Consistency conditions for sustained rapid turn inflation in two-field cosmological models

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Definition

A two-dimensional oriented scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected, oriented and borderless Riemann surface (called scalar manifold)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (this ensures conservation of energy)
2. $V > 0$ on $\mathcal{M}$ (this avoids technical problems but can be relaxed)

Each scalar triple defines a model of gravity coupled to scalar fields on $\mathbb{R}^4$:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|}\left[\frac{M^2}{2}R(g) - \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) - V\circ\varphi\right] .$$

Define the *rescaled Planck mass* $M_0 \stackrel{\mathrm{def.}}{=} \sqrt{\frac{2}{3}}M$, where M is the reduced Planck mass. Take g to describe a spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \ , \ \vec{x} = (x^1, x^2, x^3) \ , \ a(t) > 0 \ \forall t)$$

and φ to depend only on the cosmological time: $\varphi = \varphi(t)$.

Define the *Hubble parameter* $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$ and the *rescaled Hubble function*:

$$\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0} \quad , \quad \mathcal{H}(u) \stackrel{\text{def.}}{=} \frac{1}{M_0}\sqrt{||u||^2 + 2V(\pi(u))} \quad \forall u \in T\mathcal{M} \quad ,$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ is the bundle projection.

Proposition

When $H > 0$, the equations of motion are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\mathcal{H}(\dot{\varphi}(t))\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0 \quad ,$$

together with the Hubble condition:

$$H(t) = \frac{1}{3M_0}\mathcal{H}(\dot{\varphi}(t)) \quad .$$

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation are called cosmological curves. The cosmological equation defines an autonomous dissipative geometric dynamical system on $T\mathcal{M}$.

For simplicity, we take $M = 1$ i.e. $M_0 = \sqrt{\frac{2}{3}}$.

Let (T, N) be the positive Frenet frame of a cosmological curve $\varphi : I \to \mathcal{M}$:

$$T(t) \stackrel{\text{def.}}{=} \frac{\dot{\varphi}(t)}{||\dot{\varphi}(t)||} \quad , \quad N(t) = -JT(t) \quad ,$$

where $J \in \mathrm{End}(T\mathcal{M})$ is the complex structure determined on $\mathcal{M}$ by the conformal class of $\mathcal{G}$:

$$\omega(u, v) = \mathcal{G}(u, Jv) \quad , \quad \text{where} \quad \omega \stackrel{\text{def.}}{=} \mathrm{vol}_{\mathcal{G}}$$

and let σ be an increasing proper legth parameter for φ:

$$\mathrm{d}\sigma = ||\dot{\varphi}(t)||\mathrm{d}t \quad .$$

Projecting the cosmological equation along T and N gives repectively the adiabatic and *ethropic* equations:

$$\ddot{\sigma} + \frac{1}{M_0}\mathcal{H}(\sigma, \dot{\sigma})\dot{\sigma} + V_T(\sigma) = 0 \quad , \quad \Omega(\sigma) = \frac{V_N(\sigma)}{\dot{\sigma}} \quad ,$$

where

$$\mathcal{H}(\sigma, \dot{\sigma}) = \sqrt{\dot{\sigma}^2 + 2V(\sigma)} \quad ,$$

$$V_T(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(T(\sigma)) \quad , \quad V_N(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(N(\sigma))$$

and we defined the *signed turn rate* of φ through:

$$\Omega(t) \stackrel{\text{def.}}{=} -\mathcal{G}(N, \nabla_t T) \quad .$$

Definition

Consider the following functions of t associated to the cosmological curve φ:

- Define the first, second and third Hubble slow roll parameters:

$$\varepsilon = -\frac{\dot{H}}{H^2} \ , \ \eta_{\|} = -\frac{\ddot{\sigma}}{H\dot{\sigma}} \ , \ \xi = \frac{\dddot{\sigma}}{H^2\dot{\sigma}} \ .$$

- The first and second turn parameters:

$$\eta_{\perp} \stackrel{\text{def.}}{=} \frac{\Omega}{H} \ , \ \nu \stackrel{\text{def.}}{=} \frac{\dot{\eta_{\perp}}}{H\eta_{\perp}} \ .$$

- The first IR parameter κ and the conservative parameter c:

$$\kappa \stackrel{\text{def.}}{=} \frac{\dot{\sigma}^2}{2V} \ , \ c \stackrel{\text{def.}}{=} \frac{H\dot{\sigma}}{||\mathrm{d}V||} \ .$$

Remark

The *opposite relative acceleration vector* $\eta \stackrel{\text{def.}}{=} -\frac{1}{H\dot{\sigma}}\nabla_t\dot{\varphi}$ decomposes as $\eta = \eta_{\|} T + \eta_{\perp} N$ and we have:

$$\varepsilon = \frac{3\kappa}{1+\kappa} \ .$$

Slow roll and rapid turn conditions

Definition

- The first, second and third slow roll conditions are the conditions $\epsilon \ll 1$, $|\eta_{\parallel}| \ll 1$ and $|\xi| \ll 1$.
- The second order slow roll regime is defined by the joint conditions $\epsilon \ll 1$ and $|\eta_{\parallel}| \ll 1$.
- The third order slow roll regime is defined by the joint conditions $\epsilon \ll 1$, $|\eta_{\parallel}| \ll 1$ and $|\xi| \ll 1$.

Definition

- The rapid turn condition is the condition $|\eta_{\perp}| \gg 1$.
- The sustained rapid turn regime is defined by the joint conditions $|\eta_{\perp}| \gg 1$ and $|\nu| \ll 1$.

Proposition

Suppose that the second slow roll condition $|\eta_{\parallel}| \ll 1$ is satisfied. Then the rapid turn condition $|\eta_{\perp}| \gg 1$ is equivalent with the conservative condition $c \ll 1$.

The adapted frame

Let $\mathcal{M}_0 \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid (\mathrm{d}V)(m) \neq 0\}$ be the complement of the critical locus of V.

Definition

The *adapted frame* of $(\mathcal{M}, \mathcal{G}, V)$ is the oriented orthonormal frame (n, τ) of $\mathcal{M}_0$ defined by the vector fields:

$$n \stackrel{\text{def.}}{=} \frac{\mathrm{grad} V}{||\mathrm{grad} V||} \quad , \quad \tau = -Jn \quad .$$

Definition

The *characteristic angle* $\theta \in (-\pi, \pi]$ of φ is the angle of rotation from the adapted frame (n, τ) to the Frenet frame (T, N):

$$T = n \cos\theta + \tau \sin\theta \quad , \quad N = -n \sin\theta + \tau \cos\theta \quad .$$

The sign factor $s \stackrel{\text{def.}}{=} \mathrm{sign}(\sin\theta) \in \{-1, 0, 1\}$ is called the *characteristic sign* of φ.

Proposition

We have:

$$\eta_{||} = 3 + \frac{\cos\theta}{c} \quad , \quad \eta_{\perp} = -\frac{\sin\theta}{c} \quad .$$

Natural local coordinates on the phase space

For any vector fields X, Y, we use the notation:

$$V_{XY} \stackrel{\text{def.}}{=} \mathrm{Hess}(V)(X, Y) \quad ,$$

where $\mathrm{Hess}(V) \stackrel{\text{def.}}{=} \nabla \mathrm{d} V$ is the Riemannian Hessian of V.

Definition

The *nondegenerate locus* of $\mathcal{M}$ is the following subset of $\mathcal{M}_0$:

$$\mathcal{U} \stackrel{\text{def.}}{=} \{m \in \mathcal{M}_0 \;\mid\; \Phi_{n\tau}(m) \neq 0\}$$

The scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is called *nondegenerate* if $\mathcal{U} \neq \emptyset$.

Theorem

Suppose that $(\mathcal{M}, \mathcal{G}, \Phi)$ is nondegenerate. Then there exist basic natural observables $\hat{\varepsilon}, \hat{\eta}_{\|}, \hat{c}, \hat{\theta} : \dot{T}\mathcal{U} \to \mathbb{R}$ associated to $\varepsilon, \eta_{\|}, c$ and θ which provide local coordinates on a small enough neighborhood of each point of $\dot{T}\mathcal{U}$.

Proposition

We have:

$$\frac{V_{TT}}{3H^2} = \frac{\Omega^2}{3H^2} + \varepsilon + \eta_{\parallel} - \frac{\xi}{3}$$
$$\frac{V_{TN}}{H^2} = \frac{\Omega}{H}\left(3 - \varepsilon - 2\eta_{\parallel} + \nu\right) \quad .$$

Theorem

Suppose that the third order slow roll conditions $\varepsilon \ll 1$, $|\eta_{\parallel}| \ll 1$ *and* $|\xi| \ll 1$ *as well as the small rate of turn condition* $|\nu| \ll 1$ *are satisfied. In this case, we have* $\cos\theta \approx -3c, \sin\theta \approx s\sqrt{1-9c^2}$ *and:*

$$V_{TN}^2 \approx 3VV_{TT}$$
$$V_{TT} \approx 9c^2 V_{nn} - 6sc\sqrt{1-9c^2}\,V_{n\tau} + (1-9c^2)V_{\tau\tau}$$
$$V_{TN} \approx -3sc\sqrt{1-9c^2}(V_{\tau\tau} - V_{nn}) - (1-18c^2)V_{n\tau} \quad .$$

These equations admit a solution c *with* $c \ll 1$ *iff:*

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2$$

up to corrections of order one in $\varepsilon, \eta_{\parallel}, \xi$ *and* κ.

The SRRT equation

Corollary

The cosmological curve φ satisfies the sustained rapid turn conditions with third order slow roll at cosmological time t iff the following condition is satisfied at the point $m = \varphi(t)$ of $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2 \quad .$$

In particular, such cosmological curves can be found only in regions of $\mathcal{M}_0$ where this condition is satisfied.

Definition

The *SRRT equation* is the following condition which constrains target space metric $\mathcal{G}$ and scalar potential V of the model on the noncritical submanifold $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} = 3VV_{nn}^2$$

The SRRT equation can be written as nonlinear differential equation for the pair $(\mathcal{G}, V)$ of $\mathcal{M}_0$. When $\mathcal{G}$ is fixed, it can be viewed as a nonlinear second order PDE for V. When V is fixed, it can be viewed as a nonlinear first order PDE for the metric $\mathcal{G}$. Using isothermal coordinates, the latter can be writted as a *contact Hamilton-Jacobi equation* for the conformal factor of the metric $\mathcal{G}$.

Let $S \stackrel{\text{def.}}{=} \mathrm{Sym}^2(T^*\mathcal{M})$ and $S_+ \subset S$ be the fiber sub-bundle consisting of positive-definite tensors. When V is fixed, the SRRT equation has the form:

$$\mathcal{F}(j^1(\mathcal{G})) = 0 \quad ,$$

where $\mathcal{F} : j^1(S)_+ \to \mathbb{R}$ is a smooth function which depends on V.
Let $L = \det T^*\mathcal{M} = \wedge^2 T^*\mathcal{M}$ be the real determinant line bundle of $\mathcal{M}$ and L_+ be its sub-bundles of positive vectors. Fixing the complex structure J determined by $\mathcal{G}$, the map $\mathcal{G} \to \omega$ gives an isomorphism of fiber bundles $S_+ \stackrel{\sim}{\to} L_+$ which extends to an isomorphism $j^1(S)_+ \stackrel{\sim}{\to} j^1(L_+)$. Use this to transport $\mathcal{F}$ to a function $H := H_V^J : j^1(L_+) \to \mathbb{R}$. Then the SRRT equation becomes:

$$H(j^1(\omega)) = 0 \quad .$$

This is a contact Hamilton-Jacobi equation for $\omega \in \Gamma(L_+)$ relative to the Cartan contact structure of $j^1(L_+)$. H restricts to a cubic polynomial function on the fibers of the natural projection $j^1(L_+) \to L_+$.
In local isothermal coordinates (x^1, x^2) on $\mathcal{M}$ relative to J, we have:

$$\mathrm{d}s^2_{\mathcal{G}} = e^{2\varphi}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$$

and one can write this contact HJ equation as a nonlinear first order PDE for the conformal factor φ, which is cubic in the partial derivatives $\partial_1\varphi$ and $\partial_2\varphi$. The equation can be solved locally through the method of characteristics, while the Cauchy boundary value problem can be approached using the theory of viscosity solutions.