1001 Tricks with Character(s)

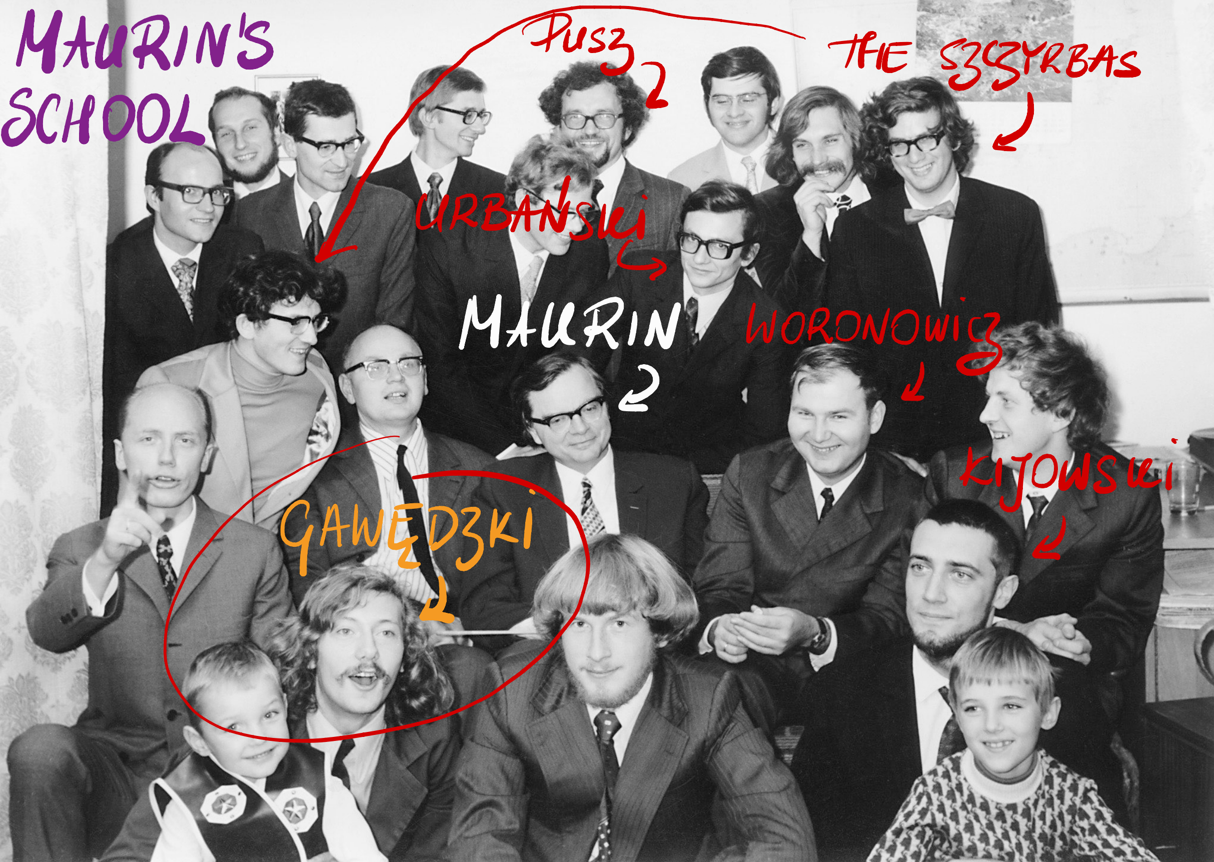
– A Case for Simplicial Mechanics sliced by Categorified Correspondences

Rafał R. Suszek

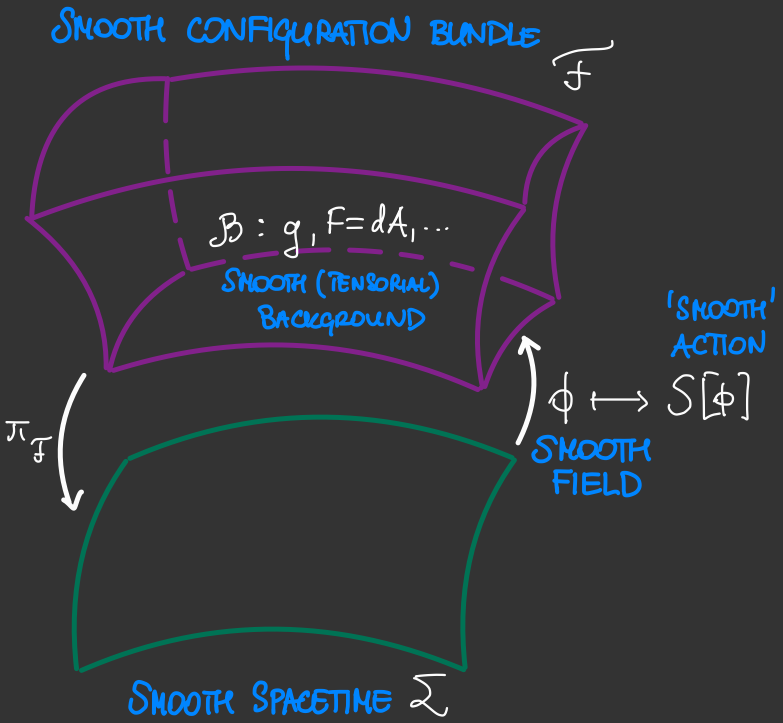
$$\begin{tikzcd} 0 \arrow[r, bend left=50, "M", ""{name=U, below}] \arrow[r, bend right=50, "F"', ""{name=D}] & 1 \arrow[Rightarrow, from=U, to=D, "K"', "M"] \end{tikzcd}$$

The Trans-Carpathian Seminar on Geometry & Physics

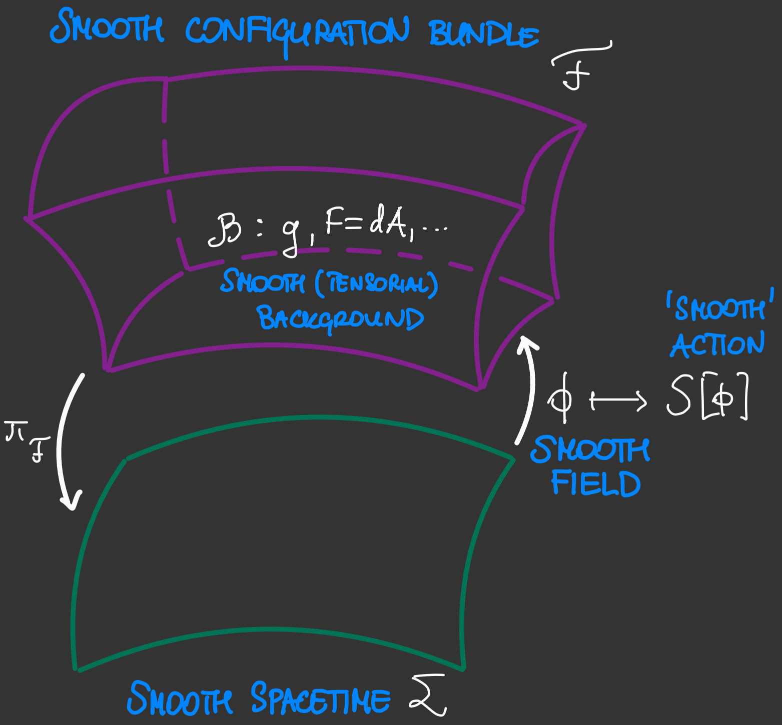
17/4/2024

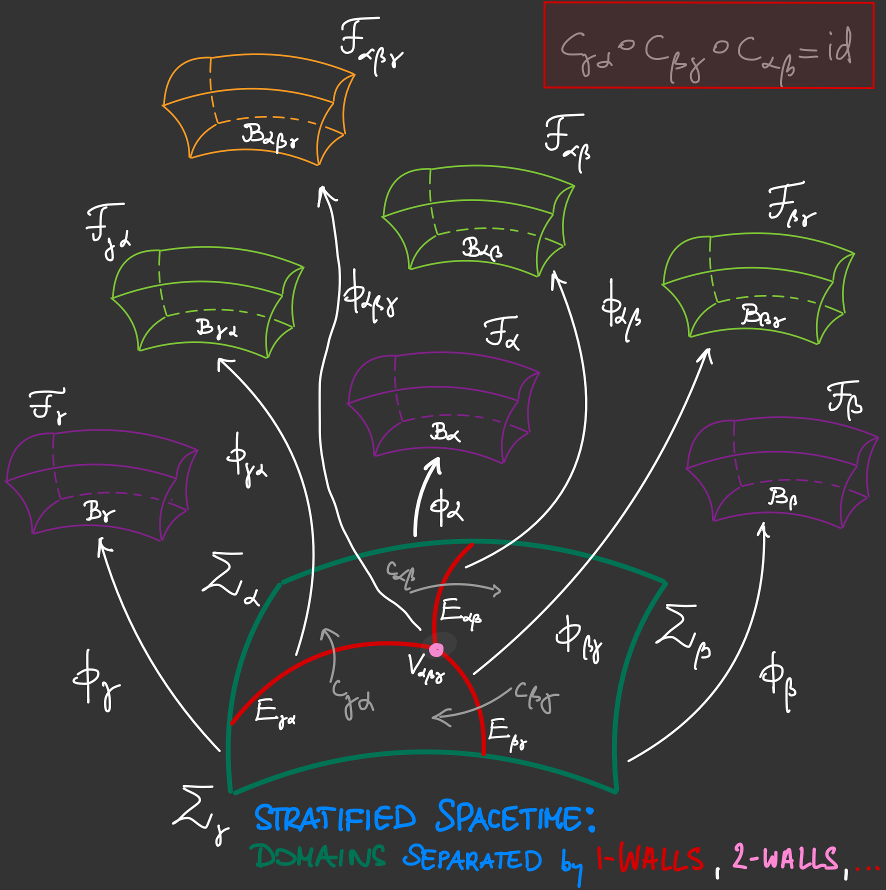

MAURIN'S SCHOOL
PUSZ
THE SZCZYRBAS
URBAŃSKI
MAURIN
WORONOWICZ
KIJOWSKI
GAWĘDZKI

THE SMOOTH PARADIGM



THE SMOOTH PARADIGM vs SIMPLICIAL DESCENT via GLUING @ DOMAIN WALLS





I THE SMOOTH PARADIGM in LFT, & THE PROSE of IT

THE PARADIGM:

iDOFs

$$\begin{tikzcd} F \arrow[r, rightsquigarrow] & \mathcal{F} \arrow[d, two heads, "\pi_{\mathcal{F}}"'] \\ & (\Sigma, \gamma) \arrow[u, bend right] \end{tikzcd}$$

CONFIGURATION BUNDLE

$f \in \Gamma(\mathcal{F}) \subset [\Sigma, \mathcal{F}]$

FIELDS

SPACETIME

$$S : \Gamma(\mathcal{F}) \longrightarrow \mathbb{R}$$

$$\llcorner = \int_{\Sigma} \mathcal{L} \circ T$$

LAGRANGEAN ACTION FUNCTIONAL

[MOLIÈRE]

I THE SMOOTH PARADIGM in LFT, & THE PROSE of IT

THE PARADIGM :

iDOFs

$$\begin{tikzcd}
F \arrow[r, squiggly] & \mathcal{F} \arrow[d, "\pi_{\mathcal{F}}"', shift right] \\
 & (\Sigma, \gamma) \arrow[u, bend right, "f"']
\end{tikzcd}$$

CONFIGURATION BUNDLE

$f \in \Gamma(\mathcal{F}) \subset [\Sigma, \mathcal{F}]$
FIELDS

(Σ, γ) SPACETIME

$S : \Gamma(\mathcal{F}) \longrightarrow \mathbb{R}$

$L = \int_\Sigma \mathcal{L} \circ T$ LAGRANGEAN ACTION FUNCTIONAL

[MOLIÈRE]

GLOBAL SYMMETRIES
(GROUP-LIKE)

$$\begin{tikzcd}
G \times \mathcal{F} \arrow[r, "\lambda"] \arrow[d, "pr_2"'] & \mathcal{F} \arrow[d, "\pi_{\mathcal{F}}"] \\
\mathcal{F} \arrow[r, "\pi_{\mathcal{F}}"'] & \Sigma
\end{tikzcd}$$

G - GROUP OBJECT
in Set, Top, Man...

$$\forall h \in G : \boxed{\Gamma \lambda_h^* S = S}$$

$$S[\lambda_h \circ \phi] = S[\phi]$$

TYPICALLY : $\mathcal{F} = \Sigma \times F$

$\Gamma(\mathcal{F}) \equiv [\Sigma, F]$

THE PROSE of IT

(i) NON-LINEAR REALISATIONS of SYMMETRY

REDUCTIVITY

HAVE $(\Sigma, \mathcal{F}, S, G)$ & $f_0 \in \Gamma(\mathcal{F})$, with $G_{f_0} = H$, ASSUME $\overset{\text{Lie}\,G}{\overset{\shortparallel}{\mathfrak{g}}} = \underset{\text{Lie}\,H}{\underset{\shortparallel}{\mathfrak{h}}} \oplus \mathfrak{t} : [\mathfrak{h}, \mathfrak{t}]_{\mathfrak{g}} \subset \mathfrak{t}$

WANT : MODEL of MANIFESTLY G-INVARIANT DYNAMICS 'around' ϕ_0, with ELEMENTARY CONFIGURATION FIBRE ~ G/H

THE PROSE of IT

(i) NON-LINEAR REALISATIONS of SYMMETRY

HAVE $(\Sigma, \mathcal{F}, S, G)$ & $\phi_0 \in \Gamma(\mathcal{F})$, with $G_{\phi_0} = H$, ASSUME $\overset{\text{Lie}\,G}{\overset{\shortparallel}{\mathfrak{g}}} = \underset{\text{Lie}\,H}{\underset{\shortparallel}{\mathfrak{h}}} \oplus \mathfrak{t} : [\mathfrak{h}, \mathfrak{t}]_{\mathfrak{g}} \subset \mathfrak{t}$ REDUCTIVITY

WANT: MODEL of MANIFESTLY G-INVARIANT DYNAMICS 'around' ϕ_0, with ELEMENTARY CONFIGURATION FIBRE $\sim G/H$

H-BASIC G-INVARIANTS on G : $\tau \circ \left(P^{\mathfrak{h}}_{\mathfrak{t}} \circ \theta_L\right)^{\otimes n}$ with $\tau \in \left(\mathfrak{g}^{* \otimes n}\right)^{H\text{-inv}}$

$\theta_L = \delta_L \log \mathrm{id}_G$

$\Sigma \times G/H$

$$\Big\Downarrow \text{ PULLBACK ALONG } \phi_\alpha$$

PIECEWISE SMOOTH FIELDS into $\bigsqcup_{i \in I} \sigma_i(\mathcal{O}_i) =: F$

$$\begin{tikzcd}
& G \leftsquigarrow H \arrow[d, two heads, "\tilde{\pi}"] \\
\mathcal{O}_{i(\alpha)} \arrow[r, hook] \arrow[ur, "\sigma_{i(\alpha)}"] & G/H \\
\Sigma_\alpha \arrow[u] \arrow[r, hook] \arrow[uur, bend left=60, "\phi_\alpha"] & \Sigma \arrow[u, "\phi"']
\end{tikzcd}$$

OBSERVATIONS : (*) CONTRIBUTIONS from the Σ_α GLUE SMOOTHLY

(**) RESULT INDEPENDENT of CHOICE σ_i

UPSHOT : WELL-DEFINED LFT for $[\Sigma, G/H]$ with GLOBAL SYMMETRY G, REALISED 'NON - LINEARLY' :

$$G \times F \longrightarrow F$$

$$(h, \sigma_i(x)) \longmapsto \sigma_{j(x,h)}\left(\tilde{\pi}(h \cdot \sigma_i(x))\right) \cdot \overset{H}{\overset{\downarrow}{h_{ji}}}(x,h)^{-1}$$

COMPENSATING GAUGE TRAFO

(ii) GAUGING via DEFECTS

IDEA : REPLACE $\mathcal{F} \circlearrowleft_{G}$ with

$$\begin{tikzcd}
F \arrow[r, squiggly] & \tilde{\mathcal{F}} \arrow[r, phantom, "\circlearrowleft"] \arrow[d, two heads] & \tilde{G} \arrow[dl, two heads] & G \arrow[l, squiggly] \\
 & \Sigma & &
\end{tikzcd}$$

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$$\begin{tikzcd}
F \arrow[r, squiggly] & \widetilde{\mathcal{F}} \arrow[d, two heads] & \circlearrowleft & \widetilde{G} \arrow[dll, two heads] & G \arrow[l, squiggly] \\
 & \Sigma & & &
\end{tikzcd}$$

TOOL: CARTAN(-BOREL) MIXING DIAGRAM:

$$\begin{tikzcd}
EG \arrow[d, two heads, "\pi_{EG}"'] \arrow[dr, squiggly, two heads] & EG \times F \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, two heads] & F \arrow[d, dashed, two heads, "\tilde{\pi}"] \arrow[dl, squiggly, two heads] \\
BG & EG \times_{\underline{\lambda}} F \arrow[l, "\pi_1"] \arrow[r, dashed, "\pi_2"'] & F /\!/ G
\end{tikzcd}$$

ORBISPACE

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BG & EG \times_{\Delta} F \arrow[l, "\pi_1"] \arrow[r, dashed, "\pi_2"'] & F/\!/G
\end{tikzcd}$$

ORBISPACE

$$\begin{tikzcd}
\Sigma \arrow[r, bend left, "\underline{\Phi}"] & BG
\end{tikzcd}$$

$$\begin{tikzcd}
P_G \arrow[d] & P_G \times F \arrow[l] \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & P_G \times G \arrow[d] \\
\Sigma & P_G \times_{\Delta} F \arrow[l, "\pi_{\tilde{\mathcal{F}}}"] \arrow[r, phantom, "\circlearrowleft"] \arrow[d, phantom, "\equiv"] & P_G \times_{Ad} G \equiv \tilde{G} \\
& \tilde{\mathcal{F}} &
\end{tikzcd}$$

ADJOINT BUNDLE $\mathrm{Ad}\, P_G \equiv P_G \times_{Ad} G$

ASSOCIATED BUNDLE

$$P_G \times_{\Delta} F \equiv (P_G \times F)/G$$

INTERPRETATION:

$$\begin{tikzcd}
\Gamma(P_G \times_{\underline{\lambda}} F) & \text{FIELDS} & & \\
 & \Gamma(\mathrm{Ad}\, P_G) \arrow[ul, bend left] & \text{GAUGE GROUP} &
\end{tikzcd}$$

UNWIELDY!

INTERPRETATION: $\Gamma(P_G \times_{\underline{\lambda}} F)$ FIELDS

$\circlearrowright \Gamma(\mathrm{Ad} P_G)$ GAUGE GROUP

UNWIELDY!

$\wr$

CONVENIENT REPACKAGING: $\mathrm{Hom}_G(P_G, F) \ni \phi$

$\circlearrowright \mathrm{Hom}_G(P_G, G) \ni \gamma$

YIELDS

LOCAL GAUGES

$\forall \sigma_\alpha \in \Gamma(P_G|_{\Sigma_\alpha})$:

$$\phi_\alpha = \sigma_\alpha^* \phi : \Sigma_\alpha \to F$$

$\underline{\lambda} \circ \circlearrowright$

$$\gamma_\alpha = \sigma_\alpha^* \gamma : \Sigma_\alpha \to G$$

LOCAL PRESENTATION

with GLUING RULES: @ $x \in \Sigma_{\alpha\beta}$

$$\phi_\alpha(x) = \underline{\lambda}_{g_{\alpha\beta}(x)}\left(\phi_\beta(x)\right)$$

$$\gamma_\alpha(x) = \mathrm{Ad}_{g_{\alpha\beta}(x)}\left(\gamma_\beta(x)\right)$$

INTERPRETATION : $\Gamma(P_G \times_{\underline{\lambda}} F)$ FIELDS

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UPSHOT : MAY SUBSTITUTE $f \rightsquigarrow \{\phi_\alpha\}_{\alpha \in A}$

IN THE ORIGINAL LFT, & CORRECT THE LATTER MINIMALLY ...

FOR DYNAMICAL FIELDS, NEED: EHRESMANN CONNECTION on $P_G \times_{\underline{\lambda}} F$

(INDUCED from PRINCIPAL ONE on P_G)

FOR $Tf \rightsquigarrow \nabla f \underset{\text{LOC.}}{\rightsquigarrow} D_{A_\alpha}\phi_\alpha = T\phi_\alpha + \underbrace{T\underline{\lambda}(t_a, 0_{T\phi_\alpha})}_{:\, K_a(\phi_\alpha)} A^a_\alpha$,

$$A_\alpha = \sigma_\alpha^* A$$
$$\equiv$$
$$A^a_\alpha \otimes t_a \in \Omega^1(\Sigma_\alpha) \otimes_{\mathbb{R}} \mathfrak{g}$$

FUNDAMENTAL VECTOR FIELDS for $\underline{\lambda}$

IF $(\Sigma, \mathcal{F}, S, G)$ IS TENSORIAL, THEN THIS IS IT...

FOR <u>DYNAMICAL</u> FIELDS, NEED: EHRESMANN CONNECTION on $P_G \times_{\underline{\lambda}} F$

(INDUCED from PRINCIPAL ONE on P_G)

FOR $Tf \rightsquigarrow \nabla f \underset{\text{LOC.}}{\rightsquigarrow} D_{A_\alpha} \phi_\alpha = \left(T\phi_\alpha + T\underline{\lambda}(t_a, 0_{T\phi_\alpha}) A^a_\alpha \right)$, $\quad A_\alpha = \sigma_\alpha^* A$

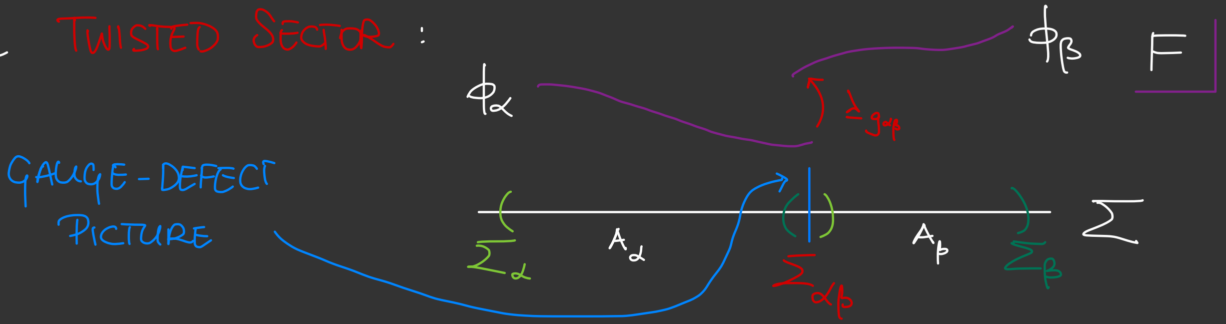
$\underline{} : \mathcal{K}_a(\phi_\alpha) \quad$ FUNDAMENTAL VECTOR FIELDS for $\underline{\lambda}$

$$A_\alpha \equiv A^a_\alpha \otimes t_a \in \Omega^1(\Sigma_\alpha) \otimes_{\mathbb{R}} \mathfrak{g}$$

IF $(\Sigma, \mathcal{F}, S, G)$ IS TENSORIAL, THEN THIS IS IT...

<u>UPSHOT</u>: MODEL of LFT with iDOFs $F /\!\!/ G$ USING FIELD IDENTIFICATIONS

& TWISTED SECTOR :



FOR DYNAMICAL FIELDS, NEED: EHRESMANN CONNECTION on $P_G \times_{\underline{\lambda}} F$

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$=: K_a(\phi_\alpha)$ $\equiv A^a_\alpha \otimes t_a \in \Omega^1(\Sigma_\alpha) \otimes_{\mathbb{R}} \mathfrak{g}$

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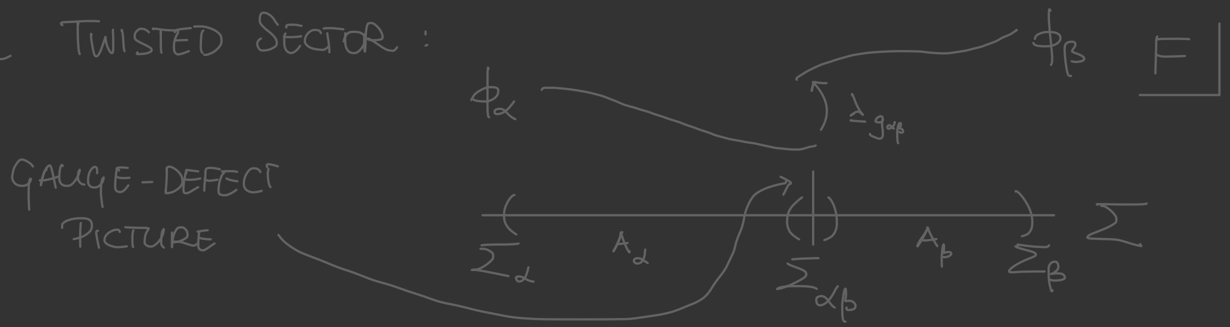

OBSERVATIONS: (*) DECOMPOSITION of Σ into DOMAINS Σ_α, DOMAIN-WISE SMOOTH FIELDS ϕ_α

(**) NEED EXTRA DATA: LOCALLY SMOOTH A_α

(A PINCH of) ABSTRACTION: THE LOCALLY SMOOTH 'BACKGROUND' IS...

ČECH NERVE of $\{\Sigma_\alpha\}_{\alpha \in A}$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4.5, "d^{(3)}_{\cdot} = pr_{\cdot,\cdot,\cdot}"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & \displaystyle\coprod_{\alpha,\beta,\gamma \in A} \Sigma_{\alpha\beta\gamma} \arrow[r, shift left=3, "d^{(2)}_{\cdot} = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=3] & \displaystyle\coprod_{\alpha,\beta \in A} \Sigma_{\alpha\beta} \arrow[r, shift left=2, "d^{(1)}_0 = pr_2"] \arrow[r, shift right=2, "d^{(1)}_1 = pr_1"'] & \displaystyle\coprod_{\alpha \in A} \Sigma_\alpha \arrow[d, two heads] \\
 & & & \Sigma
\end{tikzcd}$$

$$\Sigma_{\alpha_1 \alpha_2 \dots \alpha_n} \equiv \Sigma_{\alpha_1} \cap \Sigma_{\alpha_2} \cap \dots \cap \Sigma_{\alpha_n}$$

(A PINCH of) ABSTRACTION : THE LOCALLY SMOOTH 'BACKGROUND' IS...

SIMPLICIAL 1-FORM on ČECH NERVE of $\{\Sigma_\alpha\}_{\alpha\in A}$

$$\boxed{\breve{g} = \bigsqcup_{\alpha,\beta\in A} g_{\alpha\beta} \qquad \breve{A} = \bigsqcup_{\alpha\in A} A_\alpha}$$

$$\begin{tikzcd}
\cdots \arrow[r, shift left=4.5, "d^{(3)}_{\cdot} = pr_{\cdot,\cdot,\cdot}"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & \bigsqcup_{\alpha,\beta,\gamma\in A} \Sigma_{\alpha\beta\gamma} \arrow[r, shift left=3, "d^{(2)}_{\cdot} = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=3] & \bigsqcup_{\alpha,\beta\in A} \Sigma_{\alpha\beta} \arrow[r, shift left=2, "d^{(1)}_0 = pr_2"] \arrow[r, shift right=2, "d^{(1)}_1 = pr_1"'] & \bigsqcup_{\alpha\in A} \Sigma_\alpha \arrow[d] \\
 & & & \Sigma
\end{tikzcd}$$

$$\Sigma_{\alpha_1\alpha_2\cdots\alpha_n} \equiv \Sigma_{\alpha_1} \cap \Sigma_{\alpha_2} \cap \cdots \cap \Sigma_{\alpha_n}$$

with PROPERTIES : $\begin{cases} \Delta^{(1)} \breve{A} = i\,d\log \breve{g} \\ \Delta^{(2)} \breve{g} = 1 \end{cases}$ WRITTEN with $\Delta^{(n)} = \sum_{k=0}^{n} (-1)^k d_k^{(n)*}$

[DUPONT]

II NATURAL INGREDIENTS of SIMPLICIAL MECHANICS: TOPOLOGY, SYMMETRY & CHARGE

GOAL: RIGOROUS MODELLING of GEODESIC DYNAMICS of EXTENDED DISTRIBUTIONS of CHARGE

MOTIVATION: EFFECTIVE LFT of COLLECTIVE EXCITATIONS in SPIN CHAINS; 2d CFT; (SUPER-)STRING THEORY; EMERGENT NCG etc.; ANALOGOUS STRUCTURES in CS, BF ...

DYNAMICS: For $x \in [\Sigma, M]$ into M with $g \in \Gamma(T^*M \otimes T^*M)$ METRIC

over Σ : $\dim \Sigma = D+1$ $\qquad H \in \mathcal{Z}^{D+2}_{dR}(M)$ CHARGE FIELD

GRAVITATIONAL CHARGE $\qquad$ TOPOLOGICAL CHARGE

$$\mu \, \mathrm{Geod}(x(\Sigma), g) = q \, \mathrm{Vol}(x(\Sigma))^* \lrcorner H$$

FIELD EQUATIONS

GEODESIC (MINIMAL) EMBEDDING $\qquad$ LORENTZ-TYPE DISTORTION

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FIELD EQUATIONS

GEODESIC (MINIMAL) EMBEDDING

LORENTZ-TYPE DISTORTION

$$\mathcal{L}_g(Tx) = -\mu \sqrt{\det(x^* g)}$$

OR

$$-\frac{\mu}{2} \| Tx \|^2_{g,\gamma}$$

SET $(\mu, q) = (1,1)$

$$\mathcal{L}_H(Tx) = q \, x^* \boxed{d^{-1} H}$$

$$|||$$

$$\mathcal{L}_{WZ}(Tx)$$

AHARONOV-BOHM-TYPE PHENOMENA CALL FOR WELL-DEFINED

$$\mathcal{A}_{DF} : [\Sigma, M] \longrightarrow U(1) : x \longmapsto \exp\left(\frac{i}{\hbar}\int_{\Sigma} \mathcal{L}_g(Tx)\right)\cdot \chi_{W3}\big(x(\Sigma)\big) \quad \text{for } \Sigma \in \partial^{-1}\phi$$

[DIRAC]

SUBJECT TO CONSTRAINTS:

$$\begin{cases} \chi_{W3}\big(x_1(\Sigma_1) \sqcup x_2(\Sigma_2)\big) \overset{!}{=} \chi_{W3}\big(x_1(\Sigma_1)\big)\cdot \chi_{W3}\big(x_2(\Sigma_2)\big) \\ \chi_{W3}(\partial\Omega) \overset{!}{\underset{(ChS)}{=}} \exp\left(\frac{i}{\hbar}\int_{\Omega} H\right) \end{cases}$$

FACTORISATION of PROBABILITY

EL (NO BC's YET)

AHARONOV-BOHM – TYPE PHENOMENA CALL FOR WELL-DEFINED

$$\mathcal{A}_{DF} : [\Sigma, M] \longrightarrow U(1) : x \longmapsto \exp\left(\frac{i}{\hbar}\int_{\Sigma} \mathcal{L}_{g}(Tx)\right)\cdot \chi_{W3}\big(x(\Sigma)\big) \quad \text{for } \Sigma \in \partial^{-1}\emptyset$$

[DIRAC]

SUBJECT TO CONSTRAINTS:

$$\begin{cases} \chi_{W3}\big(x_1(\Sigma_1)\sqcup x_2(\Sigma_2)\big) \overset{!}{=} \chi_{W3}\big(x_1(\Sigma_1)\big)\cdot \chi_{W3}\big(x_2(\Sigma_2)\big) & \text{FACTORISATION of PROBABILITY} \\ \chi_{W3}(\partial\Omega) \underset{(ChS)}{\overset{!}{=}} \exp\left(\frac{i}{\hbar}\int_{\Omega} H\right) & \text{EL (NO BC's YET)} \end{cases}$$

$\Rightarrow$ χ_{W3} IS A CHEEGER-SIMONS DIFFERENTIAL CHARACTER:

$$(\chi_{W3}, H) \in \left\{ (\Psi, h) \in \mathrm{Hom}_{AbGrp}\big(\mathcal{Z}_{D+1}(M), U(1)\big) \times \mathcal{Z}^{D+2}_{dR}(M) \;\middle|\; (ChS) \right\} \equiv \boxed{\hat{H}^{D+1}\big(M, U(1)\big)},$$

TERMED VOLUME HOLONOMY, ALONG $x(\Sigma) \in \mathcal{Z}_{D+1}(M)$, OF...

D-GERBE $\mathcal{G}^{(D)}$ of CURVATURE $\mathrm{curv}(\mathcal{G}^{(D)}) = H$, e.g., FOR $D = 1$, $\mathcal{G}^{(1)} \equiv \mathcal{G}$:

$|||$

SIMPLICIAL $\mathbb{C}^\times$-BUNDLE

over

$$\begin{tikzcd}
 & & & \mathbb{C}^\times \arrow[d, rightsquigarrow] & \\
 & \Delta^{(3)}\mu_L = 1 \arrow[d, two heads] & \mu_L : \Delta^{(2)}L \simeq 1 \arrow[d, two heads] & L, A_L \arrow[d, two heads, "\pi_L"] & \pi_L^* \Delta^{(1)} B = dA_L \\
N_\bullet(Y^{[2]}M \rightrightarrows YM) : \quad \cdots & Y^{[4]}M \arrow[r, shift left=4.5, "d^{(3)}_\cdot = pr_{\cdot,\cdot,\cdot}"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & Y^{[3]}M \arrow[r, shift left=3, "d^{(2)}_\cdot = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=3] & Y^{[2]}M \arrow[r, shift left=2, "d^{(1)}_0 = pr_2"] \arrow[r, shift right=2, "d^{(1)}_1 = pr_1"'] & YM, B \arrow[d, two heads, "\pi_{YM}"] \\
 & & & \pi_{YM}^* H = dB & M, H
\end{tikzcd}$$

NERVE OF M-FIBRED PAIR GROUPOID of YM

$\left(Y^{[N]}M = \underbrace{YM \times_M YM \times_M \cdots \times_M YM}_{\text{N TIMES}} \right)$

D-GERBE $\mathcal{G}^{(D)}$ of CURVATURE $\mathrm{curv}(\mathcal{G}^{(D)}) = H$, e.g., FOR $D = 1$, $\mathcal{G}^{(1)} \equiv \mathcal{G}$:

|||

SIMPLICIAL $\mathbb{C}^\times$-BUNDLE

over

$N_\bullet(Y^{[2]}M \rightrightarrows YM)$:

NERVE OF M-FIBRED PAIR GROUPOID of YM

$$\Big(Y^{[N]}M = \underbrace{YM \times_M YM \times_M \cdots \times_M YM}_{N \text{ TIMES}}\Big)$$

$$\begin{tikzcd}
 & & \mathbb{C}^\times \arrow[dr, squiggly] & & \\
 & \Delta^{(3)}\mu_L = 1 \arrow[d, twoheadrightarrow] & \mu_L : \Delta^{(2)}L \simeq 1 \arrow[d, twoheadrightarrow] & L, A_L \arrow[d, twoheadrightarrow, "\pi_L"] & \pi_L^{*}\Delta^{(1)}B = dA_L \\
\cdots & Y^{[4]}M \arrow[r, shift left=3, "d_\cdot^{(3)} = pr_{\cdot,\cdot,\cdot}"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y^{[3]}M \arrow[r, shift left=2, "d_\cdot^{(2)} = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] & Y^{[2]}M \arrow[r, shift left=1.5, "d_0^{(1)} = pr_2"] \arrow[r, shift right=1.5, "d_1^{(1)} = pr_1"'] & YM, B \arrow[d, twoheadrightarrow, "\pi_{YM}"] \\
 & & & \pi_{YM}^{*}H = dB & M, H
\end{tikzcd}$$

e.g., $\mathcal{I}_B = (M, B, M \times \mathbb{C}^\times, pr_2^{*}\theta_L, 1)$

TRIVIAL GERBE

D-GERBE $\mathcal{G}^{(D)}$ of CURVATURE $\mathrm{curv}(\mathcal{G}^{(D)}) = H$, e.g., FOR $D=1$, $\mathcal{G}^{(1)} \equiv \mathcal{G}$:

$|||$

SIMPLICIAL $\mathbb{C}^\times$-BUNDLE

over

$$\begin{tikzcd}
 & & & \mathbb{C}^\times \arrow[d, squiggly] & \\
 & \Delta^{(3)}\mu_L = 1 \arrow[d, Rightarrow] & \mu_L : \Delta^{(2)}L \simeq 1 \arrow[d, Rightarrow] & L, A_L \arrow[d, Rightarrow, "\pi_L"] & \pi_L^*\Delta^{(1)}B = dA_L \\
N_\bullet(Y^{[2]}M \rightrightarrows YM) : \quad \cdots & Y^{[4]}M \arrow[r, shift left=4.5, "d^{(3)}_\cdot = pr_{\cdot,\cdot,\cdot}"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4.5] & Y^{[3]}M \arrow[r, shift left=3, "d^{(2)}_\cdot = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=3] & Y^{[2]}M \arrow[r, shift left=2, "d_0^{(1)} = pr_2"] \arrow[r, shift right=2, "d_1^{(1)} = pr_1"'] & YM, B \arrow[d, Rightarrow, "\pi_{YM}"] \\
 & & & \pi_{YM}^* H = dB & M, H
\end{tikzcd}$$

NERVE OF M-FIBRED PAIR GROUPOID of YM

$$\left(Y^{[N]}M = \underbrace{YM \times_M YM \times_M \cdots \times_M YM}_{N \text{ TIMES}} \right)$$

YIELDS $\boxed{\chi_{W3} \equiv \mathrm{Hol}_{\mathcal{G}}}$

e.g., $I_B = \left(M, B, M \times \mathbb{C}^\times, pr_2^*\theta_L, 1\right)$

TRIVIAL GERBE

$$\boxed{\mathcal{G} = \left(YM, B, L, A_L, \mu_L\right)}$$

GEOMETRISES...

$$\begin{tikzcd}
\Delta_\bullet(\Sigma): & \coprod\limits_{\alpha,\beta,\gamma\in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2, "d^{(2)}_{\bullet} = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "x"] & \coprod\limits_{\alpha,\beta\in A} E_{\alpha\beta} \arrow[r, shift left, "d^{(1)}_{0} = pr_2"] \arrow[r, shift right, "d^{(1)}_{1} = pr_1"'] \arrow[d, "x"] & \coprod\limits_{\alpha\in A} \Sigma_\alpha \arrow[d, shift right=2, "\iota"'] \arrow[d, shift left=2, "x"] & \Big(\longrightarrow \Sigma\Big) \\
\check{N}_\bullet(\vartheta): & \coprod\limits_{i,j,k\in I} \vartheta_{ijk} \arrow[r, shift left=2, "{}^{M}d^{(2)}_{\bullet} = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] & \coprod\limits_{i,j\in I} \vartheta_{ij} \arrow[r, shift left, "{}^{M}d^{(1)}_{0} = pr_2"] \arrow[r, shift right, "{}^{M}d^{(1)}_{1} = pr_1"'] & \coprod\limits_{i\in I} \vartheta_i & \Big(\xrightarrow[\pi_M]{} M\Big) \\
\mathcal{B}_H(I^\circ): & g_{ijk} & A_{ij} & B_i & (H) \\
\deg: & 0 & 1 & 2 &
\end{tikzcd}$$

$\hat{X} \equiv (x, \iota)$

SIMPLICIAL STRATIFIED SPACETIME

SIMPLICIAL STRATIFIED CONFIGURATION FIBRE

SIMPLICIAL STRATIFIED 2-FORM

$$\left(\pi_M^* H_i = dB_i\right)$$

$$(*)\quad \begin{cases} {}^{M}\Delta^{(1)} B_{ij} = dA_{ij} \\ {}^{M}\Delta^{(2)} A_{ijk} = i\, d\log g_{ijk} \\ {}^{M}\Delta^{(3)} g_{ijkl} = 1 \end{cases}$$

$$\begin{tikzcd}
\Delta_\bullet(\Sigma): & \bigsqcup_{\alpha,\beta,\gamma\in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2, "d^{(2)}_\cdot = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "x"] & \bigsqcup_{\alpha,\beta\in A} E_{\alpha\beta} \arrow[r, shift left, "d^{(1)}_0 = pr_2"] \arrow[r, shift right, "d^{(1)}_1 = pr_1"'] \arrow[d, "x"] & \bigsqcup_{\alpha\in A} \Sigma_\alpha \arrow[d, shift right, "\iota"'] \arrow[d, shift left, "x"] & (\longrightarrow \Sigma) \\
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\beta_H(\mathcal{I}^\bullet): & g_{ijk} & A_{ij} & B_i & (H) \\
\deg: & 0 & 1 & 2 &
\end{tikzcd}$$

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UPSHOT:

$$\boxed{\chi_{WZ}(x(\Sigma)) = \exp\left(\frac{i}{\hbar}\int_{\Delta_\bullet(\Sigma)} \hat{x}^* \beta_H(\mathcal{I}^\bullet)\right)}$$

OBSERVATION : RELATIONS (*) SATISFIED by $\beta_H(\mathcal{I}^\bullet) \equiv (B_i, A_{ij}, g_{ijk})$,

$D\beta_H(\mathcal{I}^\bullet) = 0$, D - DELIGNE DIFFERENTIAL on $\check{C}^\bullet(\mathcal{O}, \mathcal{D}(2)^\bullet)$ for

$$\mathcal{D}(2)^\bullet : \underline{U(1)}_M \xrightarrow{i\,d\log} \underline{\Omega^1(M)} \xrightarrow{d} \underline{\Omega^2(M)}$$ DELIGNE COMPLEX

IMPLY $\boxed{[\beta_H(\mathcal{I}^\bullet)] \in \mathbb{H}^2(M, \mathcal{D}(2)^\bullet)}$ BEILINSON - DELIGNE COHOMOLOGY, OBSERVATION : RELATIONS (*) SATISFIED by $\beta_H(\mathcal{I}^\bullet) \equiv (B_i, A_{ij}, g_{ijk})$,

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IMPLY $\boxed{[\beta_H(\mathcal{I}^\bullet)] \in \mathbb{H}^2(M, \mathcal{D}(2)^\bullet)}$ BEILINSON - DELIGNE COHOMOLOGY,

with CHEEGER-SIMONS MODEL : $\mathbb{H}^\bullet(M, \mathcal{D}(\bullet)^\bullet) \simeq \hat{H}^\bullet(M, U(1))$

$\Longrightarrow$ Hol_g FIXES $[g]$

UNEXPECTED BONUS: PREQUANTISATION via TRANSGRESSION

(& RAISON D'ÊTRE DE L'IDÉE)

$$\tau : \mathbb{H}^2(M, \mathcal{D}(2)^\bullet) \longrightarrow \mathbb{H}^1(LM, \mathcal{D}(1)^\bullet)$$

$$: [\beta_H(\mathcal{I}^\bullet)] \longmapsto [\alpha_{\tau H}(\Delta_\bullet(\mathbb{S}^1), \mathcal{I}^\bullet)]$$

$$\text{for } \tau H \equiv \int_{\mathbb{S}^1} ev^* H$$

$$ev : LM \times \mathbb{S}^1 \longrightarrow M$$

$$\begin{tikzcd} & & \mathbb{C}^\times \arrow[dl, squiggly] \\ {[\alpha_{\tau H}(\Delta_\bullet(\mathbb{S}^1), \mathcal{I}^\bullet)]} \arrow[r, bend right, "\text{GEOM}^{\underline{\text{N}}}"'] & \mathcal{L}_G, A_G \arrow[d, twoheadrightarrow] & \\ & LM & \end{tikzcd}$$

[GAWĘDZKI]

UNEXPECTED BONUS: PREQUANTISATION via TRANSGRESSION

(& RAISON D'ÊTRE DE L'IDÉE)

$$\begin{tikzcd}
 & & \mathbb{C}^\times \\
\tau : \mathbb{H}^2(M,\mathcal{D}(2)^\bullet) \arrow[r] & \mathbb{H}^1(LM,\mathcal{D}(1)^\bullet) & \mathcal{L}_{\mathcal{G}}, A_{\mathcal{G}} \arrow[u, squiggly, dash] \arrow[dd] \\
: [\beta_H(\mathcal{I}^\bullet)] \arrow[r, mapsto] & {[\alpha_{\tau H}(\Delta_\bullet(\mathbb{S}^1),\mathcal{I}^\bullet)]} \arrow[ur, bend right, "\text{GEOM}^{\text{n}}"'] & \\
 & & LM
\end{tikzcd}$$

for $\tau H \equiv \int_{\mathbb{S}^1} ev^* H$

$ev : LM \times \mathbb{S}^1 \longrightarrow M$

[GAWĘDZKI]

IMPLICATION: PREQUANTISABLE SYMMETRIES MUST GERBIFY

i.e., LIFT as $G \longrightarrow \mathrm{Aut}(\mathcal{G}) \underset{\cdot}{=} ?$ GERBE (AUTO-)ISOMORPHISMS

UNEXPECTED BONUS: PREQUANTISATION via TRANSGRESSION

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$$\begin{tikzcd}
\tau : \mathbb{H}^2(M, \mathcal{D}(2)^\bullet) \arrow[r] & \mathbb{H}^1(LM, \mathcal{D}(1)^\bullet) & & \mathbb{C}^\times \arrow[dl, squiggly] \\
: \big[\beta_H(\mathcal{I}^\bullet)\big] \arrow[r, mapsto] & \big[\alpha_{\tau H}(\Delta_\bullet(\mathbb{S}^1), \mathcal{I}^\bullet)\big] \arrow[r, bend right, "\text{GEOM}^{\text{ly}}"'] & \mathcal{L}_{\mathcal{G}}, A_{\mathcal{G}} \arrow[d, two heads] & \\
& & LM &
\end{tikzcd}$$

for $\tau H \equiv \int_{\mathbb{S}^1} ev^* H$

$ev : LM \times \mathbb{S}^1 \longrightarrow M$

[GAWĘDZKI]

IMPLICATION: PREQUANTISABLE SYMMETRIES MUST GERBIFY

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GERBE
(AUTO-) ISOMORPHISMS

POINT of DEPARTURE : 2-CLUTCHING THEOREM [MURRAY-STEVENSON]

$$\begin{tikzcd}
{[\mathcal{G}]} \arrow[r, bend left, "\text{TRIVIALISATION}"] & \big[\beta_H(\mathcal{I}^\bullet)\big] \arrow[l, bend left, "\text{RECONSTRUCTION}"]
\end{tikzcd}$$

NATURAL NOTION: $\mathcal{G}_1 \simeq \mathcal{G}_2 \iff [\beta^1_H(\mathcal{I}^\bullet)] = [\beta^2_H(\mathcal{I}^\bullet)]$

NATURAL NOTION: $\mathcal{G}_1 \simeq \mathcal{G}_2 \iff [\beta_H^1(\mathcal{I}^\bullet)] = [\beta_H^2(\mathcal{I}^\bullet)]$

GEOMETRISES as $\mathcal{E} = (\underbrace{Y_1M \times_M Y_2M}_{=: Y_{1,2}M}, E, A_E, \varepsilon_E)$ STABLE ISOMORPHISM [MURRAY-STEVENSON] ,

where

$$\begin{tikzcd}
 & E, A_E \arrow[d, "\pi_E"'] & \\
 & Y_{1,2}M \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
Y_1M \arrow[dr, "\pi_{Y_1M}"'] & & Y_2M \arrow[dl, "\pi_{Y_2M}"] \\
 & M &
\end{tikzcd}$$

with

$$\begin{tikzcd}
pr_{1,2}^*E \otimes pr_{2,4}^*L_2 \arrow[d] \arrow[dd, bend left=60, "\varepsilon_E", "\simeq"'] \\
Y_1M \times_M Y_2M \times_M Y_1M \times_M Y_2M \\
pr_{1,3}^*L_1 \otimes pr_{3,4}^*E \arrow[u]
\end{tikzcd}$$

$$\pi_E^*(pr_2^*B_2 - pr_1^*B_1) = dA_E$$

& COMPATIBILITY with μ_{L_1}, μ_{L_2}

WHICH LEADS TO ...

III CATEGORIFICATION of SYMMETRIES

INVARIANCE: $\exp\left(\frac{i}{\hbar} S_g\right)\cdot \mathrm{Hol}_{\mathcal{G}} \overset{!}{=} \exp\left(\frac{i}{\hbar} S_{\underline{\lambda}_h^* g}\right)\cdot \mathrm{Hol}_{\underline{\lambda}_h^* \mathcal{G}} \quad \forall h \in G$

CALLS FOR $\underline{\lambda}_h^* g \overset{!}{=} g$ & $\underline{\lambda}_h^* \mathcal{G} \overset{!}{\simeq} \mathcal{G}$ (GERBES PULL BACK NATURALLY)

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* ELEMENT-WISE PRESENTATION of G

$$\Lambda : G \longrightarrow \mathrm{Aut}(\mathcal{G})$$
$$: h \longmapsto \underline{\lambda}_h^* \mathcal{G} \underset{\Lambda_h}{\simeq} \mathcal{G}$$

OBSTRUCTION : $\underline{\lambda}_h^* H = H \Longrightarrow \underline{\lambda}_h^* \mathcal{G}$ DIFFERS from $\mathcal{G}$ by $\mathcal{G}_0 \in \mathcal{W}^3(M; H=0) \simeq \check{H}^2(M, U(1))$

$$\Longrightarrow \in \bigsqcup_{h \in G} \check{H}^2(M, U(1))$$

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$$\Lambda_\cdot : G \longrightarrow \mathrm{Aut}(\mathcal{G})$$

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QUESTION: HOW TO MAKE SENSE of 'HOMOMORPHICITY' of $\Lambda_\cdot$?

NEED: (i) COMPOSITION of 1-ISOMORPHISMS

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, bend left, "\Phi_{12}"] \arrow[rr, bend right, "\boxed{\Phi_{23} \circ \Phi_{12}}"'] & \mathcal{G}_2 \arrow[r, bend left, "\Phi_{23}"] & \mathcal{G}_3
\end{tikzcd}$$

BASES on

PILING UP

OF SURJECTIVE SUBMERSIONS

~ CORRESPONDENCES

$$\begin{tikzcd}
 & & pr_1^* E_1 \otimes pr_2^* E_2 \arrow[d, two heads] & & \\
E_1 \arrow[d, two heads] & & Y_{1,2}M \times_{Y_2 M} Y_{2,3}M \arrow[dll, "pr_1"'] \arrow[drr, "pr_2"] \arrow[dd, two heads, "pr_{1,3}"] & & E_2 \arrow[d, two heads] \\
Y_{1,2}M & & & & Y_{2,3}M \\
 & & Y_{1,3}M & &
\end{tikzcd}$$

NEED : (i) COMPOSITION of 1-ISOMORPHISMS

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\mathcal{G}_1 \arrow[r, bend left, "\mathcal{E}_{12}"] \arrow[rr, bend right, "\boxed{\mathcal{E}_{23} \circ \mathcal{E}_{12}}"'] & \mathcal{G}_2 \arrow[r, bend left, "\mathcal{E}_{23}"] & \mathcal{G}_3
\end{tikzcd}$$

BASES on
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E_1 \arrow[d, two heads] & & Y_{1,2}M \times_{Y_2 M} Y_{2,3}M \arrow[dll, "pr_1"] \arrow[drr, "pr_2"'] \arrow[dd, two heads, "pr_{1,3}"] & & E_2 \arrow[d, two heads] \\
Y_{1,2}M & & & & Y_{2,3}M \\
 & & Y_{1,3}M & &
\end{tikzcd}$$

(ii) 2-ISOMORPHISMS between 1-MORPHISMS

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r, bend left=60, "\mathcal{E}_1"{name=U}] & \mathcal{G}_2 \arrow[l, bend left=60, "\mathcal{E}_2"{name=D}]
\arrow[Rightarrow, from=U, to=D, "\simeq", "\varphi"']
\end{tikzcd}$$

BOILS DOWN to

$$\begin{tikzcd}
 & pr_1^* E_1 \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, two heads] & & pr_2^* E_2 \arrow[dl, two heads] & \\
E_1 \arrow[d, two heads] & & Y^1_{1,2}M \times_M Y^2_{1,2}M \arrow[dll, "pr_1"] \arrow[drr, "pr_2"'] & & E_2 \arrow[d, two heads] \\
Y^1_{1,2}M & & & & Y^2_{1,2}M
\end{tikzcd}$$

& COHERENCE with $\varepsilon_{E_1}, \varepsilon_{E_2}$

** HOMOMORPHIC PRESENTATION of G

$\varphi_{\cdot,\cdot} : G \times G \longrightarrow 2\text{-}\mathrm{Iso}(\mathcal{G})$ FUSION 2-ISOMORPHISM

$$\begin{tikzcd}[column sep=large, row sep=large]
: (h_1,h_2) \arrow[r, mapsto] & \mathcal{G} \arrow[r, bend left, "\Lambda_{h_2}"] \arrow[rr, bend right=40, "\underline{\lambda}^*_{h_2}\Lambda_{h_1}\circ\Lambda_{h_2}"', "/\!/" marking, ""{name=U, below}] \arrow[rrd, bend right=80, "\Lambda_{h_1h_2}"', ""{name=D, above}] & \underline{\lambda}^*_{h_2}\mathcal{G} \arrow[r, bend left, "\underline{\lambda}^*_{h_2}\Lambda_{h_1}"] & \underline{\lambda}^*_{h_2}\underline{\lambda}^*_{h_1}\mathcal{G} \arrow[d, equal] \\
 & & & \underline{\lambda}^*_{h_1h_2}\mathcal{G}
\arrow[Rightarrow, from=U, to=D, "\cong"', "\varphi_{h_1,h_2}"]
\end{tikzcd}$$

OBSTRUCTION $\in \bigsqcup_{h_1,h_2\in G} \check{H}^1(M, U(1))$

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$$\begin{tikzcd}
: (h_1,h_2) \arrow[r, mapsto] & \mathcal{G} \arrow[r, bend left, "\Lambda_{h_2}"] \arrow[rr, bend right, "\lambda^*_{h_2}\Lambda_{h_1}\circ\Lambda_{h_2}"'] \arrow[rr, bend right=80, "\Lambda_{h_1h_2}"', ""{name=B}] & \lambda^*_{h_2}\mathcal{G} \arrow[r, bend left, "\lambda^*_{h_2}\Lambda_{h_1}"] \arrow[d, phantom, "//"] & \lambda^*_{h_2}\lambda^*_{h_1}\mathcal{G} \arrow[d, equal] \\
 & & {} \arrow[d, Rightarrow, "\varphi_{h_1,h_2}", "\simeq"'] & \lambda^*_{h_1h_2}\mathcal{G} \\
 & & {} &
\end{tikzcd}$$

OBSTRUCTION $\in \bigsqcup_{h_1,h_2\in G} \check{H}^1(M,U(1))$

FINALLY,

*** ASSOCIATIVE PRESENTATION of G

$$\begin{tikzcd}
\mathcal{G} \arrow[r, bend left=60, "\lambda^*_{h_2h_3}\Lambda_{h_1}\circ\lambda^*_{h_3}\Lambda_{h_2}\circ\Lambda_{h_3}"] & \lambda^*_{h_1h_2h_3}\mathcal{G} \arrow[l, bend left=60, "\Lambda_{h_1h_2h_3}"] \arrow[l, phantom, "\overset{!}{=}"]
\end{tikzcd}$$

$$\alpha_{h_1,h_2,h_3} := [\delta_G\varphi_{h_1,h_2,h_3}] \in \bigsqcup_{h_1,h_2\in G} \check{H}^0(M,U(1)) \simeq U(1)^{\pi_0(M)}$$

OBSTRUCTION

$$[\alpha] \in H^3(G, U(1)^{\pi_0(M)})$$

ЧЕХОВ's ASSOCIATOR CLASS

A Carpathian Break ...

... on our way
Towards Higher Geometry

IV DEFECTS, SIMPLICIAL BACKGROUNDS, & ALL THAT

THE HITHERTO

$$\begin{tikzcd}
\Delta_\bullet(\Sigma): & \bigsqcup_{\alpha,\beta,\gamma\in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2, "d_\cdot^{(2)}=\mathrm{pr}_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "x"] & \bigsqcup_{\alpha,\beta\in A} E_{\alpha\beta} \arrow[r, shift left, "d_0^{(1)}=\mathrm{pr}_2"] \arrow[r, shift right, "d_1^{(1)}=\mathrm{pr}_1"'] \arrow[d, "x"] & \bigsqcup_{\alpha\in A} \Sigma_\alpha \arrow[d, shift right, "\iota"'] \arrow[d, shift left, "x"] & (\longrightarrow \Sigma) \\
\check{N}_\circ(\vartheta): & \bigsqcup_{i,j,k\in I} \vartheta_{ijk} \arrow[r, shift left=2, "{}^M d_\cdot^{(2)}=\mathrm{pr}_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] & \bigsqcup_{i,j\in I} \vartheta_{ij} \arrow[r, shift left, "{}^M d_0^{(1)}=\mathrm{pr}_2"] \arrow[r, shift right, "{}^M d_1^{(1)}=\mathrm{pr}_1"'] & \bigsqcup_{i\in I} \vartheta_i & (\xrightarrow[\pi_M]{} M) \\
\mathcal{G} \equiv \mathcal{B}_H(\mathcal{I}^\circ): & g_{ijk} & A_{ij} & B_i & (H)
\end{tikzcd}$$

$$\hat{x} \equiv (x, \iota)$$

SIMPLICIAL STRATIFIED SPACETIME

SIMPLICIAL STRATIFIED CONFIGURATION FIBRE

SIMPLICIAL STRATIFIED 2-FORM

READILY REPLACED by...

IV DEFECTS, SIMPLICIAL BACKGROUNDS, & ALL THAT

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\check{N}_\bullet(\vartheta) : & \bigsqcup_{i,j,k\in I} \vartheta_{ijk} \arrow[r, shift left=2, "{}^M d^{(2)}_\cdot = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] & \bigsqcup_{i,j\in I} \vartheta_{ij} \arrow[r, shift left, "{}^M d^{(1)}_0 = pr_2"] \arrow[r, shift right, "{}^M d^{(1)}_1 = pr_1"'] & \bigsqcup_{i\in I} \vartheta_i & \left(\xrightarrow{\pi_M} M\right) \\
\mathcal{G} \equiv \beta_H(\mathcal{I}^\circ) : & g_{ijk} & A_{ij} & B_i & (H)
\end{tikzcd}$$

$\hat{X} \equiv (X, \imath)$

SIMPLICIAL
STRATIFIED
SPACETIME

SIMPLICIAL
STRATIFIED
CONFIGURATION
FIBRE

SIMPLICIAL
STRATIFIED
2-FORM

READILY
REPLACED by...

DEFECT JUNCTIONS

DEFECT LINES

DOMAINS

DEFECT
GRAPH

$$\begin{tikzcd}
 & & & & \Gamma \arrow[d, phantom, "\cap"] \\
\Sigma_\Gamma : & \bigsqcup_{\alpha,\beta,\gamma\in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2, "d^{(2)}_\cdot = pr_{\cdot,\cdot}"] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \bigsqcup_{\alpha,\beta\in A} E_{\alpha\beta} \arrow[r, shift left, "d^{(1)}_0 = pr_2"] \arrow[r, shift right, "d^{(1)}_1 = pr_1"'] \arrow[d] & \bigsqcup_{\alpha\in A} \Sigma_\alpha \arrow[d] & \left(\longrightarrow \Sigma\right) \\
 & T_3 \arrow[r, shift left=2, "\tilde{d}^{(2)}_\cdot"] \arrow[r] \arrow[r, shift right=2] & Q \arrow[r, shift left, "\tilde{d}^{(1)}_\cdot"] \arrow[r, shift right] & M & \\
 & \varphi_3 & \Phi & \mathcal{G} & \tilde{\mathcal{G}}
\end{tikzcd}$$

$\tilde{X}$

FUSION

CORRESPONDENCE

POLY-PHASE

QUASI-SIMPLICIAL
STRATIFIED
GERBE

$$\left(\begin{array}{c} \text{COHOMOLOGICAL} \\ \text{CLASSIFICATION} \\ \text{of} \\ \text{* PHASES} \\ \text{** DEFECTS} \end{array}\right) \Longleftarrow$$

COMPONENTS of $\tilde{\mathcal{G}}$ RELATED ANALOGOUSLY to THOSE of $\mathcal{B}_H(I^\circ)$ [RUNKEL-RRS]

$$\Phi : \widetilde{\Delta}^{(1)} \mathcal{G} \xrightarrow{\simeq} I_\omega , \qquad \omega \in \Omega^2(Q)$$

EXTRA FREEDOM
(OBSTRUCTION against
TOPOLOGICALITY)

i.e., $d_0^{(1)*}\mathcal{G} \simeq d_1^{(1)*}\mathcal{G} \otimes I_\omega$

$$\varphi_3 : \widetilde{\Delta}^{(2)} \Phi \overset{\simeq}{\Longrightarrow} \mathrm{id}$$

i.e., $d_0^{(2)*}\Phi \circ d_2^{(2)*}\Phi \simeq d_1^{(2)*}\Phi$

COMPONENTS of $\mathcal{G}$ RELATED ANALOGOUSLY to THOSE of $\beta_H(\mathcal{I}^\circ)$ [RUNKEL-RRS]

$$\Phi : \tilde{\Delta}^{(1)}\mathcal{G} \xrightarrow{\simeq} I_\omega \,, \qquad \omega \in \Omega^2(Q)$$

EXTRA FREEDOM
(OBSTRUCTION against
TOPOLOGICALITY)

$$\text{i.e., } d_0^{(1)*}\mathcal{G} \simeq d_1^{(1)*}\mathcal{G} \otimes I_\omega$$

$$\varphi_3 : \tilde{\Delta}^{(2)}\Phi \overset{\simeq}{\Longrightarrow} \mathrm{id}$$

$$\text{i.e., } d_0^{(2)*}\Phi \circ d_2^{(2)*}\Phi \simeq d_1^{(2)*}\Phi$$

THESE GEOMETRISE SIMPLICIAL 3-FORM $\tilde{H} = (H, \omega)$

$$\tilde{\Delta}^{(1)} H = d\omega$$

$$\tilde{\Delta}^{(2)} \omega = 0$$

$$\Longrightarrow \tilde{\chi}_{W3}\left(\tilde{x}(\Sigma_\Gamma)\right) = \exp\left(\frac{i}{\hbar}\int_{\Delta_\cdot(\Sigma_\Gamma)} \beta_{\tilde{H}}(\hat{I}^\cdot)\right)$$

RELATIVE CHEEGER-SIMONS CHARACTER

FACTS of LIFE with DEFECTS:

(i) CURVATURE of CORRESPONDENCE ω DETERMINES (KINETIC-)MOMENTUM DISCONTINUITY @ DOMAIN WALL:

$$p_2 \circ \overline{\mathrm{T}d_1^{(1)}} - p_1 \circ \overline{\mathrm{T}d_0^{(1)}} = \mathrm{T}X(\hat{t}) \lrcorner \omega \quad (\mathrm{DGC})$$

$\Longrightarrow$ DEFINES SYMPLECTIC RELATION between PHASES, COVERED by $\tau\overline{\mathcal{D}}$

PREQUANTOMORPHISM

FACTS of LIFE with DEFECTS :

(i) CURVATURE of CORRESPONDENCE ω DETERMINES (KINETIC-)MOMENTUM DISCONTINUITY @ DOMAIN WALL :

$$p_2 \circ \mathsf{T}d_1^{(1)} - p_1 \circ \mathsf{T}d_0^{(1)} = \mathsf{T}\mathcal{X}(\hat{t}) \lrcorner \omega \quad (\mathrm{DGC})$$

$\Longrightarrow$ DEFINES SYMPLECTIC RELATION between PHASES , COVERED by $\tau\Phi$

PREQUANTOMORPHISM

(ii) CORRESPONDENCES ACCOUNT for BOUNDARIES :

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d, twoheadrightarrow] & \Phi_\partial \equiv \mathcal{J}_{\omega_\partial} : \iota^*_{Q_\partial}\mathcal{G} \simeq \mathcal{I}_{\omega_\partial} \arrow[d, twoheadrightarrow] \\
M \equiv X \sqcup \{\bullet\} & X \arrow[l, hook'] & Q_\partial \arrow[l, hook', "d_0^{(1)} \equiv \iota_{Q_\partial}"'] \arrow[dl, dashed] \\
 & \{\bullet\} \arrow[ul, hook, bend left] &
\end{tikzcd}$$

PHASE-GERBE TRIVIALISATION

FACTS of LIFE with DEFECTS:

(i) CURVATURE of CORRESPONDENCE ω DETERMINES (KINETIC-)MOMENTUM DISCONTINUITY @ DOMAIN WALL:

$$\mathrm{p}_2\circ\overline{\mathrm{T}d_1^{(1)}} - \mathrm{p}_1\circ\overline{\mathrm{T}d_0^{(1)}} = \mathrm{T}X(\hat{t})\lrcorner\,\omega \qquad (\mathrm{DGC})$$

$\Longrightarrow$ DEFINES SYMPLECTIC RELATION between PHASES, COVERED by $\tau\Phi$

PREQUANTOMORPHISM

(ii) CORRESPONDENCES ACCOUNT for BOUNDARIES:

$$\begin{tikzcd}
 & \mathcal{G} \arrow[d] & \Phi_\partial \equiv \mathcal{J}_{\omega_\partial} : \iota_{Q_\partial}^{*}\mathcal{G} \cong \mathcal{I}_{\omega_\partial} \arrow[d] \\
M \equiv X \sqcup \{\bullet\} & X \arrow[l, hook'] & Q_\partial \arrow[l, "d_0^{(1)}\equiv\iota_{Q_\partial}"'] \arrow[dl, dashed] \\
 & \{\bullet\} \arrow[ul, bend left] &
\end{tikzcd}$$

PHASE-GERBE TRIVIALISATION

(iii) AT THE OTHER EXTREME, WE FIND... SYMMETRIES $Q_G = \bigsqcup_{h\in G} M \simeq G\times M \overset{d_0^{(1)}=\mathrm{pr}_2}{\underset{d_1^{(1)}=\lambda}{\rightrightarrows}} M$ with FLAT CORRESPONDENCE $\Phi_G = \bigsqcup_{h\in G}\Phi_h$

NATURAL EXTENSION (MOTIVATED by GERBISH PRESENTATIONS of SYMMETRIES)

$$\begin{tikzcd}[column sep=large, row sep=large]
\Sigma_\Gamma : \cdots \arrow[r, shift left=4] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & \bigsqcup\limits_{\alpha_i \in A} V_{\alpha_1\alpha_2\ldots\alpha_n} \arrow[d] \arrow[r, shift left=4] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & \cdots \bigsqcup\limits_{\alpha,\beta,\gamma,\delta \in A} V_{\alpha\beta\gamma\delta} \arrow[d] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \bigsqcup\limits_{\alpha,\beta,\gamma \in A} V_{\alpha\beta\gamma} \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bigsqcup\limits_{\alpha,\beta \in A} E_{\alpha\beta} \arrow[d] \arrow[r, shift left=1] \arrow[r, shift right=1] & \bigsqcup\limits_{\alpha \in A} \Sigma_\alpha \arrow[d] \\
\tilde{x} & & & & & \\
M_\bullet : \cdots \arrow[r, shift left=4, "d_\bullet^{(n)}"] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & T_n \arrow[r, shift left=4, "d_\bullet^{(n-1)}"] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & \cdots T_4 \arrow[r, shift left=3, "d_\bullet^{(3)}"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & T_3 \arrow[r, shift left=2, "d_\bullet^{(2)}"] \arrow[r] \arrow[r, shift right=2] & Q \arrow[r, shift left=1, "d_\bullet^{(1)}"] \arrow[r, shift right=1] & M
\end{tikzcd}$$

SIMPLICIAL BACKGROUND : ? ? φ_3 Φ $\mathcal{G}$

with ...

COHERENCES FUSION CORRESPONDENCE POLY-PHASE

NATURAL EXTENSION (MOTIVATED by GERBISH PRESENTATIONS of SYMMETRIES)

$$\begin{tikzcd}
\Sigma_\Gamma : \arrow[d, phantom, "\tilde{\times}"] & \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=3] & \bigsqcup\limits_{\alpha_i \in A} V_{\alpha_1\alpha_2\ldots\alpha_n} \arrow[d] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=3] & \cdots & \bigsqcup\limits_{\alpha,\beta,\gamma,\delta \in A} V_{\alpha\beta\gamma\delta} \arrow[d] \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \bigsqcup\limits_{\alpha,\beta,\gamma \in A} V_{\alpha\beta\gamma} \arrow[d] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \bigsqcup\limits_{\alpha,\beta \in A} E_{\alpha\beta} \arrow[d] \arrow[r, shift left=1] \arrow[r, shift right=1] & \bigsqcup\limits_{\alpha \in A} \Sigma_\alpha \arrow[d] \\
M_\bullet : & \cdots \arrow[r, shift left=2, "d_\bullet^{(n)}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=3] & T_n \arrow[r, shift left=2, "d_\bullet^{(n-1)}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=3] & \cdots & T_4 \arrow[r, shift left=3, "d_\bullet^{(3)}"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & T_3 \arrow[r, shift left=2, "d_\bullet^{(2)}"] \arrow[r] \arrow[r, shift right=2] & Q \arrow[r, shift left=1, "d_\bullet^{(1)}"] \arrow[r, shift right=1] & M
\end{tikzcd}$$

SIMPLICIAL BACKGROUND with ... : ? ? φ_3 $\mathfrak{E}$ $\mathcal{G}$

COHERENCES FUSION CORRESPONDENCE POLY-PHASE

FACE MAPS

$d_\bullet^{(\cdot)} : M_\bullet \longrightarrow M_{\bullet-1}$

... SATISFYING

$$\begin{aligned}
d_i^{(n-1)} \circ d_j^{(n)} &= d_{j-1}^{(n-1)} \circ d_i^{(n)}\,, \quad i < j\,,\\
s_i^{(n+1)} \circ s_j^{(n)} &= s_{j+1}^{(n+1)} \circ s_i^{(n)}\,, \quad i \leq j\,,\\
d_i^{(n+1)} \circ s_j^{(n)} &= \begin{cases} s_{j-1}^{(n-1)} \circ d_i^{(n)} & \text{if } \ i < j\,,\\ \mathrm{id}_{X_n} & \text{if } \ i = j \text{ or } i = j+1\,,\\ s_j^{(n-1)} \circ d_{i-1}^{(n)} & \text{if } \ i > j+1\,. \end{cases}
\end{aligned}$$

SIMPLICIAL IDENTITIES

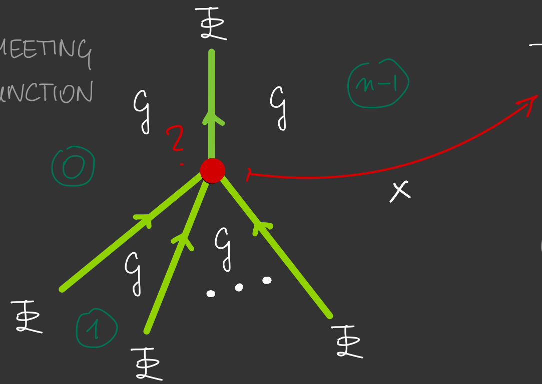
& ADDITIONAL DEGENERACY MAPS

$s_\bullet^{(\cdot)} : M_\bullet \longrightarrow M_{\bullet+1}$ for IDENTITY DEFECT $\mathrm{id}_\mathcal{G} : \mathcal{G} \simeq \mathcal{G}$

& ITS FUSION

QUESTION: HOW DOES THIS REPRODUCE THE STANDARD PICTURE?

n PHASES MEETING
@ DEFECT JUNCTION



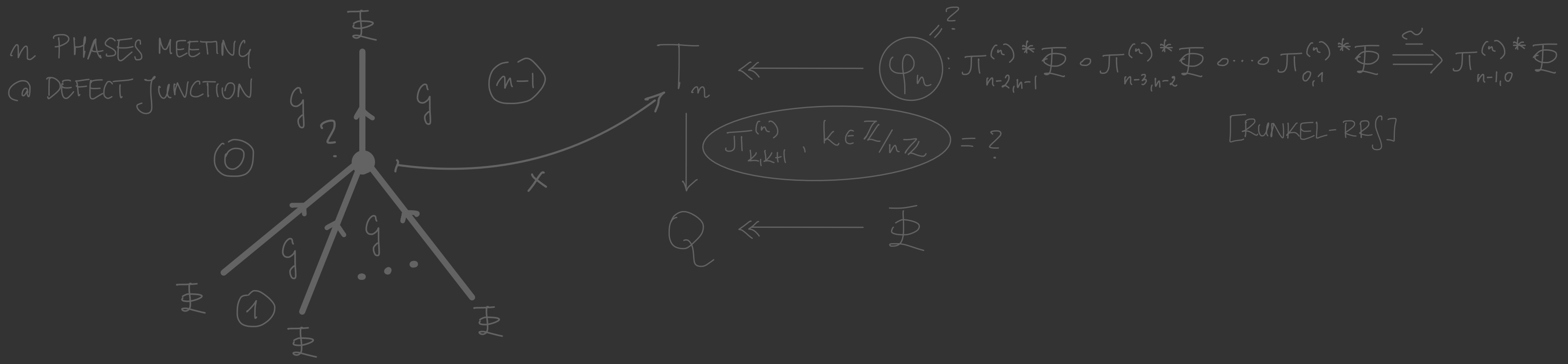
$$\begin{tikzcd} T_n \arrow[d] & \varphi_n \arrow[l, two heads] \\ Q & \Phi \arrow[l, two heads] \end{tikzcd}$$

$\varphi_n \overset{?}{=} : \pi^{(n)*}_{n-2,n-1}\Phi \circ \pi^{(n)*}_{n-3,n-2}\Phi \circ \cdots \circ \pi^{(n)*}_{0,1}\Phi \overset{\simeq}{\Longrightarrow} \pi^{(n)*}_{n-1,0}\Phi$

[RUNKEL-RRS]

$\pi^{(n)}_{k,k+1}, \ k \in \mathbb{Z}/n\mathbb{Z} \ \overset{?}{=}$

QUESTION: HOW DOES THIS REPRODUCE THE STANDARD PICTURE?



ANSWER: COMBINATORICS of $(M_\bullet, d^{(\bullet)}_\cdot)$ ENSURE:

(1) $\exists!\ \pi^{(\bullet)}_{\cdot,\cdot+1} : T_\bullet \longrightarrow Q \quad (\bullet > 3)$

$$\pi^{(n)}_{k,k+1} = d^{(2)}_2 \circ d^{(3)}_2 \circ \dots \circ d^{(n-k-1)}_2 \circ d^{(n-k)}_0 \circ d^{(n-k+1)}_0 \circ \dots \circ d^{(n-1)}_0, \quad k \in \overline{0,n-2}$$

$$\pi^{(n)}_{n-1,0} = d^{(2)}_1 \circ d^{(3)}_1 \circ \dots \circ d^{(n-1)}_1$$

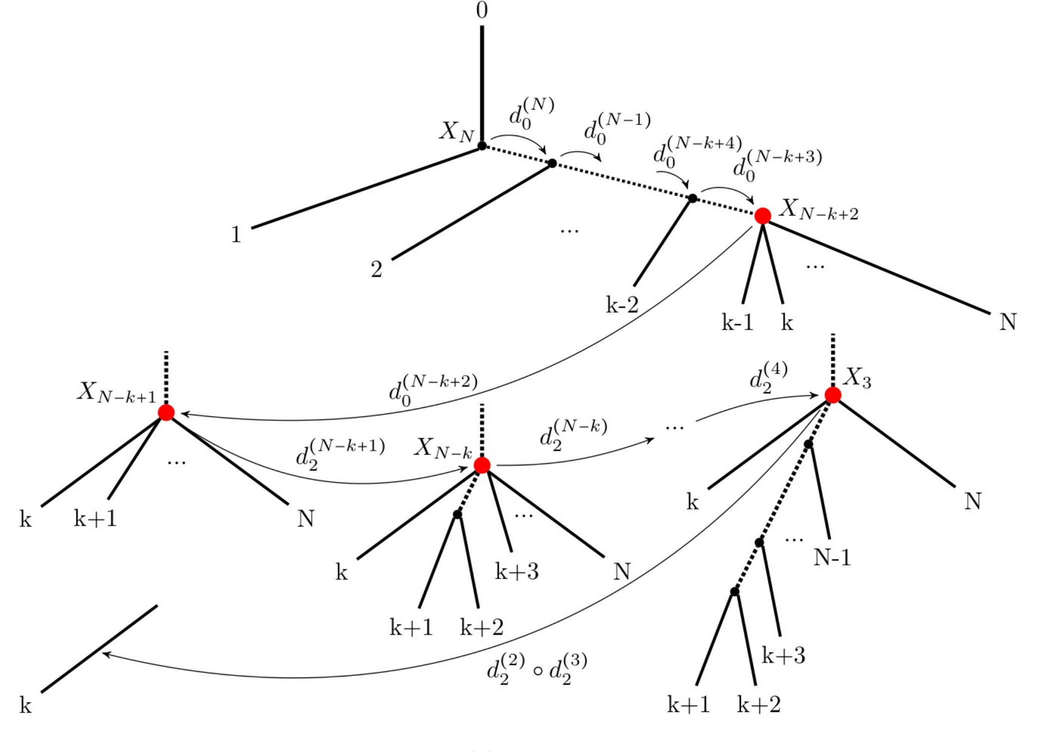
(2) $\exists!\ \varphi_n$ CANONICALLY INDUCED from φ_3 for (G, Φ, φ_3)

SIMPLICIAL

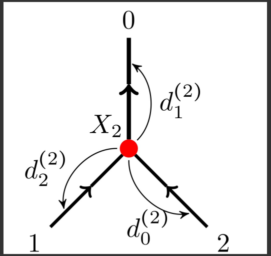
...

INDUCTION through TERNARY RESOLUTIONS of BINARY TREES

A∂(1) e.g., for $\pi^{(N+1)}_{k-1,k}$



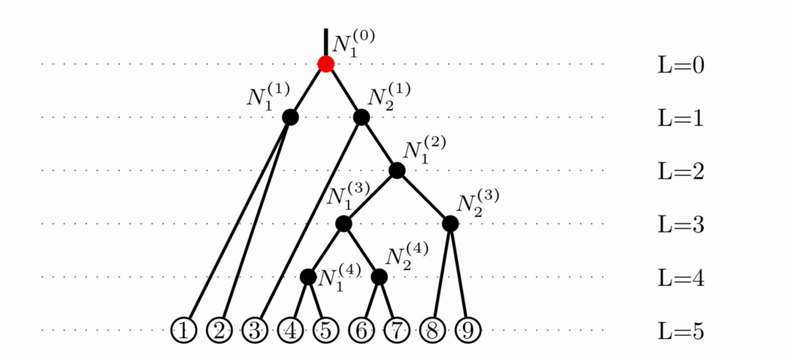
USING

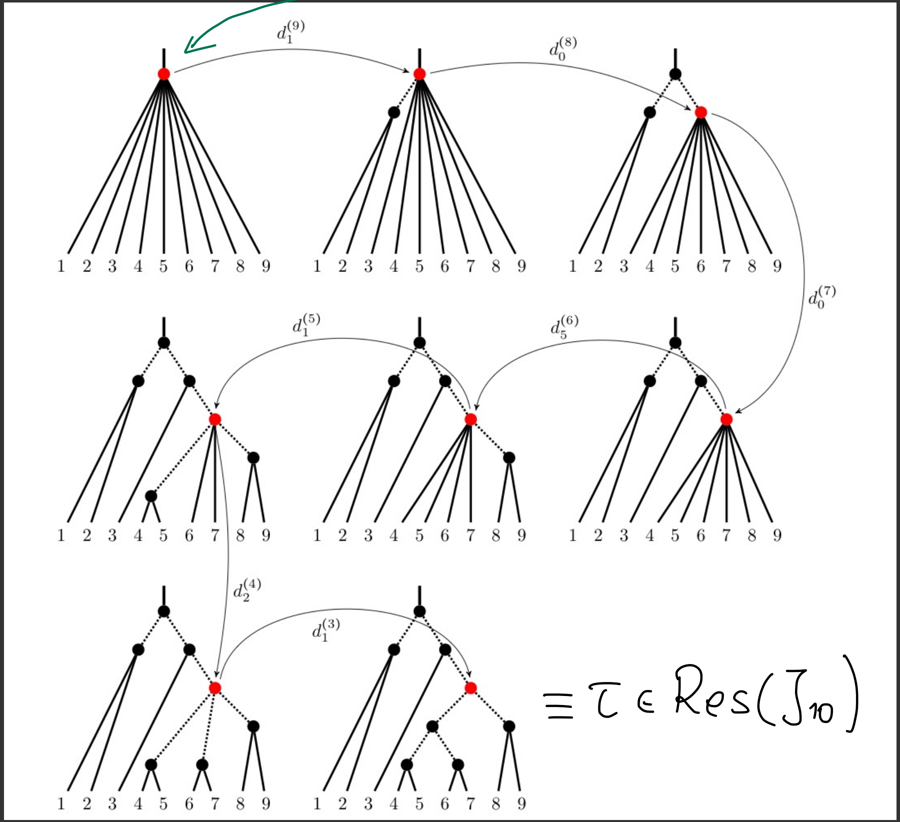


etc.

$A\partial(2)$ e.g., for $n=10$ RESOLVE JUNCTION

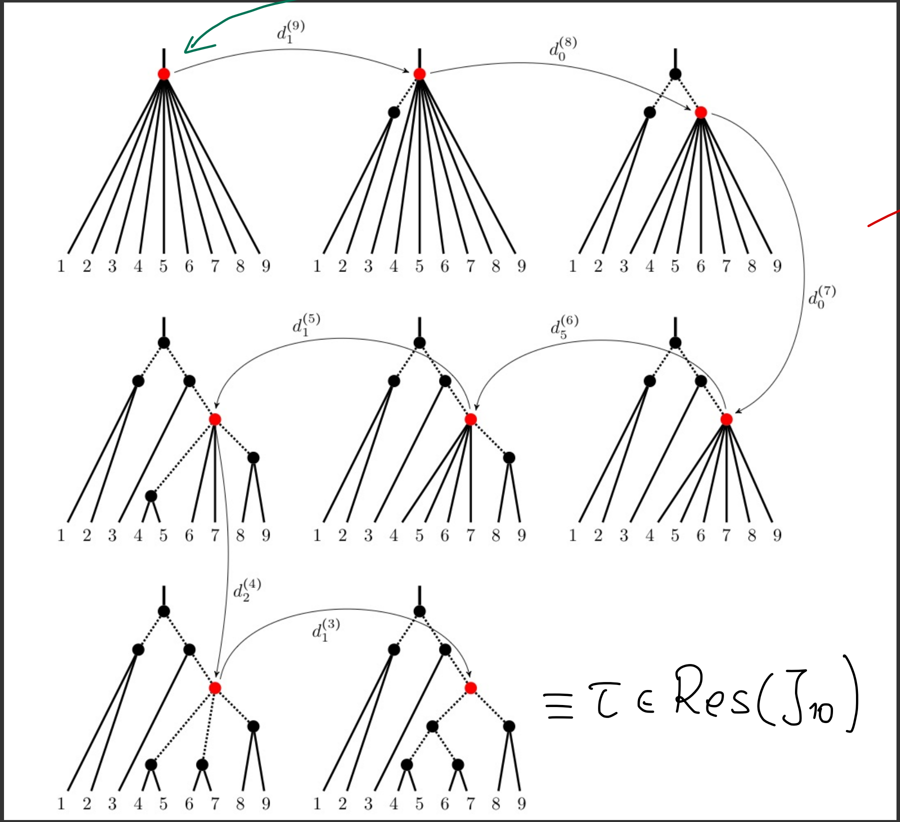
as $\hookrightarrow$



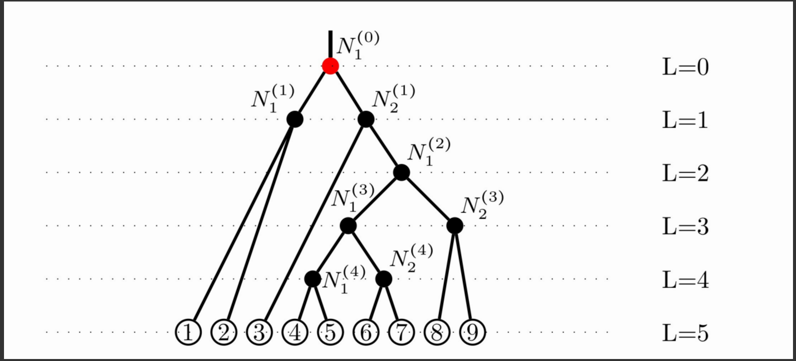


$\equiv \tau \in Res(J_{10})$

A∂(2) e.g., for $n=10$ RESOLVE JUNCTION as ↪



$\equiv \tau \in Res(J_{10})$



OBTAINING

$N_1^{(2)} \quad : \quad d_1^{(3)} \circ d_2^{(4)} \circ d_1^{(5)} \circ d_5^{(6)} \circ d_0^{(7)} \circ d_0^{(8)} \circ d_1^{(9)} \equiv D_1^{(2)}$

AND — ANALOGOUSLY —

$N_1^{(0)} \quad : \quad d_2^{(3)} \circ d_3^{(4)} \circ d_3^{(5)} \circ d_3^{(6)} \circ d_5^{(7)} \circ d_7^{(8)} \circ d_1^{(9)} \equiv D_1^{(0)}$

$N_1^{(1)} \quad : \quad d_3^{(3)} \circ d_3^{(4)} \circ d_4^{(5)} \circ d_4^{(6)} \circ d_4^{(7)} \circ d_6^{(8)} \circ d_8^{(9)} \equiv D_1^{(1)}$

$N_2^{(1)} \quad : \quad d_2^{(3)} \circ d_2^{(4)} \circ d_3^{(5)} \circ d_2^{(6)} \circ d_6^{(7)} \circ d_0^{(8)} \circ d_1^{(9)} \equiv D_2^{(1)}$

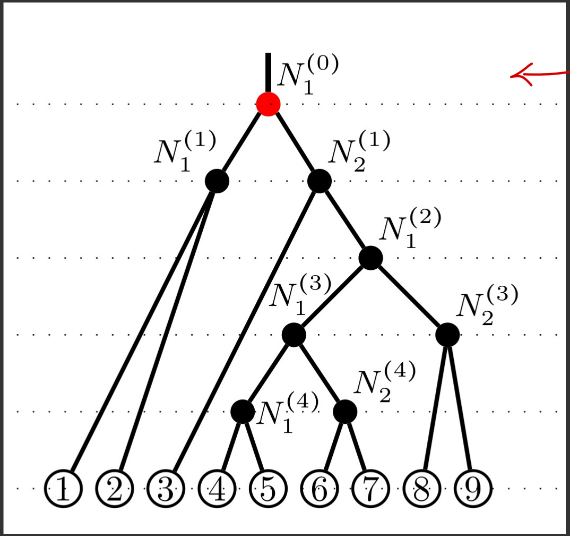
$N_1^{(3)} \quad : \quad d_2^{(3)} \circ d_4^{(4)} \circ d_1^{(5)} \circ d_0^{(6)} \circ d_6^{(7)} \circ d_0^{(8)} \circ d_1^{(9)} \equiv D_1^{(3)}$

$N_2^{(3)} \quad : \quad d_0^{(3)} \circ d_1^{(4)} \circ d_2^{(5)} \circ d_1^{(6)} \circ d_0^{(7)} \circ d_0^{(8)} \circ d_1^{(9)} \equiv D_2^{(3)}$

$N_1^{(4)} \quad : \quad d_0^{(3)} \circ d_4^{(4)} \circ d_3^{(5)} \circ d_0^{(6)} \circ d_0^{(7)} \circ d_7^{(8)} \circ d_1^{(9)} \equiv D_1^{(4)}$

$N_2^{(4)} \quad : \quad d_0^{(3)} \circ d_4^{(4)} \circ d_1^{(5)} \circ d_0^{(6)} \circ d_0^{(7)} \circ d_7^{(8)} \circ d_1^{(9)} \equiv D_2^{(4)}$

USING THESE MAPS, DEFINE



$$\tau \longmapsto \varphi_{10}[\tau] := D_1^{(0)*}\varphi_3$$

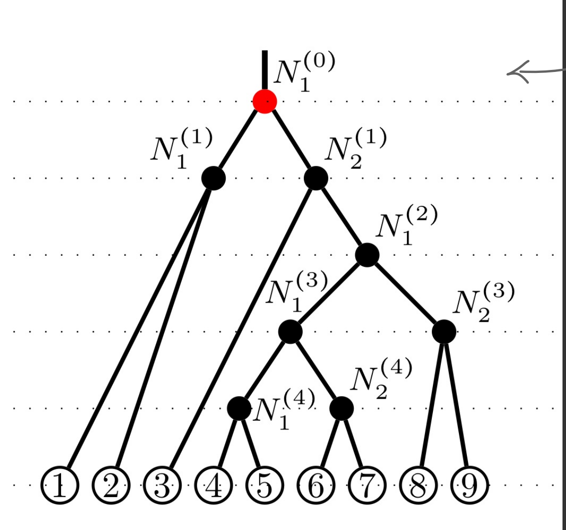
$$\bullet \left(D_2^{(1)*}\varphi_3 \circ D_1^{(1)*}\varphi_3\right)$$

$$\bullet \left(D_1^{(2)*}\varphi_3 \circ Id_{\pi_{2,3}^{(10)*}\Phi} \circ Id_{\pi_{1,2}^{(10)*}\Phi} \circ Id_{\pi_{0,1}^{(10)*}\Phi}\right)$$

$$\bullet \left(D_2^{(3)*}\varphi_3 \circ D_1^{(3)*}\varphi_3 \circ Id_{\pi_{2,3}^{(10)*}\Phi} \circ Id_{\pi_{1,2}^{(10)*}\Phi} \circ Id_{\pi_{0,1}^{(10)*}\Phi}\right)$$

$$\bullet \left(Id_{\pi_{8,9}^{(10)*}\Phi} \circ Id_{\pi_{7,8}^{(10)*}\Phi} \circ D_2^{(4)*}\varphi_3 \circ D_1^{(4)*}\varphi_3 \circ Id_{\pi_{2,3}^{(10)*}\Phi} \circ Id_{\pi_{1,2}^{(10)*}\Phi} \circ Id_{\pi_{0,1}^{(10)*}\Phi}\right)$$

USING THESE MAPS, DEFINE



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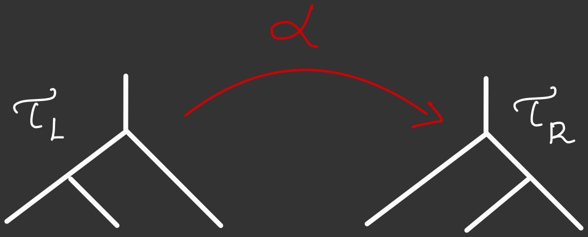
$$\bullet \left(Id_{\pi_{8,9}^{(10)*}\Phi} \circ Id_{\pi_{7,8}^{(10)*}\Phi} \circ D_2^{(4)*}\varphi_3 \circ D_1^{(4)*}\varphi_3 \circ Id_{\pi_{2,3}^{(10)*}\Phi} \circ Id_{\pi_{1,2}^{(10)*}\Phi} \circ Id_{\pi_{0,1}^{(10)*}\Phi}\right)$$

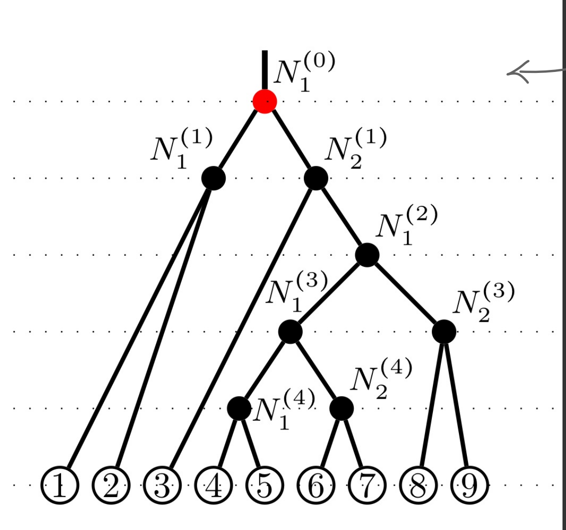
OBSERVATION: $\forall \tau, \tau' \in Res(J_N)$

$\exists < \infty$ SEQUENCE of ASSOCIATOR MOVES

WHICH SENDS $\tau' \longmapsto \tau$



USING THESE MAPS, DEFINE



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$$\bullet \left(D_2^{(3)*}\varphi_3 \circ D_1^{(3)*}\varphi_3 \circ \mathrm{Id}_{\pi_{2,3}^{(10)*}\Phi} \circ \mathrm{Id}_{\pi_{1,2}^{(10)*}\Phi} \circ \mathrm{Id}_{\pi_{0,1}^{(10)*}\Phi}\right)$$

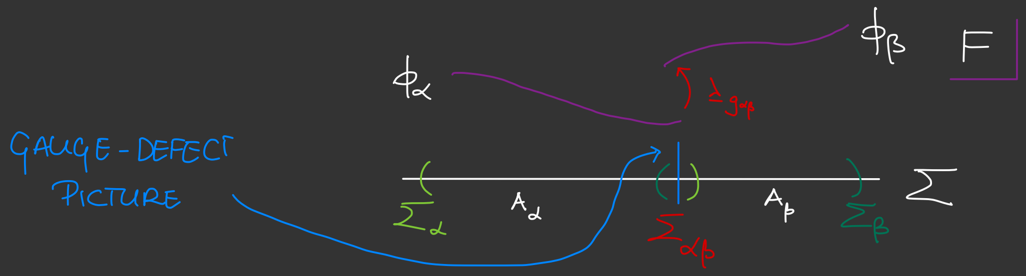
$$\bullet \left(\mathrm{Id}_{\pi_{8,9}^{(10)*}\Phi} \circ \mathrm{Id}_{\pi_{7,8}^{(10)*}\Phi} \circ D_2^{(4)*}\varphi_3 \circ D_1^{(4)*}\varphi_3 \circ \mathrm{Id}_{\pi_{2,3}^{(10)*}\Phi} \circ \mathrm{Id}_{\pi_{1,2}^{(10)*}\Phi} \circ \mathrm{Id}_{\pi_{0,1}^{(10)*}\Phi}\right)$$

OBSERVATION: $\forall \tau, \tau' \in \mathrm{Res}(J_N)$ $\exists < \infty$ SEQUENCE of ASSOCIATOR MOVES WHICH SENDS $\tau' \longmapsto \tau$

CONCLUSION: IMPOSE $\varphi_4[\tau_L] \overset{!}{=} \varphi_4[\tau_R]$ for UNIQUENESS, but...

... THIS IS PRECISELY COHERENCE for $(\mathcal{G}, \Phi, \varphi_3)$!

A CONCRETE & USEFUL EXAMPLE : RECALL THE MOTIVATING



WE READILY RECOVER SIMPLICIAL GAUGE-DEFECT CONFIGURATION SPACE :

$$\begin{tikzcd}
\Sigma_\Gamma : \cdots \arrow[r, shift left=2] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r] \arrow[r, shift right=2] & \coprod\limits_{\alpha_i\in A} V_{\alpha_1\alpha_2\dots\alpha_n} \arrow[r, shift left=2] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "{(g_{\alpha_n\alpha_{n-1}}, g_{\alpha_{n-1}\alpha_{n-2}}, \dots, g_{\alpha_2\alpha_1}, x_{\alpha_1})}"'] & \cdots \coprod\limits_{\alpha,\beta,\gamma,\delta\in A} V_{\alpha\beta\gamma\delta} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "{(g_{\alpha_4\alpha_3}, g_{\alpha_3\alpha_2}, g_{\alpha_2\alpha_1}, x_{\alpha_1})}"'] & \coprod\limits_{\alpha,\beta,\gamma\in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "{(g_{\gamma\beta}, g_{\beta\alpha}, x_\alpha)}"'] & \coprod\limits_{\alpha,\beta\in A} E_{\alpha\beta} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "{(g_{\beta\alpha}, x_\alpha)}"'] & \coprod\limits_{\alpha\in A} \Sigma_\alpha \arrow[d, "x_\alpha"'] \\
N_\bullet(G\ltimes_\lambda M) : \cdots \arrow[r, shift left=2, "\tilde{d}_\bullet^{(n)}"] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r] \arrow[r, shift right=2] & G^{\times n-1}\times M \arrow[r, shift left=2, "\tilde{d}_\bullet^{(n-1)}"] \arrow[r, phantom, shift left=1, "\vdots"] \arrow[r] \arrow[r, shift right=2] & \cdots G^{\times 3}\times M \arrow[r, shift left=3, "\tilde{d}_\bullet^{(3)}"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{\times 2}\times M \arrow[r, shift left=2, "\tilde{d}_\bullet^{(2)}"] \arrow[r] \arrow[r, shift right=2] & G\times M \arrow[r, shift left, "\tilde{d}_\bullet^{(1)}"] \arrow[r, shift right] & M
\end{tikzcd}$$

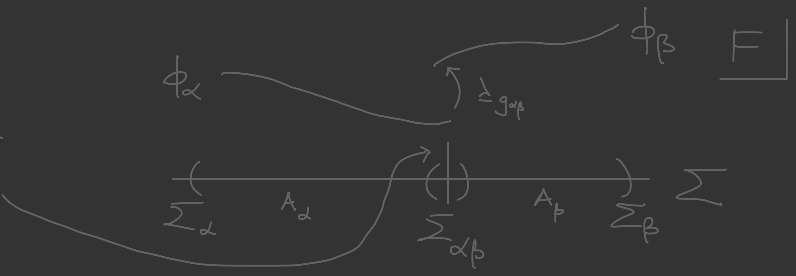
NERVE
of ACTION GROUPOID

$$\underbrace{G\times M \rightrightarrows M}_{G\ltimes_\lambda M}$$

ACTION GROUPOID

A CONCRETE & USEFUL EXAMPLE: RECALL THE MOTIVATING

GAUGE-DEFECT PICTURE



WE READILY RECOVER SIMPLICIAL GAUGE-DEFECT CONFIGURATION SPACE:

$$\begin{tikzcd}
\Sigma_\Gamma : \cdots \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & \bigsqcup_{\alpha_i \in A} V_{\alpha_1\alpha_2\ldots\alpha_n} \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] \arrow[d, "(g_{\alpha_n\alpha_{n-1}}, g_{\alpha_{n-1}\alpha_{n-2}}, \ldots, g_{\alpha_2\alpha_1}, x_{\alpha_1})"'] & \cdots & \bigsqcup_{\alpha,\beta,\gamma,\delta \in A} V_{\alpha\beta\gamma\delta} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[d, "(g_{\alpha_4\alpha_3}, g_{\alpha_3\alpha_2}, g_{\alpha_2\alpha_1}, x_{\alpha_1})"'] & \bigsqcup_{\alpha,\beta,\gamma \in A} V_{\alpha\beta\gamma} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "(g_{\gamma\beta}, g_{\beta\alpha}, x_\alpha)"'] & \bigsqcup_{\alpha,\beta \in A} E_{\alpha\beta} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(g_{\beta\alpha}, x_\alpha)"'] & \bigsqcup_{\alpha \in A} \Sigma_\alpha \arrow[d, "x_\alpha"'] \\
N_\bullet(G \ltimes_\lambda M) : \cdots \arrow[r, shift left=2, "d_\cdot^{(n)}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & G^{\times n-1} \times M \arrow[r, shift left=2, "d_\cdot^{(n-1)}"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[r, shift right=4] & \cdots & G^{\times 3} \times M \arrow[r, shift left=3, "d_\cdot^{(3)}"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{\times 2} \times M \arrow[r, shift left=2, "d_\cdot^{(2)}"] \arrow[r] \arrow[r, shift right=2] & G \times M \arrow[r, shift left, "d_\cdot^{(1)}"] \arrow[r, shift right] & M
\end{tikzcd}$$

NERVE of ACTION GROUPOID

$$\underbrace{G \times M \rightrightarrows M}_{G \ltimes_\lambda M}$$

ACTION GROUPOID

FIXING THE SIMPLICIAL GERBE:

(*) IDENTIFICATION of TANGENTIAL RIGID SYMMETRIES

(**) MINIMAL CORRECTION of $\mathcal{A}_{DF}$ for P_G TRIVIAL

⊛ CONTINUOUS RIGID SYMMETRIES $\subset \mathrm{Isom}(M,g)$

$$\forall_{\substack{x\in[\Sigma,M] \\ a\in\overline{1,\dim_{\mathbb{R}}\mathfrak{g}}}} \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_{DF}\left[\Phi_{\mathcal{K}_a}(t,x(\cdot))\right] = 0 \overset{}{\underset{\text{ChS}}{\iff}} \int_{x(\Sigma)} \mathcal{K}_a \lrcorner H = 0 \underset{\partial\Sigma=\emptyset}{\iff} \exists\, \underset{\text{COMOMENTUM}}{\kappa_A \in \Omega^1(M)} : \mathcal{K}_A \lrcorner H = -d\kappa_A$$

$\mathcal{K}_a(\cdot) \equiv T_{(e,\cdot)}\lambda(-t_a, 0_{TM}(\cdot))$, $\{t_a\}_{a\in\overline{1,\dim_{\mathbb{R}}\mathfrak{g}}}$ BASIS of $\mathfrak{g}$

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UPSHOT: TANGENTIAL SYMMETRIES : $K_a := (\mathcal{K}_a, \kappa_a) \in \Gamma(E^{1,1}M)$, where $E^{1,1}M = TM \oplus T^*M$ HITCHIN's GENERALISED TANGENT BUNDLE

with ALGEBRAIC STRUCTURE :

$$[\![\cdot,\cdot]\!]_H : \Gamma(E^{1,1}M)^{\times 2} \longrightarrow \Gamma(E^{1,1}M)$$

$$: \big((\mathcal{V}_1,\omega_1),(\mathcal{V}_2,\omega_2)\big) \longmapsto \Big([\mathcal{V}_1,\mathcal{V}_2]_{\Gamma(TM)} \,,\, \pounds_{\mathcal{V}_1}\omega_2 - \pounds_{\mathcal{V}_2}\omega_1 - \tfrac{1}{2}d(\mathcal{V}_1\lrcorner\omega_2 - \mathcal{V}_2\lrcorner\omega_1) + \mathcal{V}_1\lrcorner\mathcal{V}_2\lrcorner H\Big)$$

H-TWISTED COURANT BRACKET [ŠEVERA-WEINSTEIN]

WHICH CLOSES on TANGENTIAL SYMMETRIES ! (cp. [ALEKSEEV-STROBL])

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H-TWISTED
COURANT BRACKET
[ŠEVERA-WEINSTEIN]

WHICH CLOSES on TANGENTIAL SYMMETRIES! (cp. [ALEKSEEV-STROBL])

(**) MINIMAL CORRECTION:

$$\begin{tikzcd} \mathcal{G} \arrow[d] & pr_2^*\mathcal{G} \otimes \mathcal{I}_{\varrho_{A_\alpha}} \arrow[d] \\ M \arrow[ur] & \Sigma_\alpha \times M \end{tikzcd}$$

with $\varrho_{A_\alpha} = pr_2^*\kappa_a \wedge pr_1^* A^a_\alpha - \frac{1}{2} pr_2^*(\mathcal{K}_a \lrcorner \kappa_b)\, pr_1^*(A^a_\alpha \wedge A^b_\alpha)$

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\mathcal{G} \arrow[d] & & \mathrm{pr}_2^*\mathcal{G}\otimes\mathcal{I}_{\varrho_{A_\alpha}} \arrow[d] \\
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with $\varrho_{A_\alpha}=\mathrm{pr}_2^*\kappa_a\wedge\mathrm{pr}_1^*A^a_\alpha-\frac{1}{2}\mathrm{pr}_2^*(\mathcal{K}_a\lrcorner\kappa_b)\,\mathrm{pr}_1^*(A^a_\alpha\wedge A^b_\alpha)$

MUST BE SUCH THAT

$(\mathrm{SGA}=0)$

ACTION ALGEBROID

$$\left(\langle\mathfrak{K}_a,\ a\in\overline{1,\dim_{\mathbb{R}}\mathfrak{g}}\rangle_{C^\infty(M,\mathbb{R})},\ [\![\cdot,\cdot]\!]_H,\ \mathrm{pr}_1\right)\simeq\mathfrak{g}_\lambda\ltimes M$$

INVARIANCE under LARGE GAUGE TRANSFORMATIONS CALLS for

G- EQUIVARIANT STRUCTURE on $\mathcal{G}$ $\equiv$ SIMPLICIAL GERBE

$$\begin{tikzcd}(\mathcal{G}, \mathcal{I}, \upsilon) \arrow[d, two heads] \\ N_\bullet(G \ltimes_\lambda M)\end{tikzcd},$$

to BE PULLED BACK to GAUGE-SYMMETRY DEFECT $\Gamma \subset \Sigma$ [RRS]

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THE GLOBAL MECHANISM [GAWĘDZKI-WALDORF-RRS]:

$$\mathrm{BGrb}_\nabla(X/G) \simeq \mathrm{BGrb}_\nabla(X)^{G\text{-equiv}}_{\varrho=0}$$

CURVATURE of Υ

WORKS for $X = P_G \times M$ & $\mathcal{G}_\mathcal{A} = \mathrm{pr}_2^*\mathcal{G} \otimes \mathcal{I}_{\varrho_\mathcal{A}}$, where $\mathcal{A} \in \Omega^1(P_G) \otimes_\mathbb{R} \mathfrak{g}$ PRINCIPAL CONNECTION 1-FORM

PRINCIPAL G-BUNDLE

OBSTRUCTION CLASS in $H^2(G \times M, U(1))$ & $H^1(G^{\times 2} \times M, U(1))$

CLASSIFYING GROUP $H^1(G \times M, U(1)) \times H^0(G^{\times 2} \times M, U(1))$ INEQUIVALENT GAUGINGS

$\underline{\overline{\mathrm{V}}}$ BARELY CLOSE, & YET – A HABANO

1) THE GEOMETRY: NATURAL SETTING for DISCUSSION of IMPLEMENTATION of SYMMETRY by DEFECTS

$$\begin{tikzcd}
\mathrm{LieGrp}_{\mathrm{CS1C}} \ni G \equiv M \arrow[d, phantom, "\vert\vert\vert" description] & & Q \subset G \times G \arrow[ll, shift right, "d_0^{(1)} = \mathrm{pr}_1"'] \arrow[ll, shift left, "d_1^{(1)} = r \equiv m"] \arrow[d, phantom, "\vert\vert\vert" description] & \text{RIGHT-TRANSLATIONAL SYMMETRY} \\
\mathrm{Ob}(G \rtimes_r G) \arrow[dr, bend left] & & \mathrm{Mor}(G \rtimes_r G) \arrow[dl, bend right] & \\
& M_\bullet \subset N_\bullet(G \rtimes_r G) & &
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RIGHT-TRANSLATIONAL SYMMETRY

BUT M IS A $G^{\times 2}$-MANIFOLD, with $\lambda : G^{\times 2} \times M \longrightarrow M : ((a,b),h) \longmapsto a \cdot h \cdot b^{-1}$

$\Longrightarrow$ AUGMENTATION: SIMPLICIALITY $\nearrow$ $G^{\times 2}$-SYMMETRIC SIMPLICIALITY: by DECLARING $d_\cdot^{(\cdot)}$ $G^{\times 2}$-EQUIVARIANT

PROPAGATE $\lambda^{(0)} \equiv \lambda$

$$\Longrightarrow \lambda^{(n)} : G^{\times 2} \times M_n \longrightarrow M_n : \left((a,b),h,k_1,k_2,\ldots,k_n\right) \longmapsto \left(\lambda((a,b),h), Ad_b(k_1), Ad_b(k_2), \ldots, Ad_b(k_n)\right)$$

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$$\Longrightarrow \lambda^{(n)} : G^{\times 2} \times M_n \longrightarrow M_n : ((a,b),h,k_1,k_2,\ldots,k_n) \longmapsto (\lambda((a,b),h), \mathrm{Ad}_b(k_1), \mathrm{Ad}_b(k_2), \ldots, \mathrm{Ad}_b(k_n))$$

FURTHER NATURAL AUGMENTATION: $G^{\times 2}$-SYMMETRIC SIMPLICIALITY $\nearrow$ SEMISIMPLE $G^{\times 2}$-SYMMETRIC SIMPLICIALITY $(3 \times S)$

$$\Longrightarrow M_0 = G, \quad M_1 \subset G \times \bigsqcup_{\lambda \in P_+(\mathfrak{g})} C_\lambda, \quad M_n \subset G \times \bigsqcup_{\vec{\lambda} \in P_+(\mathfrak{g})^{\times n}} \bigsqcup_{\mu \in P_+(\mathfrak{g})} \bigsqcup_{\delta \in \mathcal{D}^\mu_{\vec{\lambda}}} \mathcal{J}^\mu_{\vec{\lambda}}[\delta] \quad : \quad \mathcal{J}^\mu_{\vec{\lambda}}[\delta] \subset \mathop{\times}_{\vec{\lambda} \in P_+(\mathfrak{g})^{\times n}} C_{\lambda_i} \cap m^{-1}(C_\mu)$$

2) THE DIFFERENTIAL GEOMETRY (TENSORIAL BACKGROUND)

CANONICAL PHASE BACKGROUND : $$g_k = \frac{k}{8\pi}\underbrace{\delta_{ab}\,\Theta_L^a \otimes \Theta_L^b}_{\text{CARTAN - KILLING METRIC}} \,, \qquad H_k = -\frac{k}{48\pi}\underbrace{f_{abc}\,\Theta_L^a \wedge \Theta_L^b \wedge \Theta_L^c}_{\text{CARTAN 3-FORM}} \,, \qquad k \in \mathbb{R}_+$$

$\Longrightarrow$ WESS-ZUMINO-NOVIKOV-WITTEN σ-MODEL of 2d RCFT (EFFECTIVE LFT of SLOW SPINONS in G-SPIN CHAINS ...)

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OBSERVATIONS : (*) g_k & H_k ARE λ-INVARIANT

(**) CONORMALISATION $\Rightarrow$ $(G^{\times 2}, \lambda) \nearrow (LG^{\times 2}, L\lambda)$ in WZNW

(***) H_1 GENERATES $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$!

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DEMAND THAT $L\lambda^{(1)} (\nwarrow \lambda^{(1)})$ PRESERVE (DGC) on $G \times C_\lambda \supset Q_\lambda$ TO FIX

$$\omega_k\big|_{Q_\lambda} = \varrho_k\big|_{Q_\lambda} - pr_2^* \omega^\partial_{k,\lambda} \, , \quad \text{where} \quad \begin{aligned} &(m^* - pr_1^* - pr_2^*) H_k = -d\varrho_k \\ &\iota_\lambda^* H_k = d\omega^\partial_{k,\lambda} \end{aligned}$$

POLYAKOV-WIEGMANN IDENTITY

TRIVIALISATION $\sim \partial$

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$$\iota_\lambda^* H_k = d\omega_{k,\lambda}^{\partial}$$ TRIVIALISATION $\sim \partial$

QUESTION : WHAT about CURVATURE CONSTRAINT (CC) $\tilde{\Delta}^{(2)} \omega_k \overset{!}{=} 0$ in $G \times C_{\lambda_1} \times C_{\lambda_2}$?

'TECHNICAL' FACT: (CC) on $T_n \underset{[RRS]}{\Longleftrightarrow} \Omega_{k,\vec{\lambda}}\Big|_{\bigtimes_{\vec{\lambda}\in P_+(\mathfrak{g})^{\times n}} C_{\lambda_i} \cap m^{-1}(C_\mu)} = 0$

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$\underline{\subset}: \mathrm{CS}(\vec{\lambda};\mu)$

STRAY OBSERVATION: $\mathrm{CS}(\vec{\lambda};\mu)$ IS THE PHASE SPACE of ...

$$S_{\mathrm{CS|W}}[\mathcal{A},\{\gamma_i\}] = -\frac{\mathsf{k}}{4\pi}\int_{\mathbb{R}\times\mathbb{S}^2}\mathrm{tr}_{\mathfrak{g}}\big(\mathcal{A}\wedge\mathsf{d}\mathcal{A}+\tfrac{2}{3}\,\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)+\mathsf{k}\sum_{i=1}^{\overset{3}{\cancel{n+1}}}\int_{\ell_i}\mathrm{tr}_{\mathfrak{g}}\big[\overset{\lambda_3=\mu}{\lambda_i}\big(\mathsf{T}_e\mathrm{Ad}_{\gamma_i}\circ\mathcal{A}+\gamma_i^*\theta_{\mathrm{L}}\big)\big]$$

TFT for $\begin{cases} \mathcal{A} \text{ - PRINCIPAL CONNECTION 1-FORM on TRIVIAL } P_G\text{, with } \delta\text{-SOURCES of CURVATURE @ the } \ell_i \equiv \mathbb{R}\times\{P_i\} \\ \gamma_i \in [\mathbb{R}\times\{P_i\}, G] \end{cases}$

3d CHERN-SIMONS TFT on $\mathbb{R}\times\mathbb{S}^2$ with WILSON LINES through $P_1, P_2, P_3 \in \mathbb{S}^2$

[WITTEN & al.; GAWĘDZKI]

& $\Omega_{k,\vec{\lambda}}$ IS... ITS PRESYMPLECTIC FORM, PARTIALLY SYMPLECTICALLY REDUCED wrt. $\Gamma(\mathrm{Ad}P_G) \searrow G$ À LA [ALEKSEEV-MALKIN]

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CONCLUSION: THE G-ORBITS $\mathcal{J}_{\vec{\lambda}}^{\mu}[\delta] \subset CS(\vec{\lambda};\mu)$ ARE MAXIMAL (PHASE-STRIPPED) FUSION-&-COHERENCE STRATA

FURTHER CONSTRAINTS COME FROM...

3) THE HIGHER GEOMETRY

UN-OBSTRUCTED & UNIQUE GEOMETRISATION of THE $\lambda^{(\cdot)}$-INVARIANT SIMPLICIAL 3-FORM $(H_k, \omega_k, 0)$ $\exists$ ONLY for $k \in \underset{\text{LEVEL}}{\mathbb{Z}}$:

(*) $H_k = \mathrm{curv}\big(\underset{=: \mathcal{G}_k}{\mathcal{G}_b^{\otimes k}}\big)$ for $\boxed{\mathcal{G}_b}$ THE GAWĘDZKI-HITCHIN-MEINRENKEN BASIC GERBE

with $\lambda^*_{(a,b)} \mathcal{G}_b \cong \mathcal{G}_b$

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$\exists$ ONLY for $k \in \mathbb{Z}$:

LEVEL

(*) $H_k = \mathrm{curv}\big(\underbrace{\mathcal{G}_b^{\otimes k}}_{=:\, \mathcal{G}_k}\big)$ for $\mathcal{G}_b$ THE GAWĘDZKI-HITCHIN-MEINRENKEN BASIC GERBE with $\lambda_{(a,b)}^{*}\mathcal{G}_b \simeq \mathcal{G}_b$

(**) $\mathcal{E}_{k,\lambda}$:

$$\begin{tikzcd}[column sep=small]
d_0^{(1)*}\mathcal{G}_k \equiv \mathrm{pr}_1^*\mathcal{G}_k \equiv \mathrm{pr}_1^*\mathcal{G}_k \otimes \mathrm{pr}_2^*\mathcal{G}_k \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \arrow[d, "\mathcal{M}\otimes \mathrm{id}"', "\simeq"] \\
m^*\mathcal{G}_k \otimes \mathcal{I}_{\rho_k} \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \equiv \\
d_1^{(1)*}\mathcal{G}_k \otimes \mathcal{I}_{\omega_{k,\lambda}} \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \otimes \mathcal{I}_{\mathrm{pr}_2^*\omega^{\partial}_{k,\lambda}} \arrow[d, "\simeq"', "\mathrm{id}\otimes \mathrm{pr}_2^*\mathcal{J}_{k,\lambda}^{-1}\otimes \mathrm{id}"] \\
d_1^{(1)*}\mathcal{G}_k \otimes \mathcal{I}_{\omega_{k,\lambda}}
\end{tikzcd}$$

$\exists!$

MULTIPLICATIVE STRUCTURE

(SIMPLICIAL) CATEGORIFICATION of m

3) THE HIGHER GEOMETRY

UN-OBSTRUCTED & UNIQUE GEOMETRISATION of THE $\lambda^{(\cdot)}$-INVARIANT SIMPLICIAL 3-FORM $(H_k, \omega_k, 0)$

∃ ONLY for $k \in \mathbb{Z}$:

LEVEL

(*) $H_k = \mathrm{curv}\left(\mathcal{G}_b^{\otimes k}\right) =: \mathcal{G}_k$ for $\mathcal{G}_b$

THE GAWĘDZKI-HITCHIN-MEINRENKEN BASIC GERBE

with $\lambda_{(a,b)}^* \mathcal{G}_b \simeq \mathcal{G}_b$

(**) $\Phi_{k,\lambda}$:

$$\begin{tikzcd}
d_0^{(1)*}\mathcal{G}_k \equiv \mathrm{pr}_1^*\mathcal{G}_k \equiv \mathrm{pr}_1^*\mathcal{G}_k \otimes \mathrm{pr}_2^*\mathcal{G}_k \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \arrow[d, "\simeq", "\mathcal{M}\otimes\mathrm{id}"'] \\
m^*\mathcal{G}_k \otimes \mathcal{I}_{\rho_k} \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \equiv \\
d_1^{(1)*}\mathcal{G}_k \otimes \mathcal{I}_{\omega_{k,\lambda}} \otimes \mathrm{pr}_2^*\mathcal{G}_k^\vee \otimes \mathcal{I}_{\mathrm{pr}_2^*\omega^2_{k,\lambda}} \arrow[d, "\mathrm{id}\otimes\mathrm{pr}_2^*\mathcal{J}_{k,\lambda}^{-1}\otimes\mathrm{id}", "\simeq"'] \\
d_1^{(1)*}\mathcal{G}_k \otimes \mathcal{I}_{\omega_{k,\lambda}}
\end{tikzcd}$$

∃!

MULTIPLICATIVE STRUCTURE

(SIMPLICIAL) CATEGORIFICATION of m

∃!

ONLY for $\lambda \in P_+^k(\mathfrak{g})$

FUNDAMENTAL AFFINE WEYL ALCOVE

$\Downarrow$

$$\Phi_k = \bigsqcup_{\lambda \in P_+^k(\mathfrak{g})} \Phi_{k,\lambda}$$

over

$$Q = G \times \bigsqcup_{\lambda \in P_+^k(\mathfrak{g})} C_\lambda$$

(***) $\exists\, \varphi_{n+1}\big|_{G \times \mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \Longleftrightarrow \exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta]$ on $\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$, where

$$\varphi^{\mu}_{(\lambda_1,\lambda_2)}[\delta] : pr_1^* \mathcal{T}_{k,\lambda_1} \otimes pr_2^* \mathcal{T}_{k,\lambda_2}\big|_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \xRightarrow{\cong} (m^* \mathcal{T}_{k,\mu} \otimes id) \circ \mathcal{M}\big|_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]}$$ etc. (cp. [CAREY-WONG])

$(***)$ $\exists\, \varphi_{n+1}\restriction_{G\times\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \iff \exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta]$ on $\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$, where

$$\varphi^{\mu}_{(\lambda_1,\lambda_2)}[\delta] : \mathrm{pr}_1^*\mathcal{J}_{k,\lambda_1}\otimes\mathrm{pr}_2^*\mathcal{J}_{k,\lambda_2}\restriction_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \xRightarrow{\cong} \left(m^*\mathcal{J}_{k,\mu}\otimes\mathrm{id}\right)\circ\mathcal{M}\restriction_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]}$$

etc. (cp. [CAREY-WONG])

UPSHOT : GERBE THEORY SELECTS $M_n = G\times\bigsqcup_{(\vec{\lambda},\mu)\in\mathcal{F}(n+1)}\ \bigsqcup_{\delta\in\Delta^{\mu}_{\vec{\lambda}}}\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$ with $\mathcal{F}(n+1)\subset P_+^k(\mathfrak{g})^{\times n+1}$

& $\Delta^{\mu}_{\vec{\lambda}}\subset\mathcal{D}^{\mu}_{\vec{\lambda}}$ SUCH THAT

$$\exists\, \varphi^{\mu}_{\vec{\lambda}}[\delta]\ !$$

(***) $\exists\, \varphi_{n+1}\restriction_{G\times\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \Longleftrightarrow \exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta]$ on $\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$, where

$$\varphi^{\mu}_{(\lambda_1,\lambda_2)}[\delta] : \mathrm{pr}_1^*\mathcal{J}_{k,\lambda_1}\otimes \mathrm{pr}_2^*\mathcal{J}_{k,\lambda_2}\restriction_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \xRightarrow{\cong} (m^*\mathcal{J}_{k,\mu}\otimes \mathrm{id})\circ\mathcal{M}\restriction_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]}$$

etc. (cp. [CAREY-WONG])

UPSHOT: GERBE THEORY SELECTS $M_n = G \times \bigsqcup_{(\vec{\lambda},\mu)\in\mathcal{F}(n+1)} \bigsqcup_{\delta\in\Delta_{\vec{\lambda}}^{\mu}} \mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$ with $\mathcal{F}(n+1)\subset P_+^k(\mathfrak{g})^{\times n+1}$

& $\Delta_{\vec{\lambda}}^{\mu} \subset \mathcal{D}_{\vec{\lambda}}^{\mu}$ SUCH THAT

$\exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta]$!

HYPOTHESIS I: $\begin{cases} (\vec{\lambda},\mu)\in\mathcal{F}(n+1) \Longleftrightarrow \mathcal{N}_{\vec{\lambda}}^{\mu}\neq 0 \\ |\Delta_{\vec{\lambda}}^{\mu}| = \mathcal{N}_{\vec{\lambda}}^{\mu} \end{cases}$ VERLINDE FUSION RULES

HINTS: (i) $G = SU(2)$ ✓ [RUNKEL-RRS]

(ii) FUNCTORIAL QUANTISATION of d W3NW
[FRÖHLICH-FUCHS-RUNKEL-SCHWEIGERT]

(iii) CS vs bW3NW [WITTEN; GAWĘDZKI]

(***) $\exists\, \varphi_{n+1}\big|_{G \times \mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \Longleftrightarrow \exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta]$ on $\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$, where

$$\varphi_{(\lambda_1,\lambda_2)}^{\mu}[\delta] : pr_1^* \mathcal{J}_{k,\lambda_1} \otimes pr_2^* \mathcal{J}_{k,\lambda_2}\big|_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]} \xrightarrow{\cong} (m^* \mathcal{J}_{k,\mu} \otimes \mathrm{id}) \circ \mathcal{M}\big|_{\mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]}$$ etc. (cp. [CAREY-WONG])

UPSHOT: GERBE THEORY SELECTS $M_n = G \times \bigsqcup_{(\vec{\lambda},\mu) \in \mathcal{F}(n+1)} \bigsqcup_{\delta \in \Delta_{\vec{\lambda}}^{\mu}} \mathcal{T}_{\vec{\lambda}}^{\mu}[\delta]$ with $\mathcal{F}(n+1) \subset P_+^k(\mathfrak{g})^{\times n+1}$

& $\Delta_{\vec{\lambda}}^{\mu} \subset \mathcal{D}_{\vec{\lambda}}^{\mu}$ SUCH THAT

$$\exists\, \varphi_{\vec{\lambda}}^{\mu}[\delta] \,!$$

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HYPOTHESIS II: α-PHASES $\sim$ MOORE-SEIBERG FUSING MATRICES of W3NW

HINT: $C_{\lambda_i} \subset Z(G)$ ✓ [RUNKEL-RRS]

SUMMARY: SIMPLICIAL HIGHER-GEOMETRIC STRUCTURES over SIMPLICIAL CONFIGURATION FIBRES

MORAL: MODEL – UPON PULLBACK TO DEFECT-STRATIFIED SPACETIME of THE σ-MODEL — DYNAMICS ON SYMMETRY/DUALITY QUOTIENTS

$$``(M, g, \mathcal{G}) / (Q, \mathfrak{D})"$$

(FOR ACTUAL DESCENT, WE NEED TOPOLOGICALITY OF THE DEFECT. . . .)

LONG SHOT: THE ABOVE IS THE EMERGENT GEOMETRY of THE TARGET SPACETIME AS PROBED with THE CHARGED ELEMENTARY CONFIGURATIONS (LOOPS) of THE σ-MODEL.

(e.g., T-FOLDS)

FURTHER DEVELOPMENTS :

(i) CONSTRUCTION & CLASSIFICATION of PHASES & BOUNDARY DEFECTS FOR NON-1-CONNECTED LIE GROUPS [GAWĘDZKI-REIS] (NON-ABELIAN BD's)

(ii) CONSTRUCTION & CLASSIFICATION of WZNW PHASES OVER NON-ORIENTABLE Σ [SCHREIBER-SCHWEIGERT-WALDORF; GAWĘDZKI-WALDORF-RRS]

(iii) DUALITIES [RRS; NIKOLAUS-WALDORF; ...]

(iv) CONSTRUCTIONS IN THE $\mathbb{Z}/2\mathbb{Z}$-GRADED CATEGORY [DI FIORENZA-SATI-SCHREIBER-STASHEFF; RRS]

(v) GERBE-TWISTED EMERGENT SPECTRAL NCG [RECKNAGEL-RRS]

FUTURE DIRECTIONS :

(i) GAUGE PRINCIPLE for GROUPOIDAL SYMMETRIES

(LOOP MECHANICS on FOLIATIONS) [COLLABO w/ STROBL]

(ii) T-DUALITY EXPLAINED via GAUGING

(T-FOLDS) [COLLABO w/ LAZAROIU]

(iii) DEABELIANISATION of GERBES

[COLLABO w/ LAZAROIU]


СЛАВА УКРАЇНІ
CECI (N')EST (PAS) LA FIN!