geq 2$
The lattice: $\Gamma \cong \mathbb{Z}^d \subset \mathcal{X}$

The potential: $V \in BC^\infty(\mathcal{X}; \mathbb{R})$
$V(\cdot + \gamma) = V(\cdot) \, \forall \gamma \in \Gamma$

$$\boxed{H_\Gamma := -\Delta + V(\cdot).} \qquad \Delta := \sum_{1 \leq j \leq d} \partial_j^2.$$

The magnetic field:
$B \in BC^\infty(\mathcal{X}; \bigwedge^2 \mathcal{X}^*)$
$B = dA, \; A \in C^\infty_{\mathtt{pol}}(\mathcal{X}; \mathcal{X}^*).$

The magnetic Hamiltonian:

$$\boxed{H_\Gamma^A := \sum_{1 \leq j \leq d} \big(-i\partial_j - A_j(\cdot) \big)^2 + V(\cdot).}$$

The Bloch-Floquet representation.

$$H_\Gamma = \mathfrak{U}_\Gamma^{-1}\left(\int_{\mathbb{T}_*}^{\oplus} \widehat{H}_\Gamma(\theta)\,d\theta\right)\mathfrak{U}_\Gamma$$ where:

- $\mathbb{T}_* := \mathcal{X}^* / \Gamma_*;\ \Gamma_* := \left\{\gamma^* \in \mathcal{X}^*\,,\, <\gamma^*, \gamma> \in 2\pi\mathbb{Z}\,\forall\gamma \in \Gamma\right\}$,
- $\mathscr{F}_\theta := \left\{f \in L^2_{\mathtt{loc}}(\mathcal{X})\,,\, f(\cdot + \gamma) = e^{i<\theta,\gamma^*>} f(\cdot),\ \forall\gamma \in \Gamma\right\}$,
  $\|f\|_\theta^2 := \int_{\{x: x_j \in [-1/2, 1/2), 1 \leq j \leq d]\}} dx\,|f(x)|^2$.
- $\mathfrak{U}_\Gamma : L^2(\mathcal{X}) \overset{\sim}{\to} \int_{\mathbb{T}_*}^{\oplus} \mathscr{F}_\theta\,d\theta : \left(\mathfrak{U}_\Gamma f\right)_\theta(x) := \sum_{\gamma \in \Gamma} e^{-i<\theta,\gamma>} f(x + \gamma)$.

$$\sigma\left(\widehat{H}_\Gamma(\theta)\right) = \{\lambda_n\}_{n \in \mathbb{N}}$$ where:

- $-\infty < \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} < \infty,\ \lim_{n \nearrow \infty} \lambda_n = +\infty$,
- $\forall n \in \mathbb{N}:\ \mathbb{T}_* \ni \theta \mapsto \lambda_n(\theta) \in \mathbb{R}$, is continuous

  and smooth on open sets with constant multiplicity.

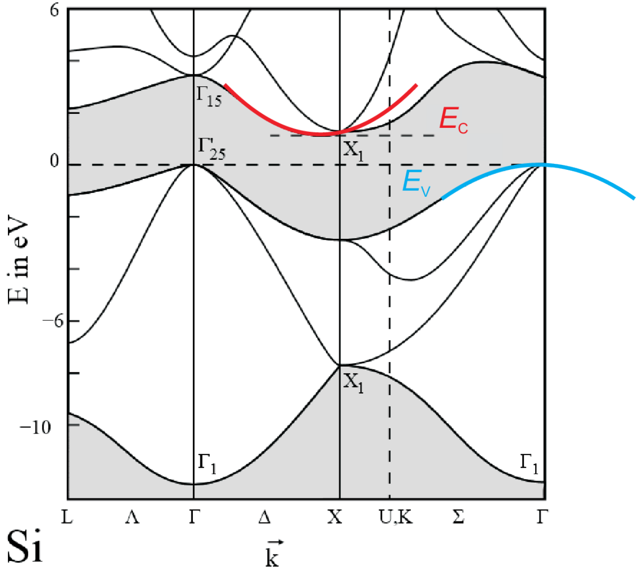
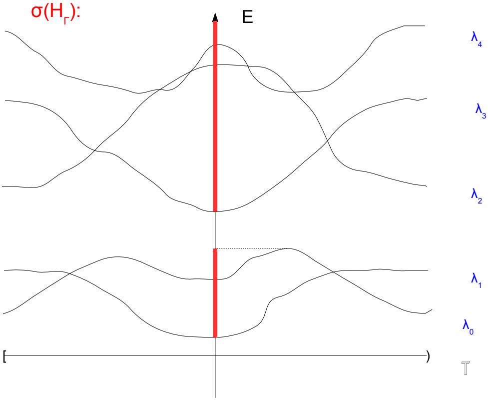


image provenant de:

https://www.tf.uni-kiel.de/matwis/amat/td_kin_ii/index.html

The spectrum of H_Γ.

- spectral islands with absolutely continuous spectrum.
- possible spectral gaps

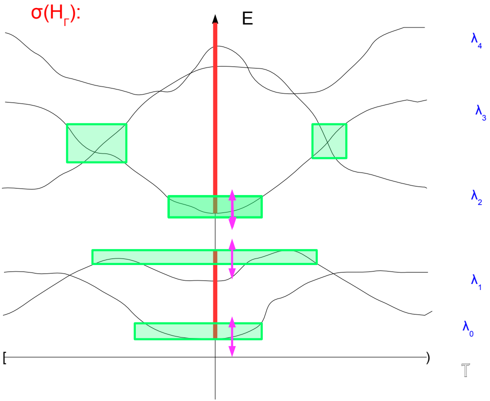


Spectral changes induced by the magnetic field.

Questions:

A. variation of the spectral islands.

B. apparition of spectral gaps.



Giving precise mathematical answers to these two questions proved to be a rather difficult matter!

Let us begin by mentioning some basic references that in our opinion laid the basis for the study of the above two questions:

- Elliott, G.: *Gaps in the spectrum of an almost periodic Schrodinger operator*. C.R. Math. Rep. Acad. Sci. Canada 4, 255-259 **(1982)**.
- Avron, J.E., Simon, B.: *Stability of gaps for periodic potentials under variation of a magnetic field*. J. Phys. A: Math. Gen. 18, 2199-2205 **(1985)**.
- Nenciu, G.: *Stability of energy gaps under variation of the magnetic field*. Lett. Math. Phys. 11, 127-132 **(1986)**.
- Bellissard, J.: *C*-algebras in solid state Physics. 2D electrons in a uniform magnetic field* (July **1987**). In: "Operator algebras and applications," Vol. II. Cambridge: University Press, 1988.
- Helffer, B., Sjötrand, J.: *Analyse semi-classique pour l'équation de Harper. I - III.* (SEDP, MSMF) **1986** - **1990**.

In these papers several approaches to the above problems have been initiated and the spectral continuity with respect to the intensity of the magnetic field has been proven.

More recently, the papers:

- Nenciu, G.: *On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory*. J. Math. Phys. 43 (3), 1273– 1298 **(2002)**.
- Nenciu, G.: *On the smoothness of gap boundaries for generalized Harper operators*. In "Advances in operator algebras and mathematical physics", Theta Ser. Adv. Math. 5, 173-182, Theta, Bucharest, 2005. arXiv:math-ph/**03**09009v2

brought some important extensions to the existing results concerning questions A. and B.

by putting into evidence the importance of gauge invariance

and a factorization of the integral kernel of the resolvent of the magnetic Schrödinger Hamiltonian isolating the 'bad behaviour' of the magnetic perturbation in an oscillating phase factor.

Starting from the results of George Nenciu in the above two papers and noticing the strong connection between his integral kernel factorization and the magnetic Weyl calculus developed in our papers with Viorel Iftimie and Marius Măntoiu, in:

- Cornean, H. D.; Purice, R.: *On the Regularity of the Hausdorff Distance Between Spectra of Perturbed Magnetic Hamiltonians*. Spectral Analysis of Quantum Hamiltonians, in Operator Theory: Advances and Applications Volume 224, **2012**, pp 55-66, Springer Basel.
- Cornean, H. D.; Purice, R.: *Spectral edge regularity of magnetic Hamiltonians*, Journal of the London Mathematical Society, **(2015)**, 16 p.

we have obtained some rather general results concerning question A. above for a large class of periodic pseudo-differential operators.

We studied the problem B. in:

- Horia D. Cornean, Bernard Helffer and Radu Purice: *Spectral analysis near a Dirac type crossing in a weak non-constant magnetic field*. Transactions of the American Mathematical Society 374 (10), (2021), p. 7041–7104.
- Horia Cornean, Bernard Helffer, Radu Purice: *Peierls' substitution for low lying spectral energy windows*, Journal of Spectral Theory 9 (4)(2019) pag. 1179 – 1222.
- Horia D. Cornean, Bernard Helffer, Radu Purice: Low lying spectral gaps in- duced by slowly varying magnetic fields, Journal of Functional Analysis 273, (1), (2017), pp. 206–282

We consider the following two situations.

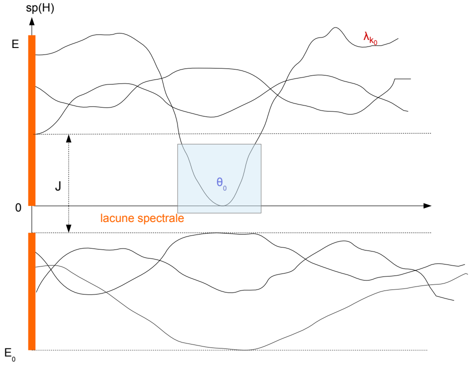
Situation (a)

- It exists $k_0 \in \mathbb{N}$ and $\theta_0 \in \mathbb{S}^2$ such that:

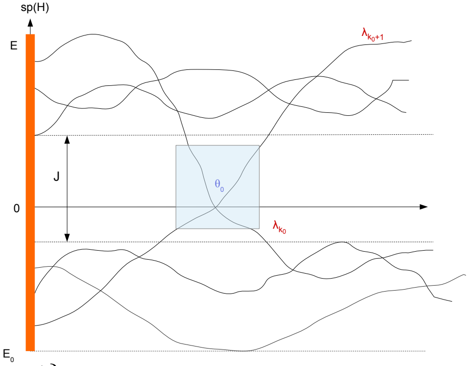
$$\lambda_{k_0}(\theta_0) = 0 \text{ is a global minimum for } \lambda_{k_0}; \mathbb{S}^2 \to \mathbb{R}$$

- it exists $\mathrm{d}_0 > 0$ such that:

$$\sup_{\theta \in \mathbb{S}^2} \lambda_{k_0 - 1}(\theta) < -\mathrm{d}_0.$$



Situation (b)



- $\exists J := [-\Lambda_-, \Lambda_+] \subset \mathbb{R}$, $0 \in \mathring{J}$,
- $\exists (k_0, \theta_0) \in \mathbb{N} \times \mathbb{S}^2$,
- $\exists \Sigma_J \ni \theta_0$ contractible compact neighbourhood, such that:
  - $J \cap \lambda_k(\mathbb{S}^2) \neq \emptyset$ iff $k \in \{k_0, k_0 + 1\}$;
  - $[-\Lambda_-, 0] = \Sigma_J \cap \lambda_{k_0}(\mathbb{S}^2)$, $[0, \Lambda_+] = \Sigma_J \cap \lambda_{k_0+1}(\mathbb{S}^2)$;
  - $\lambda_{k_0}(\theta) = \lambda_{k_0+1}(\theta)$ iff $\theta = \theta_0$;
  - the map $\mathbb{S}^2 \ni \theta \mapsto D_0 := \lambda_{k_0}(\theta)\lambda_{k_0+1}(\theta)$ vanishes with non-zero derivatives at $\theta_0 \in \mathbb{S}^2$.

Thank you for your attention !