Dynamical aspects of classical and quantum multifield cosmological models

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Overview

> All civilizations have an origin myth.
> We are the first to get it right.
>
> David Tong

Drawing from observational data, it's reasonable to conclude that the Universe exhibits homogeneity and isotropy on large scales.

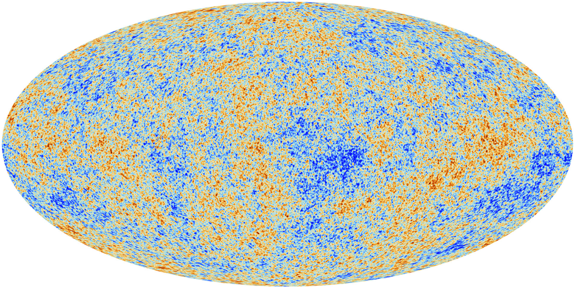


Figure: The anisotropies of the Cosmic microwave background (CMB) as observed by Planck. The CMB is a snapshot of the oldest light in our Universe, imprinted on the sky when the Universe was just 380 000 years old. It shows tiny temperature fluctuations that correspond to regions of slightly different densities, representing the seeds of all future structure: the stars and galaxies of today. Source: ESA and the Planck Collaboration

A homogeneous & isotropic universe is described by the FLRW metric, first derived by Friedmann in 1922:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2) \quad ,$$

where $a(t) > 0$ is the so-called scale factor, whose time evolution is determined by the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad .$$

The 00 and 11 components of Einstein's equation give the *Friedmann equation*:

$$H^2(t) \stackrel{\mathrm{def.}}{=} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{tot}} - \frac{k}{a^2} \quad ,$$

with $\rho_{\mathrm{tot}} = \rho_{\mathrm{m}} + \rho_{\mathrm{rad}} + \rho_{\mathrm{vac}}$, and the equation:

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p \quad .$$

From Einstein's equations one can also derive the First Law of Thermodynamics for Cosmology:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\rho(t) a^3(t)] = -p(t)\frac{\mathrm{d}}{\mathrm{d}t}[a^3(t)] \quad .$$

Depending on the content of the Universe, one can approximate:

$$\rho \sim \frac{1}{a^4} \qquad p = \frac{\rho}{3} \quad \text{(radiation domination)}$$
$$\rho \sim \frac{1}{a^3} \qquad p = 0 \quad \text{(matter domination)}$$
$$\rho \sim \text{const} \quad p = -\rho \quad \text{(vacuum domination)} \quad .$$

Solving for these cases one finds:

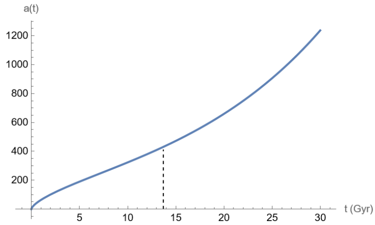


Figure: Stages of evolution in a flat FLRW model for $\Omega_r = \Omega_\nu = \Omega_m = \frac{1}{3}$ (source: Hartle, J.B. (2021) Gravity: An Introduction to Einstein's General Relativity. Cambridge: Cambridge University Press.)

$$a(t) \sim \sqrt{t} \quad \text{(radiation domination)}$$
$$a(t) \sim t^{2/3} \quad \text{(matter domination)}$$
$$a(t) \sim e^{Ht} \quad \text{(vacuum domination)}$$

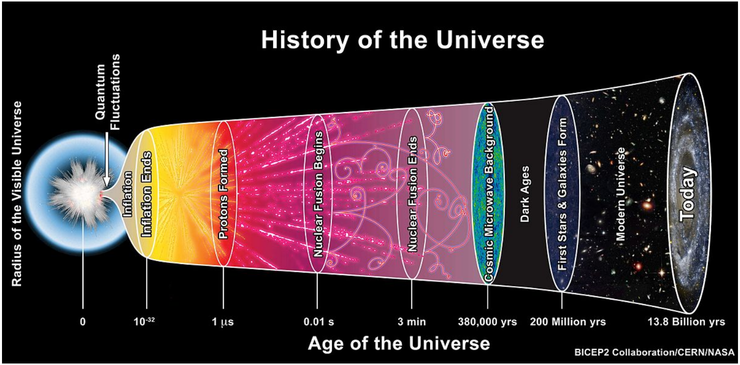
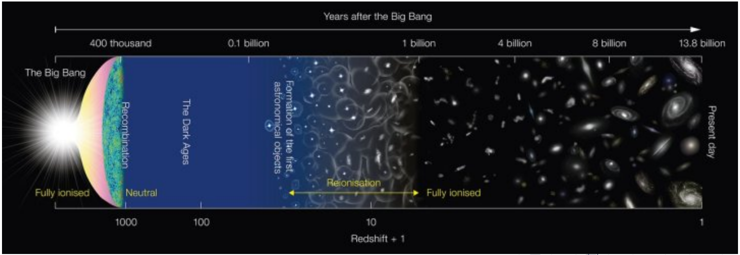


Figure: A rough history of the universe according to standard Λ-CDM cosmology



Definition

A d-dimensional oriented scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected, **oriented** and borderless Riemannian manifold of dimension d (called scalar manifold)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (this ensures conservation of energy)
2. $V > 0$ on $\mathcal{M}$ (this avoids technical problems but can be relaxed)

Each such triple defines a model of gravity coupled to d real scalar fields ("inflatons") on $\mathbb{R}^4$:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x\,\sqrt{|g|}\left[\frac{M^2}{2}R(g) - \frac{1}{2}\mathrm{Tr}_g\varphi^*(\mathcal{G}) - V\circ\varphi\right]$$

where:

$$\mathrm{Tr}_g\varphi^*(\mathcal{G}) \overset{\mathrm{def.}}{=} g^{\mu\nu}\mathcal{G}_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j \quad , \quad \mu,\nu\in\{0,..,3\}\ ,\ i,j\in\{1,d\}$$

Define the *rescaled Planck mass* $M_0 \overset{\mathrm{def.}}{=} \sqrt{\frac{2}{3}}M$, where M is the reduced Planck mass.
Take metric g to describe a spatially flat FLRW universe of scale factor $a(t)$ and the scalar fields $\varphi : \mathbb{R}^4 \to \mathcal{M}$ to depend only on the cosmological time: $\varphi = \varphi(t)$.

Define the *Hubble parameter* $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$ and the *rescaled Hubble function*:

$$\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0} \quad , \quad \mathcal{H}(u) \stackrel{\text{def.}}{=} \frac{1}{M_0}\sqrt{||u||^2 + 2V(\pi(u))} \quad \forall u \in T\mathcal{M} \quad ,$$

where $||u||^2 \stackrel{\text{def.}}{=} \mathcal{G}_{ij} u^i u^j$ (squared norm of u) and $\pi : T\mathcal{M} \to \mathcal{M}$ (bundle projection).

Proposition

When $H > 0$, the equations of motion are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\mathcal{H}(\dot{\varphi}(t))\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0 \quad ,$$

together with the Hubble condition:

$$H(t) = \frac{1}{3M_0}\mathcal{H}(\dot{\varphi}(t)) \quad .$$

$$\nabla_t \dot{\varphi}^i = \ddot{\varphi}^i + \Gamma^i_{jk}\dot{\varphi}^j\dot{\varphi}^k$$

$$\mathrm{grad}_{\mathcal{G}} V = \mathcal{G}^{ij}(\partial_j V)\partial_i \quad , \quad \partial_i := \frac{\partial}{\partial \varphi^i} \quad .$$

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation are called cosmological curves. The cosmological equation defines an autonomous dissipative geometric dynamical system on $T\mathcal{M}$.

For general multifield models

1. Dynamical renormalization and universality (scaling behavior of classical multifield cosmological models by introducing a dynamical renormalization group action which relates their UV and IR limits)
2. Natural observables and dynamical approximations (geometric construction of certain first order natural dynamical observables)
3. Hesse manifolds and Hessian symmetries (Mathematical theory of Noether symmetries which decompose into visible and Hessian ('hidden') symmetries.

For two-field models

1. IR behavior in tame hyperbolizable two-field models
2. Generalized two-field α-attractor models from geometrically finite hyperbolic surfaces (as well as) from hyperbolic triply-punctured sphere
3. Natural coordinates and horizontal approximations in two-field cosmological model (natural local coordinate systems on phase space)
4. A differential consistency condition for slow-roll inflation with rapid-turn (SRRT).

Future prospects:

1. Classical dynamics
   a. Mean field theory for two field cosmological models
   b. IR and UV expansions in two field models
2. Quantum dynamics
   a. Geometric minisuperspace quantization of multifield cosmological models.
   b. Applications to two field models whose scalar manifold is a hyperbolic surface
   c. Applications to the mixed state formalism for minisuperspace.

Definition

A *two-field cosmological model* is a multifield model with $d = 2$, so the target space $\mathcal{M}$ is a (generally non-compact) connected Riemann surface without boundary. We assume that $\mathcal{M}$ is oriented for simplicity.

Slow roll and rapid turn (SRRT) conditions

For simplicity, we take $M = 1$ i.e. $M_0 = \sqrt{\frac{2}{3}}$. Let J be the complex structure determined by $\mathcal{G}$ on $\mathcal{M}$ through the equation:

$$\omega(u, v) = \mathcal{G}(Ju, v) \quad \forall (u, v) \in T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} \quad ,$$

where ω is the volume form of $(\mathcal{M}, \mathcal{G})$. Let $\mathcal{M}_0 \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid (\mathrm{d}V)(m) \neq 0\}$ be the complement of the critical locus.

Definition

The *adapted frame* of $(\mathcal{M}, \mathcal{G}, V)$ is the oriented orthonormal frame (n, τ) of $\mathcal{M}_0$ defined by the vector fields:

$$n \stackrel{\text{def.}}{=} \frac{\mathrm{grad} V}{||\mathrm{grad} V||} \quad , \quad \tau = Jn \quad .$$

In the adapted frame (n, τ) of $(\mathcal{M}, \mathcal{G}, V)$:

Theorem (Anguelova & Lazaroiu, 2022)

A cosmological curve $\varphi : I \to \mathcal{M}_0$ satisfies the sustained rapid turn conditions with third order slow roll at cosmological time $t \in I$ iff the following condition is satisfied at the point $m = \varphi(t)$ of $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2 \quad .$$

$$V_{ij} \stackrel{\text{def.}}{=} \mathrm{Hess}\,V(\partial_i \partial_j) = \partial_i \partial_j V - \Gamma_{ij}^k \partial_k V$$

$$\textit{where} \quad \mathrm{Hess}\,V = \nabla \mathrm{d} V \quad , \quad \nabla = \textit{LC connection on } \mathcal{M}$$

Definition

The *SRRT equation* is the following condition which constrains the target space metric $\mathcal{G}$ and scalar potential V on the noncritical submanifold $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} = 3VV_{nn}^2$$

A metric $\mathcal{G}$ on $\mathcal{M}_0$ which satisfies this equation for a fixed scalar potential V is called an *SRRT metric relative to V*.

The SRRT equation can be written as a nonlinear differential equation for the pair $(\mathcal{G}, V)$ on $\mathcal{M}_0$. When $\mathcal{G}$ is fixed, it can be viewed as a nonlinear second order PDE for V. When V is fixed, it can be viewed as a nonlinear first order PDE for $\mathcal{G}$.

Fixing the conformal class of $\mathcal{G}$

Let $S \stackrel{\text{def.}}{=} \mathrm{Sym}^2(T^*\mathcal{M})$ and $S_+ \subset S$ be the fiber sub-bundle consisting of positive-definite tensors. When V is fixed, the SRRT equation has the form:

$$\mathcal{F}(j^1(\mathcal{G})) = 0 \quad ,$$

where $\mathcal{F} : j^1(S_+) \to \mathbb{R}$ is a smooth function which depends on V.
Let $L = \det T^*\mathcal{M} = \wedge^2 T^*\mathcal{M}$ be the real determinant line bundle of $\mathcal{M}$ and L_+ be its sub-bundle of positive vectors. Fixing the complex structure J determined by $\mathcal{G}$, the map $\mathcal{G} \to \omega$ gives an isomorphism of fiber bundles $S_+ \stackrel{\sim}{\to} L_+$ which extends to an isomorphism $j^1(S_+) \stackrel{\sim}{\to} j^1(L_+)$. Use this to transport $\mathcal{F}$ to a function $F := F_V^J : j^1(L_+) \to \mathbb{R}$. Then the SRRT equation becomes:

$$F(j^1(\omega)) = 0 \quad .$$

This is a contact Hamilton-Jacobi equation for $\omega \in \Gamma(L_+)$ relative to the Cartan contact structure of $j^1(L_+)$. F restricts to a cubic polynomial function on the fibers of the natural projection $j^1(L_+) \to L_+$.

In local isothermal coordinates (U, x^1, x^2) on $\mathcal{M}$ relative to J, we have:

$$\mathrm{d}s_{\mathcal{G}}^2 = e^{2\varphi}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \quad , \quad \omega = e^{2\varphi}\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and one can write the contact HJ equation as a nonlinear first order PDE for the conformal factor φ, which is cubic in the partial derivatives $\partial_1\varphi$ and $\partial_2\varphi$. A change of local isothermal coordinates corresponds to a contact transformation.

The contact Hamiltonian in isothermal Liouville coordinates

Let $\mathcal{G}_0$ be the locally-defined flat metric with squared line element $\mathrm{d}s_0^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2$ and define the *modified Euclidean gradient* of V through:

$$\mathrm{grad}_0^J V \overset{\mathrm{def.}}{=} J\mathrm{grad}_0 V \quad ,$$

where $\mathrm{grad}_0 V = \mathrm{grad}_{\mathcal{G}_0} V = \partial_1 V \partial_1 + \partial_2 V \partial_2$ is the ordinary Euclidean gradient. Let $\cdot$ denote the Euclidean scalar product defined by $\mathcal{G}_0$, thus $\partial_i \cdot \partial_j = \delta_{ij}$. Let:

$$H_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, \mathrm{grad}_0 V) = \partial_i \partial_j V \partial_i V \partial_j V \quad ,$$
$$\tilde{H}_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, J\mathrm{grad}_0 V) = -\partial_i \partial_j V \partial_i V \varepsilon_{jk} \partial_k V \quad .$$

Let $U_0 \subset \mathbb{R}^2$ be the image of U in the isothermal chart (U, x^1, x^2). The isothermal Liouville coordinates $(U, x^1, x^2, u, p_1, p_2)$ induce an isomorphism of fiber bundles $j^1(L_+)|_U \simeq U_0 \times \mathbb{R} \times \mathbb{R}^2$. Consider the smooth functions $A, B : U_0 \times \mathbb{R}^2 \to \mathbb{R}$ defined through:

$$A(x,p) \overset{\mathrm{def.}}{=} (\partial_i V)(x) p_i \quad , \quad B(x,p) \overset{\mathrm{def.}}{=} -\epsilon_{ij} (\partial_j V)(x) p_i \quad .$$

The linear transformation $\mathbb{R}^2 \ni (p_1, p_2) \to (A(x), B(x)) \in \mathbb{R}^2$ is nondegenerate for $x \in U_0$, with inverse:

$$p_1 = \frac{\partial_1 V A - \partial_2 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad , \quad p_2 = \frac{\partial_2 V A + \partial_1 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad .$$

Theorem

In isothermal Liouville coordinates (x^1, x^2, u, p_1, p_2) on $j^1(L_+)|_U$, the contact Hamiltonian is given by the smooth function $F : U_0 \times \mathbb{R}^3 \to \mathbb{R}$ given by:

$$F(x,u,p) \overset{\text{def.}}{=} [B(x) - \tilde{H}_0(x)]^2[A(x,p) + (\Delta_0 V)(x) - H_0(x)] - 3e^{2u}V[A(x,p) - H_0(x)]^2$$

and the contact Hamilton-Jacobi equation takes the form:

$$F(x_1, x_2, \varphi, \partial_1\varphi, \partial_2\varphi) = 0 \quad .$$

Remark

- The contact HJ equation can be solved *locally* through the method of characteristics.
- The contact Hamiltonian is regular in the sense that it depends monotonously on u. Hence the Dirichlet problem can be approached *globally* using the theory of viscosity solutions.

We have:

$$F = AB^2 - 3Ve^{2u}A^2 + (\Delta_0 V - H_0)B^2 - 2\tilde{H}_0 AB + (6Ve^{2u}H_0 + \tilde{H}_0^2)A + 2\tilde{H}_0(H_0 - \Delta_0 V)B + F_0 \quad ,$$

where:

$$F_0 = \tilde{H}_0^2[(\Delta_0 V) - H_0] - 3Ve^{2u}H_0^2 \quad .$$

Define $P_1 \overset{\text{def.}}{=} A - H_0$ and $P_2 = B - \tilde{H}_0$, which are related to p_1 and p_2 by an x-dependent affine transformation. Then F can be written as:

$$F = P_1 P_2^2 - 3Ve^{2u}P_1^2 + (\Delta_0 V)P_2^2 \quad .$$

The momentum curve is the curve $C_{x,u}$ defined by the condition $F(x,u,p) = 0$ in the p-plane. This curve passes through the origin of P-plane, i.e. through the point with coordinates:

$$\begin{aligned} p_1 &:= p_{01} \stackrel{\mathrm{def.}}{=} -\frac{\mathrm{grad} V \cdot (-H_0, \tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_1 V H_0 - \partial_2 V \tilde{H}_0}{(\partial_1 V)^2 + (\partial_2 V)^2} \\ p_2 &:= p_{02} \stackrel{\mathrm{def.}}{=} \frac{\mathrm{grad}_J V \cdot (-H_0, \tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_2 V H_0 + \partial_1 V \tilde{H}_0}{(\partial_1 V)^2 + (\partial_2 V)^2} \end{aligned}$$

in the p-plane. The singular points of the momentum curve coincide with the characteristic points of the contact HJ equation.

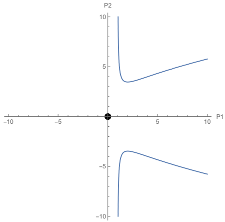
Proposition

The origin of the P-plane is the only singular point of the momentum curve. When $(\Delta_0 V)(x) = 0$, the curve is reducible and F factorizes as:

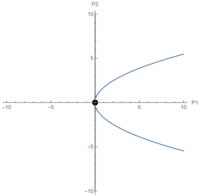
$$F = P_1(P_2^2 - 3Ve^{2u}P_1) \quad .$$

The curve is symmetric under reflection in the P_1-axis. When $(\Delta_0 V)(x) > 0$, it is connected and contained in the half-space $P_1 \geq -(\Delta V)(x)$, being the union of two embedded curves which intersect each other at the origin of the P-plane. When $(\Delta_0 V)(x) < 0$, it has three connected components, namely the origin of the (P_1, P_2)-plane (which is its only singular point) and two connected components which are nonsingular and contained in the half-space $P_1 > -(\Delta_0 V)(x)$.

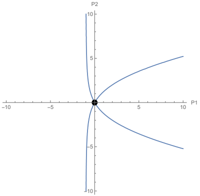
The momentum curve



() $(\Delta_0 V)(x) = -1.$

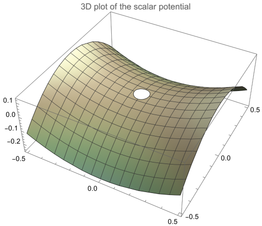


() $(\Delta_0 V)(x) = 0.$

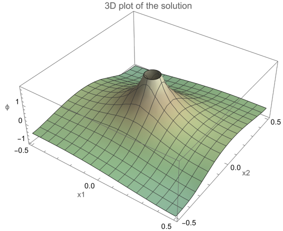


() $(\Delta_0 V)(x) = 1.$

Figure: The momentum curve for $V(x)e^{2u(x)} = 1$ in the cases $(\Delta_0 V)(x) = -1, 0, 1$. The singular point of the curve is shown as a black dot.



() Quadratic potential V with principal values $\lambda_1 = 1, \lambda_2 = -2$.



() Numerical solution φ for this quadratic potential.

Quasilinear approximation near an isolated critical point

Let $c \in U_0$ be an isolated critical point of V and λ_1, λ_2 be the principal values of $\mathrm{Hess}(V)(c)$. In principal isothermal coordinates centered at c, we have:

$$V(x) = V(c) + \frac{1}{2}(\partial_i \partial_j V)(c) x^i x^j + \mathcal{O}(||x||_0^3) = V(c) + \frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2) + \mathcal{O}(||x||_0^3) \ .$$

Consider the following homogeneous polynomial functions of degree two in the variables x_1 and x_2, where $k \in \mathbb{Z}_{>0}$:

$$s_k(x) \stackrel{\text{def.}}{=} \lambda_1^k x_1^2 + \lambda_2^k x_2^2 \ .$$

Proposition

We have:

$$F(x, u, p) = \frac{a_1(x, u) x^1 p_1 + a_2(x) x^2 p_2 - b(x, u)}{s_2(x)^3} + \mathcal{O}(||x||_0^2) \ ,$$

where a_i and b are homogeneous polynomial functions of degree six in x_1 and x_2 (whose coefficients depend on u) given by:

$$a_i(x, u) = \lambda_i s_2(x) \left[t_i(x) + 6V(c) e^{2u} s_2(x) s_3(x) \right]$$

with:

$$t_1(x) = \lambda_1 \lambda_2^2 (\lambda_1 - \lambda_2) x_2^2 [s_2(x) - 3\lambda_2 s_1(x)]$$
$$t_2(x) = \lambda_2 \lambda_1^2 (\lambda_2 - \lambda_1) x_1^2 [s_2(x) - 3\lambda_1 s_1(x)] \ .$$

and:

$$b(x, u) = -\lambda_1^3 \lambda_2^3 (\lambda_1 - \lambda_2)^2 x_1^2 x_2^2 s_1(x) + 3V(c) e^{2u} s_2(x) s_3(x)^2 \ .$$

Corollary

The contact HJ equation is approximated to first order in $||x||_0$ by the following quasilinear first order PDE:

$$a_1(x,\varphi)x^1\partial_1\varphi + a_2(x,\varphi)x^2\partial_2\varphi = b(x,\varphi) \quad . \tag{1}$$

This quasilinear PDE can be studied by the Lagrange-Charpit method. Its scale-invariant solutions can be studied by reduction to a nonlinear ODE for a function defined on the unit circle.

Proposition

Suppose that φ satisfies the quasilinear equation (1) and that we have $\varphi(x) \gg 1$. Then φ is an approximate solution of the following linear first order PDE:

$$2s_2(x)\lambda_i x^i \partial_i\varphi = s_3(x) \quad , \tag{2}$$

which it satisfies up to corrections of order $\mathcal{O}\left(\frac{e^{-2\varphi}}{3V(c)}\right)$.

Solutions which blow up at an isolated critical point

Consider the polar coordinate system (r, θ) defined though:

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad .$$

Proposition

Suppose that $\lambda_1 \neq \lambda_2$. Then the general smooth solution of the linear equation (2) is:

$$\varphi(r,\theta) = \varphi_0(\theta) + Q_0\left(\frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}\log r + \frac{1}{\lambda_1}\log|\cos\theta| - \frac{1}{\lambda_2}\log|\sin\theta|\right) \quad , \tag{3}$$

where:

$$\varphi_0(\theta) = \frac{1}{4}\log(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta) - \frac{1}{2}\frac{\lambda_2\log|\cos\theta| - \lambda_1\log|\sin\theta|}{\lambda_2 - \lambda_1}$$

and Q_0 is an arbitrary smooth function of a single variable.

Proposition

Suppose that $\lambda_1 = \lambda_2 := \lambda$. Then the linear equation (2) reduces to:

$$x^i\partial_i\varphi = \frac{1}{2} \quad ,$$

whose general solution is:

$$\varphi(r,\theta) = \frac{1}{2}\log r + Q_0(\theta) \quad , \tag{4}$$

where $Q_0 \in \mathcal{C}^\infty(\mathrm{S}^1)$ is an arbitrary smooth function.

Solutions which blow up at an isolated critical point

Suppose that $\lambda_1 \neq \lambda_2$. Defining $Q(w) = Q_0(\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} w)$, the general solution (3) reads:

$$\varphi(r,\theta) = \varphi_0(\theta) + Q\left(\log r + \frac{\lambda_2 \log|\cos\theta| - \lambda_1 \log|\sin\theta|}{\lambda_2 - \lambda_1}\right)$$

and satisfies $\lim_{r\to 0}\varphi(r,\theta) = +\infty$ iff $\lim_{w\to -\infty} Q(w) = +\infty$. In this case, we have:

$$\varphi \approx Q(\log r) \ \text{ for } \ r \ll 1 \ \ ,$$

so φ is rotationally-invariant near c. The corresponding SRRT metric is asymptotically rotationally-invariant at c, with Gaussian curvature:

$$K \approx -e^{-2\varphi}\Delta\varphi \approx -e^{-2Q(\log r)} Q''(\log r) \ \text{ for } \ r \ll 1 \ \ .$$

Requiring $K = K_c$ for some constant K_c gives:

$$e^{-2Q(w)} Q''(w) = K_c \ .$$

Also require that $\mathcal{G}$ is geodesically complete at c. For $K_c = 0$, we can take $Q(w) = -w$, which gives $\varphi(r,\theta) \approx_{r\ll 1} -\log r$ and:

$$\mathrm{d}s^2 \approx_{r\ll 1} \frac{1}{r^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) = \mathrm{d}\rho^2 + \mathrm{d}\theta^2 \ \ , \ \ \text{where} \ \ \rho \overset{\text{def.}}{=} \log r \ \ .$$

so $\mathcal{G}$ asymptotes at c to the metric on a flat cylinder. For $K_c = -1$, the SRRT metric $\mathcal{G}$ asymptotes to the hyperbolic cusp metric at c:

$$\mathrm{d}s^2 \approx \frac{1}{(r\log r)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) \ \text{ for } \ r \ll 1 \ \ .$$

Multifield cosmological models whose target space $\mathcal{M}$ is a d-dimensional torus T^d can be obtained from orientifold compactifications of IIB string theory. When $d = 2$, the target space becomes a smooth elliptic curve $E = (\mathrm{T}^2, J)$ when endowed with the complex structure J determined by the conformal class of the target space metric $\mathcal{G}$. We have $E = \mathbb{C}/\Lambda$ for some full lattice $\Lambda \subset \mathbb{C}$ and hence E admits a periodic complex coordinate induced by the complex coordinate z of $\mathbb{C}$, subject to identifications given by Λ-translations. The scalar potential lifts to a real and Λ-periodic function which has a Fourier expansion of the form:

$$V(z,\bar{z}) = \mathrm{Re} \sum_{q\in\Lambda^*} V_q e^{2\pi i (q,z)} = \sum_{q\in\Lambda^*} A_q \cos(2\pi[(q,z) + \alpha_q]) =$$

$$= A_0 + \sum_{q\in\Lambda^*\setminus\{0\}} A_q \cos(2\pi[(q,z)\alpha_q]) \quad , \quad \text{with} \quad A_q \in \mathbb{R} \quad ,$$

where $\Lambda^* \subset \mathbb{C}$ is the dual lattice. The condition that V is positive amounts to:

$$A_0 > - \sum_{q\in\Lambda^*\setminus\{0\}} A_q \cos(2\pi[(q,z) + \alpha_q]).$$

In lattice-adapted coordinates θ^1, θ^2 determined by a basis (e_1, e_2) of Λ, the expansion above reads:

$$V(z,\bar{z}) = \sum_{q\in\Lambda^*} A_q \cos(2\pi[q_1\theta^1 + q_2\theta^2 + \alpha_q])$$

and the metric $\mathcal{G}$ takes the form:

$$\mathrm{d}s^2 = f(\theta)[|e_1|^2 d\theta_1^2 + |e_2|^2 d\theta_2^2 + (e_1\bar{e}_2 + e_2\bar{e}_1)d\theta_1 d\theta_2].$$