Discrete integrable dynamical systems: geometry of invariants and symmetries

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- Complexity index and algebraic entropy; linearisable systems

What is integrability?

Well known fact:
Hamiltonian systems!
Definition:
Let M be a Poisson manifold (a manifold endowed with a bivector field - Poisson bracket) and $H \in C^{\infty}(M)$. A classical system is called **integrable** if the commutant of the hamiltonian H in the algebra of observables contains an abelian subalgebra of maximal possible rank.

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All integrable systems can be directly formulated in this setting?
NO! Examles:

- some non-autonomous systems (coming from isomonodromic deformations of linear opeartors
- some dissipative systems (Lorentz attractor-integrable case)
- some linearisable systems (Burgers equation which is also a dissipative PDE)
- discrete equations etc.

It was observed that various reductions and limits of completely integrable hamiltonian systems exhibit other interesting properties:

- existence of invariants (conservation laws) and symmetries

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- etc.

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From nonlinear discrete equations(mappings) to surface theory

The main motivation! Extend the singularity analysis to **discrete equations**
Example:

$$E(n) \equiv x_{n+1} + x_n + x_{n-1} = \frac{a}{x_n}$$

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Singularity pattern $(f, 0, \infty, \infty, 0, -f)$. So after a **finite** number of steps the singularities are confined and **initial information is recovered**- *singularity confinement criterion*

Singularity confinement useful for getting exact solutions of the equation: The pattern suggests the following substitution:

$$x_n = \frac{F_{n-1}F_{n+2}}{F_n F_{n+1}}$$

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Introducing in the *discrete derivative* of the mapping we get the follwing quadrilinear expression:

$$E(n+1) - E(n) \equiv F_{n-1}(F_{n+4}F_n - aF_{n+2}^2) - F_{n+3}(F_{n+2}F_{n-2} - aF_n^2) = 0$$

giving the following bilinear form:

$$F_{n+2}F_{n-2} - aF_n^2 - F_{n+1}F_{n-1} = 0$$

solvable in terms of Riemann theta functions (particular form of Fay identity). This aspect is extremely useful even in the case of partial differential-difference systems since it blends the confining singularities with Painleve property

Singularity Confinement-Painleve property

Question: how do we study these types of equations? They are differential-difference form and hard to apply methods of hamiltonian mechanics (to get invariants for example).

In order to avoid *butterfly effect* on the Riemann sheets of some branch points of the solutions we impose that in t the singular part to be at most *poles*

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which can be written as a 2-point mapping,

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni (u_n, v_n) \to (u_{n+1}, v_{n+1}) \in \mathbb{P}^1 \times \mathbb{P}^1$$

whose points are depeding on t:

$$u_{n+1} = v_n \tag{1}$$

$$v_{n+1} = \frac{\dot{v}_n}{v_n} + u_n \tag{2}$$

It is obvious that if (u_n, v_n) have no movable critical singularities, then the same will be true for (u_{n+1}, v_{n+1}). Let us consider the simplest case, in a neighbourhood of t, to have a simple zero for v_n and regular u_n. Thus the curve $(u_n, 0)$ goes to a point $(0, \infty)$ which means loosing a degree of freedom (curve blow-down process). More precisely, starting as above from $(\tau = t - t_0)$,

$$u_n = a_0 + a_1\tau + O(\tau^2), v_n = \alpha\tau + \beta\tau^2 + O(\tau^3)$$

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$$\begin{pmatrix} a_0 \\ \alpha\tau + ... \end{pmatrix} \to \begin{pmatrix} \alpha\tau + ... \\ \tau^{-1} + \beta/\alpha + a_0 + ... \end{pmatrix} \to$$

$$\to \begin{pmatrix} \tau^{-1} + \beta/\alpha + a_0 + ... \\ -\tau^{-1} + \beta/\alpha + a_0 + ... \end{pmatrix} \to \begin{pmatrix} -\tau^{-1} + \beta/\alpha + a_0 + ... \\ \gamma(a_0, \alpha, \beta)\tau + ... \end{pmatrix} \to \begin{pmatrix} \gamma(a_0, \alpha, \beta)\tau + ... \\ f(a_0, \alpha, \beta) + ... \end{pmatrix}$$

where γ, f are some finite expressions containing the parameters a_0, α, β etc. So in a small neighbourhood of t_0 (where $\tau \approx 0$) we can write

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$$... \to \text{regular} \to \left(\begin{array}{c} a_0 \\ 0 \end{array} \right) \to \left(\begin{array}{c} 0 \\ \infty \end{array} \right) \to \left(\begin{array}{c} \infty \\ -\infty \end{array} \right) \to \left(\begin{array}{c} -\infty \\ 0 \end{array} \right) \to \left(\begin{array}{c} 0 \\ f(a_0, \alpha, \beta) \end{array} \right) \to \text{regu}$$

So the initial curve blows down to three points and then blows up to another curve containing initial parameters. In this way the singularity confinement is satisfied.

The backward evolution shows exactly regular evolution. Namely if

$$u_{n-1} = \quad v_n - \frac{\dot{u}_n}{u_n} \tag{3}$$

$$v_{n-1} = \quad u_n \tag{4}$$

then

$$\ldots \rightarrow \text{regular} \rightarrow \text{regular} \rightarrow \begin{pmatrix} a_0 \\ 0 \end{pmatrix} \rightarrow \ldots$$

For higher order starting singularities $v \sim \alpha_0 \tau^2$ we have bigger length:

$$\begin{pmatrix} a_0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \infty \end{pmatrix} \rightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \rightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \rightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \rightarrow \begin{pmatrix} \infty \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0^2 \\ * \end{pmatrix}$$

This singularity pattern is crucial. It helps us to *find a substitution* which *solves explicitly the equation*
Indeed one can see immediately that for both u_n, v_n the orbit pattern is

$$u_n(t) : \text{...regular} \rightarrow 0 \rightarrow \infty \rightarrow \infty \rightarrow 0 \rightarrow \text{regular...}$$

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So exist a function F_n which is *holomorphic* and u_n, v_n are expressed as ratios of products of such functions. Hence let us consider that u_n has a function F_n in the numerator and this F_n passes through 0, so $u_n = 0$. Because u_{n+1}, u_{n+2} are infinite then the denominator of u_n must have F_{n-1}, F_{n-2}. Then $u_{n+3} = 0$ so at the numerator we have F_{n-3}. Accordingly one can write

$$u_n = \frac{F_n F_{n-3}}{F_{n-1} F_{n-2}}$$

and introducing in the equation we find the bilinear form:

$$(\partial_t F_{n-1}) F_{n-2} - F_{n-1}(\partial_t F_{n-2}) - F_n F_{n-3} + F_{n-1} F_{n-2} = 0$$

which admits general *multi-soliton solution* i.e. general multiple collision of arbitrary solitons and has the form:

$$F_n(t) = \sum_{\mu_1,\dots,\mu_M \in \{0,1\}} \exp\left(\sum_{i=1}^{M} \mu_i(k_i n + \omega_i t) + \sum_{i<j}^{M} A_{ij}\mu_i\mu_j \right) \tag{5}$$

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with the dispersion relation and interaction phase given by

$$\omega_i \equiv \omega(k_i) = 2\sinh(k_i)$$

$$\exp A_{ij} = \frac{-\cosh((k_i - k_j)/2) + \cosh(3/2(k_i - k_j)) + (-\omega_i + \omega_j)\sinh((k_i - k_j)/2)}{\cosh((k_i + k_j)/2) - \cosh(3/2(k_i + k_j)) + (\omega_i + \omega_j)\sinh((k_i + k_j)/2)}$$

More general in the case of periodic solutions we have expressed using again Riemann Theta function (the so called g-phase solution)

$$F_n(t) = \Theta(k_1 n + \omega_1 t, \ldots, k_g n + \omega_g t|\mathbb{B})$$

with the disperion relation given by

$$\omega_i = \omega(k_i) = \frac{\Theta[1,1](k_i|\mathbb{B})}{\partial_{k_i}\Theta[1,1](0|\mathbb{B})}$$

The matrix $\mathbb{B}$ has a more complicated structure, being the period matrix for a suitable Riemann surface of genus g

Let us go back to our 2-dimensional example. It can be written as:

$$\phi : \begin{cases} x_{n+1} & = y_n \\ y_{n+1} & = \quad -x_n - y_n + \frac{a}{y_n} \end{cases} . \tag{6}$$

seen as a chain of birational mappings ... $\to (\underline{x}, \underline{y}) \to (x, y) \to (\bar{x}, \bar{y}) \to$ ... where $\underline{x} = x_{n-1}, x = x_n, \bar{x} = x_{n+1}$ and so on.
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Each step is an automorphism of the field of rational functions $\mathbb{C}(x, y)$
Singularity confinement:

$$\underbrace{(f, 0)}_{(x_0, y_0)} \to \underbrace{(0, \infty)}_{(x_1, y_1)} \to \underbrace{(\infty, -\infty)}_{(x_2, y_2)} \to \underbrace{(-\infty, 0)}_{(x_3, y_3)} \to \underbrace{(0, -f)}_{(x_4, y_4)}$$

and the mechanism is the follwing:
If $(x_0, y_0) = (f, \epsilon)$ then the following products are *finite*

$$x_1 y_1 = a + O(\epsilon), \quad \frac{x_2}{y_2} = -1 + O(\epsilon), \quad x_3 y_3 = -a + O(\epsilon)$$

So lets construct a surface by glueing

$$\mathbb{C}^2 \cup \mathbb{C}^2 = \left(\frac{1}{x_2}, \frac{x_2}{y_2}\right) \cup \left(\frac{y_2}{x_2}, \frac{1}{y_2}\right)$$

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But this is nothing but blow up of the affine space $\mathrm{Spec}\mathbb{C}[X, Y]$ with the center $(X, Y) = (0, 0)$ which gives the surface $(Y = 1/y, X = 1/x)$:

$$\mathcal{X} = \{(X, Y, [z_0 : z_1]) \in \mathrm{Spec}\mathbb{C}[X, Y] \times \mathbb{P}^1 | X z_0 = Y z_1\} =$$

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So by blowing up $\mathbb{C}^2$ in the points
$(x_1, y_1) = (0, \infty), (x_2, y_2) = (\infty, \infty), (x_3, y_3) = (\infty, 0)$ the equation then make sense on this new surface.
Accordingly we do analize any discrete order two nonlinear equation by identifying the singularities and blow them up.
From now on we shall replace $\mathbb{C}^2$ with $\mathbb{P}^1 \times \mathbb{P}^1$ and any nonlinear equation will be a birational mapping $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ After blowing up the singular points we get a surface X and our mapping is lifted to a regular mapping:

$$\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

Algorithm for analysing mappings

- check if $\varphi : X \to X$ is free from singularities. If no, then do another series of blow ups and so on, until we get finally a new final surface S and the final mapping $\varphi : S \to S$ without any singularity

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Rational elliptic surface:

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A complex surface X is called a rational elliptic surface if there exists a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbb{P}^1$ the fibre $\pi^{-1}(k)$ is an elliptic curve
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Halphen surface of index m: A rational surface X is called a *Halphen surface of index m* if the anticanonical divisor class $-K_X$ is decomposed into prime divisors as $[-K_X] = D = \sum m_i D_i (m_i \geq 1)$ such that $D_i \cdot K_X = 0$ Halphen surfaces can be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by succesive 8 blow-ups. In addition the dimension of the linear system $|-kK_X|$ is zero for $k = 1, \ldots, m-1$ and 1 for $k = m$. Here, the linear system $|-mK_X|$ is the set of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2m, 2m)$ passing through each point of blow-up with multiplicity m.

Singularities, surfaces and invariants

Basic example:

$$x_{n+1} = -x_{n-1} \frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)} \tag{7}$$

$$\begin{aligned} \overline{x} &= y \\ \overline{y} &= -x \frac{(y - a)(y - 1/a)}{(y + a)(y + 1/a)} \end{aligned} \tag{8}$$

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Indeterminate points for ϕ and ϕ^{-1}:

$$\begin{aligned} E_1 : (x,y) &= (0,-a), \quad & E_2 : (x,y) &= (0,-1/a), \\ E_3 : (X,y) &= (0,a), \quad & E_4 : (X,y) &= (0,1/a), \\ E_5 : (x,y) &= (a,0), \quad & E_6 : (x,y) &= (1/a,0), \\ E_7 : (x,Y) &= (-a,0), \quad & E_8 : (x,Y) &= (-1/a,0). \end{aligned}$$

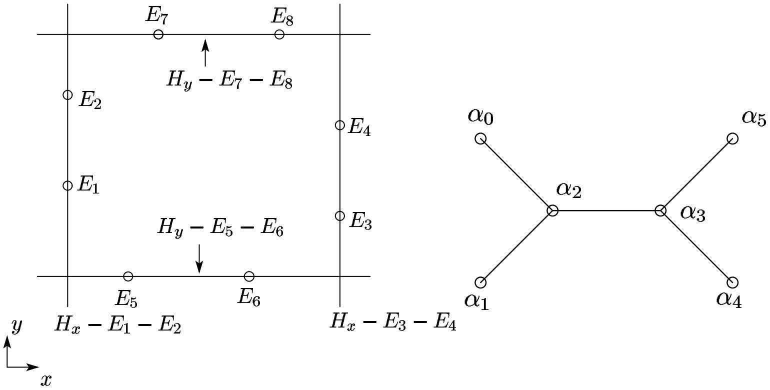

Figure: Space of initial conditions and orthogonal complement

Blow up $(x, y)|_{E_1=(0,-a)} \leftarrow (x, (y + a)/x) \cup (x/(y + a), y + a) := (u_1, v_1) \cup (U_1, V_1)$
The exceptional divizor E_1 is given by the equation $u_1 = 0$ or $V_1 = 0$

The Picard group of $\mathcal{X}$ is a $\mathbb{Z}$-module

$$\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}\mathcal{H}_x \oplus \mathbb{Z}\mathcal{H}_y \oplus \bigoplus_{i=1}^{8} \mathbb{Z}\mathcal{E}_i,$$

$\mathcal{H}_x$, $\mathcal{H}_y$ are divizzor *classes* of horizontal and vertical lines $x = \mathrm{const.}$, $y = \mathrm{const.}$
$\mathcal{E}_i$ is the class of the exceptional divizor. Elements of the divizor classes are written with normal characters e.g. $H_{x=0} \in \mathcal{H}_x$ is the total transform of the line $x = 0$
The intersection form:

$$\mathcal{H}_x \cdot \mathcal{H}_y = 1, \quad \mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij}, \quad \mathcal{H}_x \cdot \mathcal{E}_k = \mathcal{H}_y \cdot \mathcal{E}_k = \mathcal{H}_x \cdot \mathcal{H}_x = 0$$

. Anti-canonical divisor of X (the pole-divisor of invariant symplectic form $\omega = dx \wedge dy/xy$):

$$-K_{\mathcal{X}} = 2\mathcal{H}_x + 2\mathcal{H}_y - \sum_{i=1}^{8} \mathcal{E}_i.$$

Singularity confining:

$$(f, a) \to (a, 0) \to (0, -a) \to (-a, f) \Longleftrightarrow H_y - E_3 \to E_5 \to E_1 \to H_x - E_7$$

$$(f, 1/a) \to (1/a, 0) \to (0, -1/a) \to (-1/a, f) \Longleftrightarrow H_y - E_4 \to E_6 \to E_2 \to H_x - E_8$$

On the Picard lattice Pic($\mathcal{X}$) of the surface, Pic($\mathcal{X}$) $=$ Span$_{\mathbb{Z}}\{\mathcal{H}_x, \mathcal{H}_y, \mathcal{E}_1, \ldots, \mathcal{E}_8\}$, the push-forward and the pull-back actions of the mapping are given by (here $\phi_* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\bar{\mathcal{X}}), \phi^* : \mathrm{Pic}(\bar{\mathcal{X}}) \to \mathrm{Pic}(\mathcal{X})$)

$$\varphi_* : \begin{array}{l} \mathcal{H}_x \mapsto \bar{\mathcal{H}}_y, \quad \mathcal{H}_y \mapsto \bar{\mathcal{H}}_x + 2\bar{\mathcal{H}}_y - \bar{\mathcal{E}}_1 - \bar{\mathcal{E}}_2 - \bar{\mathcal{E}}_5 - \bar{\mathcal{E}}_6, \\ \mathcal{E}_1 \mapsto \bar{\mathcal{E}}_4, \quad \mathcal{E}_2 \mapsto \bar{\mathcal{E}}_3, \quad \mathcal{E}_3 \mapsto \bar{\mathcal{H}}_y - \bar{\mathcal{E}}_1, \quad \mathcal{E}_4 \mapsto \bar{\mathcal{H}}_y - \bar{\mathcal{E}}_2, \\ \mathcal{E}_5 \mapsto \bar{\mathcal{E}}_8, \quad \mathcal{E}_6 \mapsto \bar{\mathcal{E}}_7, \quad \mathcal{E}_7 \mapsto \bar{\mathcal{H}}_y - \bar{\mathcal{E}}_5, \quad \mathcal{E}_8 \mapsto \bar{\mathcal{H}}_y - \bar{\mathcal{E}}_6 \end{array} \tag{9}$$

$$\varphi^* : \begin{array}{l} \bar{\mathcal{H}}_x \mapsto 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8, \quad \bar{\mathcal{H}}_y \mapsto \mathcal{H}_x \\ \bar{\mathcal{E}}_1 \mapsto \mathcal{H}_x - \mathcal{E}_3, \quad \bar{\mathcal{E}}_2 \mapsto \mathcal{H}_x - \mathcal{E}_4, \quad \bar{\mathcal{E}}_3 \mapsto \mathcal{E}_2, \quad \bar{\mathcal{E}}_4 \mapsto \mathcal{E}_1 \\ \bar{\mathcal{E}}_5 \mapsto \mathcal{H}_x - \mathcal{E}_7, \quad \bar{\mathcal{E}}_6 \mapsto \mathcal{H}_x - \mathcal{E}_8, \quad \bar{\mathcal{E}}_7 \mapsto \mathcal{E}_6, \quad \bar{\mathcal{E}}_8 \mapsto \mathcal{E}_5. \end{array} \tag{10}$$

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One can see that both φ^* and φ_* are *linear* mappings on the divisor classes. So, the eigenvector with eigenvalue 1, will be the divisor corresponding to a conservation law (only if its dimension in NOT zero). Indeed one can see imediately that:

$$2\bar{H}_x + 2\bar{H}_y - \sum_{i=1}^{8} \bar{E}_i = 2H_x + 2H_y - \sum_{i=1}^{8} E_i \equiv -K_{\mathcal{X}}$$

It preserves a decomposition of $-K_X = \sum_{i=0}^{3} D_i$ (which is not unique):

$$D_0 = H_x - E_1 - E_2,\ D_1 = H_y - E_5 - E_6$$
$$D_2 = H_x - E_3 - E_4,\ D_3 = H_y - E_7 - E_8$$

there are many elliptic curves corresponding to the this anti-canonical class (these curves pass through all E_i for any $k = \alpha/beta$).

$$\begin{aligned} F \equiv & \alpha xy + \beta a^{-1}(axy + x + y - a)(xy + ax + ay - 1) = 0 \\ \Leftrightarrow & \; kxy + a^{-1}(axy + x + y - a)(xy + ax + ay - 1) = 0. \end{aligned}$$

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So the conservation law will be:

$$I = \left(\frac{a^{-1}(axy + x + y - a)(xy + ax + ay - 1)}{xy} \right)^2$$

The curve F can be transformed to a Weierstrass form and from the zeros of elliptic discriminant we get the singular fibers. From them one can write **various decompositions of anti-canonical divisor**. THis is used for deautonomisation.

Symmetries

$$\mathrm{rank}\mathrm{Pic}(X) = \mathrm{rank} < H_0, H_1, E_1, ...E_8 >_{\mathbb{Z}} = 10$$

Define:

$$< D >= \sum_{i=0}^{3} \mathbb{Z} D_i, \quad < D >^{\perp} = \{\alpha \in \mathrm{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3\}$$

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which have 6-generators:

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Lattice $< D >$ is called *surface sub-lattice* and $< D >^{\perp}$ is called *symmetry sub-lattice*. With respect to intersection form the symmetry sub-lattice can be viewed as a Weyl group with the roots α_i and Cartan matrix $c_{ji} = 2(\alpha_j \cdot \alpha_i)/(\alpha_i \cdot \alpha_i)$.
Elementary reflections:

$$w_i : \text{Pic}(\mathcal{X}) \to \text{Pic}(\mathcal{X}), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

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Because w_i does not affect the surface, it preserves the surface so it is a *symmetry* and accordingly the mapping can be expressed as a combinations of elementary reflections in the Weyl group (in our case $D_5^{(1)}$)

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Accordingly our mapping *lives* in a Weyl group and has the following decomposition in elementary reflections:

$$\phi_* = \sigma_{tot} \circ w_3 \circ w_5 \circ w_4 \circ w_3$$

All elements $\omega \in \widetilde{W}(D_5^{(1)})$ which commutes with ϕ_*, namely $(\omega \circ \phi_* = \phi_* \circ \omega)$ form the **symmetries** of the mapping.
The equation is related to the translations in this affine Weyl group. In general for an affine Weyl group with null vector δ the traslation of an element D with respect to the root α_i is given by

$$t_{\alpha_i} : D \to D - (D, \delta)\alpha_i + (D, \alpha_i + \delta)\delta$$

and our mapping is "the fourth root" of a translation:

$$\phi_*^4 \equiv t_{\alpha_3} \circ t_{\alpha_3} \circ t_{\alpha_4} \circ t_{\alpha_5} = t_{2\alpha_3 + \alpha_4 + \alpha_5}$$

Moving blowing-up points = deautonomisation

This extended affine Weyl group can be realized as an automorphism of a family of generalized Halphen surfaces which are obtained by **allowing the points of blow-ups to move** so that they **preserve the decomposition of -K_X and the action on the Picard group**.

$$\begin{aligned} E_1 &: (x,y) = (0,a_1), & E_2 &: (x,y) = (0,a_2), \\ E_3 &: (X,y) = (0,a_3), & E_4 &: (X,y) = (0,a_4), \\ E_5 &: (x,y) = (a_5,0), & E_6 &: (x,y) = (a_6,0), \\ E_7 &: (x,Y) = (a_7,0), & E_8 &: (x,Y) = (a_8,0), \end{aligned}$$

which can be normalized as $a_1a_2a_3a_4 = a_5a_6a_7a_8 = 1$. Accordingly, our mapping lives in an extended affine Weyl group $\tilde{W}(D_5^{(1)})$ and deautonomized as

$$(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,q) \to \left(\frac{-1}{a_6\sqrt{q}}, \frac{-1}{a_5\sqrt{q}}, \frac{-\sqrt{q}}{a_8}, \frac{-\sqrt{q}}{a_7}, a_3, a_4, a_1, a_2, q\right)$$

and $q = (a_1a_2a_7a_8)/(a_3a_4a_5a_6)$. The mapping turns into a q-Painleve equation:

$$x_{n+1} = a_1(n)a_2(n)y_n$$

$$y_{n+1} = -x_n\frac{(y_n - a_3(n))(y_n - a_4(n))}{(y_n - a_1(n))(y_n - a_2(n))}$$

Singular fibers; "symmetry-phase transitions"

The elliptic curve:

$$F \equiv \alpha xy + \beta a^{-1}(axy + x + y - a)(xy + ax + ay - 1) = 0$$

can be put in a Weierstarss form (using Möbius tranformations):

$$Y^2 = 4X^3 - g_1 X - g_2$$

where g_1, g_2 depends on $k = \alpha/beta$. The values of k which are solutions of $\Delta(k) \equiv g_1^3 - 27g_2^2 = 0$ give the *singlar fibers* of the elliptic curve. For any of these values of k the *surface sub-lattice is different* and, accordingly the *symmetry group will be different*. So for these initial conditions of the dynamical system $x(0), y(0)$ which give the value of k singular fibers, the symmetry group is changed $\equiv$ "phase transition" In our case the singular fibers are (with multiplicities):

$$k = 0, k = \infty, k = \pm 4i(a + a^{-1}), k = (a - a^{-1})^2$$

So for $k = 0$ we have $F = (axy + x + y - a)(xy + ax + ay - 1)$ and accordingly

$$-K_{\mathcal{X}} = D_0 + D_1, \mathrm{with} D_0 = H_x + H_y - E_1 - E_3 - E_6 - E_8, D_1 = H_x + H_y - E_2 - E_4 - E_5 - E_7$$

So, $< D >= \mathbb{Z}D_0 + \mathbb{Z}D_1, \quad < D >^{\perp} = \mathrm{Span}_{\mathbb{Z}}\{\alpha_0, ..., \alpha_7\} \equiv E_7^{(1)}$ and the roots of $E_7^{(1)}$ are

$$\alpha_0 = H_x - H_y, \alpha_1 = E_6 - E_8, \alpha_2 = E_3 - E_6, \alpha_3 = E_1 - E_3$$

$$\alpha_4 = H_y - E_1 - E_2, \alpha_5 = E_2 - E_4, \alpha_6 = E_4 - E_5, \alpha_7 = E_5 - E_7$$

and the pull-back mapping

$$(\varphi^4)^* : (\bar{\alpha_0}, ..., \bar{\alpha_7}) \to (\alpha_0, ..., \alpha_7) + (2, 1, -1, 1, -2, 1, -1, 1)K_{\mathcal{X}}$$

is a translation in $E_7^{(1)}$ and the push-forward can be written in terms of reflections as

$$\varphi_* = \sigma w_1 w_2 w_3 w_2 w_0 w_5 w_4 w_3 w_7 w_6 w_5 w_6 w_4 w_3 w_2$$

and the reflection symmetry given by the exchanging $\sigma = (17)(26)(35)$
In general here we have:

- generic k and $k = \pm 4i(a + a^{-1})$: $-K_{\mathcal{X}} = D_0 \quad < D >^{\perp} = E_8^{(1)}$
- $k = 0$ and $k = \pm(a - a^{-1})^2$: $\quad -K_{\mathcal{X}} = D_0 + D_1 \quad < D >^{\perp} = E_7^{(1)}$
- $k = \infty$: $\quad -K_{\mathcal{X}} = D_0 + D_1 + D_2 + D_3 \quad < D >^{\perp} = D_5^{(1)}$

Higher order invariants

Let us consider the following mapping

$$\bar{x} = \frac{(x-t)(x+t)}{y(x-1)}$$
$$\bar{y} = x \tag{11}$$

We blow it up at the following points:

$$P_1 : (x, y) = (t, 0), P_2 : (x, y) = (-t, 0), P_3 : (x, y) = (0, t), P_4 : (x, y) = (0, -t)$$
$$P_5 : (x, y) = (1, \infty), P_6 : (x, y) = (\infty, 1), P_7 : (x, y) = (\infty, \infty), P_8 : (x, x/y) = (\infty, 1)$$

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Anti-canonical divisor class: $-K_X = 2H_x + 2H_y - E_1 - \cdots - E_8$ and the corresponding curve is $xy = 0$ trivial; $\dim|-K_X| = 0$, but $\dim|-2K_X| = 1$. Indeed, we have

$$|-2K_X| = \alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 -$$
$$-2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv \alpha f + \beta g$$

and the invariant is ($k = \alpha/\beta$):

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$
$$+ \frac{x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4)}{x^2y^2}$$

Discrete Nahm equations: non-minimal elliptic surfaces

Let us conider a different class of integrable discrete mappings, obtained by discretisations of Nahm equations.

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with the integral of motion:

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$$K(\epsilon) = \frac{y(2x + 3y)(x - y)^2}{1 - 10\epsilon^2(x^2 + 4y^2) + \epsilon^4(9x^4 + 272x^3y - 352xy^3 + 696y^4)}$$

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- Icosahedral symmetry

$$\bar{x} - x = \epsilon(2x\bar{x} - y\bar{y})$$
$$\bar{y} - y = -\epsilon(5y\bar{x} + 5x\bar{y} - y\bar{y})$$

with the integral of motion:

$$K(\epsilon) = \frac{y(3x-y)^2(4x+y)^3}{1+\epsilon^2 c_2 + \epsilon^4 c_4 + \epsilon^6 c_6}$$

with

$$c_2 = -35x^2 + 7y^2$$
$$c_4 = 7(37x^4 + 22x^2y^2 - 2xy^3 + 2y^4)$$
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The tetrahedral symmetry (simple can be brought to QRT (a general mapping having elliptic functions as solutions)):

$$\bar{x} - x = \epsilon(x\bar{x} - y\bar{y})$$

$$\bar{y} - y = -\epsilon(y\bar{x} + x\bar{y})$$

use the substitution $u = (1-\epsilon x)/y,\ v = (1+\epsilon x)/y$ and we get **QRT-mapping** $(\bar{u} = v)$ and

$$3\bar{u}\underline{u} - u(\bar{u} + \underline{u}) - u^2 + 4\epsilon^2 = 0$$

with the invariant

$$K = \frac{-3(u-\bar{u})^2 + 4\epsilon^2}{2\epsilon^2(u+\bar{u})(u\bar{u}-\epsilon^2)} \equiv \frac{3x^2y - y^3}{1-\epsilon^2(x^2+y^2)}$$

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The red substitution looks like curves corresponding to divisor classes of some blow-down structure.
The cases of octahedral and icosahedral symmetry cannot be transformed to QRT forms by these type of substitutions.
So we need to analyse carefully the singularity structure. What is seen is that we have more singularities and apparently some of them are useless making the corresponding rational elliptic surface to be more complicated.

Analytical stability and blowing-down structure

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Let $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be a birational automorphism.
For any such automorphism we can blow up $\mathbb{P}^1 \times \mathbb{P}^1$ and construct a rational surface X such that: $\tilde{\phi} : X \to X$ with $\phi = \tilde{\phi}$ in general and $\tilde{\phi}$ is analytically stable which means: $(\tilde{\phi^*})^n = (\tilde{\phi^n})^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$
Analitical stability is equivalent with the following: There is no divisor D such that exist $k > 0$ and $\tilde{\phi}(D)$ =point, $\tilde{\phi}^k(D) =$ indeterminate

$$D \to \bullet \to \bullet \to \ldots \bullet \to D'$$

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\phi}} & X \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

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- Blow down the (-1) curves in the following way: Let C be the (-1) divisor class and F_1, F_2 two divisor classes such that

$$F_1 \cdot F_1 = F_2 \cdot F_2 = 0, \quad F_1 \cdot F_2 = 1, \quad C \cdot F_1 = C \cdot F_2 = 0$$

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- all the above procedure is allowed by the Castelnuovo theorem (1902), and if $\dim|F_1| = \dim|F_2| = 1$ we can put $|F_1| = \alpha_1 x' + \beta_1 y', |F_2| = \alpha_2 x'' + \beta_2 y''$

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- the genus formula is helping here $g = 1 + \frac{1}{2}(F^2 + F \cdot K_X)$ which must be zero
- then we have a new coordinate system where X is minimal given by the following transformation:

$$\mathbb{C}^2 \ni (x, y) \longrightarrow \left(\frac{y'}{x'}, \frac{y''}{x''}\right) \in \mathbb{P}^1 \times \mathbb{P}^1$$

The case of octahedral symmetry:

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$x = \frac{1}{3}(\chi - 2y), \quad \bar{x} = \frac{1}{3}(\bar{\chi} - 2\bar{y}), u = (1 - \epsilon\chi)/y, v = (1 + \epsilon\chi)/y$ to the following system:

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$$\begin{cases} \bar{u} & = v \\ \bar{v} & = \quad \dfrac{(u + 2v - 20\epsilon)(v + 10\epsilon)}{4u - v + 10\epsilon} \end{cases} \quad . \tag{12}$$

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The space of initial conditions is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following nine points:

$$E_1 : (u, v) = (-10\epsilon, 0),\ E_2(0, 10\epsilon),\ E_3(10\epsilon, 5\epsilon),$$
$$E_4(5\epsilon, 0),\ E_5(0, -5\epsilon),\ E_6(-5\epsilon, -10\epsilon)$$
$$E_7(\infty, \infty),\ E_8 : (1/u, u/v) = (0, -1/2),\ E_9 : (1/u, u/v) = (0, -2).$$

The action on the Picard group:

$$\bar{H}_u = 2H_u + H_v - E_1 - E_3 - E_7 - E_8,\ \bar{H}_v = H_u$$
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$$\alpha_0 = H_u + H_v - E_1 - E_2 - E_7,\ \alpha_1 = H_u + H_v - E_1 - E_2 - E_8 - E_9,$$
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The curve corresponding to α_0 is a (-1) curve which must be blown down. $E_1 \to H_a = H_u + H_v - E_2 - E_7$ and $E_2 \to H_b = H_u + H_v - E_1 - E_7$, 0-curves intersecting each other: The corresponding curves are given by:

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$$a_1 u + a_2(v - 10\epsilon) = 0, \quad b_1(u + 10\epsilon) + b_2 v = 0$$

So if we set $a = (v - 10\epsilon)/u \quad b = (u + 10\epsilon)/v$ our dynamical system becomes

$$\begin{cases} \bar{a} = \dfrac{3ab - 2a + 2}{a - 4} \\ \bar{b} = \dfrac{4 - a}{2a + 1} \end{cases} . \tag{13}$$

This system has the following space of initial conditions which define a minimal rational elliptic surface:

$$\begin{aligned}
&F_1 : (a, b) = (0, \infty),\ \ F_2 : (a, b) = (\infty, 0),\\
&F_3 : (a, b) = (-1/2, 4),\ \ F_4 : (a, b) = (-2, \infty)\\
&F_5 : (a, b) = (\infty, -2),\ \ F_6 : (a, b) = (4, -1/2),\\
&F_7 : (a, b) = (-2, -1/2),\ \ F_8 : (a, b) = (-1/2, -2).
\end{aligned}$$

This system has the following space of initial conditions which define a minimal rational elliptic surface:

$$F_1 : (a, b) = (0, \infty),\ \ F_2 : (a, b) = (\infty, 0),$$
$$F_3 : (a, b) = (-1/2, 4),\ \ F_4 : (a, b) = (-2, \infty)$$
$$F_5 : (a, b) = (\infty, -2),\ \ F_6 : (a, b) = (4, -1/2),$$
$$F_7 : (a, b) = (-2, -1/2),\ \ F_8 : (a, b) = (-1/2, -2).$$

The invariant is nothing but the proper transform of the anti-canonical divisor:

$$K_X = 2H_a + 2H_b - \oplus_{i=1}^{8} F_i$$

namely

$$K = \frac{(ab-1)(ab+2a+2b-5)}{4ab+2a+2b+1}$$

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$$
\begin{aligned}
&F_1 : (a,b) = (0,\infty),\ \ F_2 : (a,b) = (\infty,0),\\
&F_3 : (a,b) = (-1/2,4),\ \ F_4 : (a,b) = (-2,\infty)\\
&F_5 : (a,b) = (\infty,-2),\ \ F_6 : (a,b) = (4,-1/2),\\
&F_7 : (a,b) = (-2,-1/2),\ \ F_8 : (a,b) = (-1/2,-2).
\end{aligned}
$$

The invariant is nothing but the proper transform of the anti-canonical divisor:

$$K_X = 2H_a + 2H_b - \oplus_{i=1}^{8} F_i$$

namely

$$K = \frac{(ab-1)(ab+2a+2b-5)}{4ab+2a+2b+1}$$

which is the same as the one given at the beginning [Suris et al. 2012]

$$K(\epsilon) = \frac{y(2x+3y)(x-y)^2}{1-10\epsilon^2(x^2+4y^2)+\epsilon^4(9x^4+272x^3y-352xy^3+696y^4)}$$

Higher dimensional systems

Let us consider the following 4-dimensional mappings
$\varphi : \mathbb{C}^4 \to \mathbb{C}^4; (q_1, q_2, p_1, p_2) \mapsto (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$:

$$\left\{ \begin{array}{lcl} \bar{q}_1 & = & -p_2 - q_2 + aq_2^{-1} + b \\ \bar{p}_1 & = & q_2 \\ \bar{q}_2 & = & -q_1 - p_1 + aq_1^{-1} + b \\ \bar{p}_2 & = & q_1 \end{array} \right. . \qquad (14)$$

and its slight modification:

$$\left\{ \begin{array}{lcl} \bar{q}_1 & = & -q_1 - p_2 + aq_2^{-1} + b_1 \\ \bar{p}_1 & = & q_2 \\ \bar{q}_2 & = & -q_2 - p_1 + aq_1^{-1} + b_2 \\ \bar{p}_2 & = & q_1 \end{array} \right. , \qquad (15)$$

It turns out that deautonomized version of Mapping (14) is a Bäcklund transformation of a direct product of the fourth Painlevé equation, which has two continuous variables and $A_2^{(1)} + A_2^{(1)}$ Weyl group type symmetry, while that of Mapping (15) is a Bäcklund transformation of Noumi-Yamada's $A_5^{(1)}$ equation, which has only one continuous variable and Weyl group $A_5^{(1)}$ type symmetry. These systems provide typical models for geometric studies on higher dimensional Painlvé systems.

Singularity Confinement

The method of singularity confinement in higher dimension;
The idea is that a hypersurface in some compactification X of $\mathbb{C}^n$ which is contracted to lower dimensional variety (singularity) in by f is recovered to a hypersurface after a finite number of iterations of f and the memory of initial conditions is recovered generically. To see this we introduce the exceptional set, given by the zeros of the Jacobian

$$\mathcal{E}(f) = \{D \subset X : \text{hypersurface} \mid \det(\partial f/\partial x) = 0 \text{ on } D \text{ in generic}\},$$

where zero of the Jacobian implies contraction to a lower dimensional variety. If every D in $\mathcal{E}(f)$ is not contracted by f^n for some positive integer n, we say that the initial data is not lost and the map f satisfies the singularity confinement criterion.
It is well-known that this test provides a necessary condition for the mapping to be integrable but is not sufficient. For its complement, the notion of algebraic entropy was introduced by Hietarinta and Viallet [HV] and studied geometrically [Takenawa01, Mase18]. This entropy is essentially the same with topological entropy [Gromov, Yomdin] for algebraic stable mappings.
In this section we consider the mappings on compactified space $(\mathbb{P}^1)^4 = (\mathbb{CP}^1)^4$ and apply the singularity confinement test to them.

Algebraic stability

The following proposition is fundamental to our study.

Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be dominant rational maps. Then $f^* \circ g^* = (g \circ f)^*$ holds if and only if there does not exist a divisor D on $\mathcal{X}$ such that $f(D \setminus I(f)) \subset I(g)$.

In our case we have the following corolary:

A rational map φ from a smooth projective variety $\mathcal{X}$ to itself is algebraically stable if and only if there does not exist a positive integer k and a divisor D on $\mathcal{X}$ such that $f(D \setminus I(f)) \subset I(f^k)$.

If we take $q_1 = \varepsilon$ with $|\varepsilon| \ll 1$ and the others are generic, the principal terms of the Laurent series with respect to ε in the trajectories are

$$\begin{aligned}&(\varepsilon, p_1^{(0)}, q_2^{(0)}, p_2^{(0)})\text{: 3 dim}\\ &\quad\to (q_1^{(1)}, p_1^{(1)}, a\varepsilon^{-1}, \varepsilon)\text{: 2 dim } \boxed{14}\\ &\quad\to (-a\varepsilon^{-1}, a\varepsilon^{-1}, q_2^{(2)}, p_2^{(2)})\text{: 2 dim } \boxed{4}\\ &\quad\to (q_1^{(3)}, p_1^{(3)}, -\varepsilon, -a\varepsilon^{-1})\text{: 2 dim } \boxed{16}\\ &\quad\to (q_1^{(4)}, -\varepsilon, q_2^{(4)}, p_2^{(4)})\text{: 3 dim,}\end{aligned}$$

where $x_i^{(j)}$ denotes a generic value in $\mathbb{C}$, "k dim" denotes the dimension of corresponding subvariety in $(\mathbb{P}^1)^4$ and $\boxed{n}$ denotes the order of blowing up. Similarly, starting with $q_2 = \varepsilon$ and the others being generic, we get

$$\begin{aligned}&(q_1^{(0)}, p_1^{(0)}, \varepsilon, p_2^{(0)})\text{: 3 dim}\\ &\quad\to (a\varepsilon^{-1}, \varepsilon, q_2^{(1)}, p_2^{(1)})\text{: 2 dim } \boxed{6}\\ &\quad\to (q_1^{(2)}, p_1^{(2)}, -a\varepsilon^{-1}, a\varepsilon^{-1})\text{: 2 dim } \boxed{12}\\ &\quad\to (-\varepsilon, -a\varepsilon^{-1}, q_2^{(3)}, p_2^{(3)})\text{: 2 dim } \boxed{8}\\ &\quad\to (q_1^{(4)}, p_1^{(4)}, q_2^{(4)}, -\varepsilon)\text{: 3 dim.}\end{aligned}$$

In both two cases information on the initial values $x_i^{(0)}$ is recovered after finite steps.

For the second mapping we find the following two singularity sequences:

$$\begin{aligned}
&(\varepsilon, p_1^{(0)}, q_2^{(0)}, p_2^{(0)}): \text{ 3 dim} \\
&\quad \to (-p_2^{(0)} + a/q_2^{(0)} + b_1, q_2^{(0)}, a\varepsilon^{-1}, \varepsilon): \text{ 2 dim } \boxed{6} \\
&\quad \to (p_2^{(0)} - a/q_2^{(0)}, a\varepsilon^{-1}, -a\varepsilon^{-1}, -p_2^{(0)} + a/q_2^{(0)} + b_1): \text{ 1 dim } \boxed{4} \\
&\quad \to (-\varepsilon, -a\varepsilon^{-1}, q_2^{(3)}, p_2^{(0)} - a/q_2^{(0)}): \text{ 2 dim } \boxed{8} \\
&\quad \to (q_1^{(4)}, p_1^{(4)}, q_2^{(4)}, -\varepsilon): \text{ 3 dim,}
\end{aligned}$$

and

$$\begin{aligned}
&(q_1^{(0)}, p_1^{(0)}, \varepsilon^{-1}, p_2^{(0)}): \text{ 3 dim} \\
&\quad \to (-p_2^{(0)} - q_1^{(0)} + b_1, \varepsilon^{-1}, -\varepsilon^{-1}, q_1^{(0)}): \text{ 2 dim } \boxed{2} \\
&\quad \to (p_2^{(0)}, -\varepsilon^{-1}, q_2^{(2)}, -p_2^{(0)} - q_1^{(0)} + b_1): \text{ 3 dim} \\
&\quad \to (q_1^{(3)}, p_1^{(3)}, \varepsilon^{-1}, p_2^{(3)}): \text{ Returned}
\end{aligned}$$

General blow-up procedure

In this section we construct a space of initial conditions by blowing up the defining variety along singularities of the previous section. Recall that in local coordinates $U \subset \mathbb{C}^N$, blowing up along a subvariety V of dimension $N-k$, $k \geq 2$, written as

$$x_1 - h_1(x_{k+1}, \dots x_N) = \dots = x_k - h_k(x_{k+1}, \dots x_N) = 0,$$

where h_i's are holomorphic functions, is a birational morphism $\pi : X \to U$ such that X is an open variety given by

$$U_i = \{(u_1^{(i)}, \dots, u_k^{(i)}, x_{k+1}, \dots x_N) \in \mathbb{C}^N\} \quad (i = 1, \dots, k)$$

with $\pi : U_i \to U$:

$$\begin{aligned}(x_1, \dots, x_N) =&(u_1^{(i)} u_i^{(i)} + h_1, \dots, u_{i-1}^{(i)} u_i^{(i)} + h_{i-1}, u_i^{(i)} + h_i, \\ & u_{i+1}^{(i)} u_i^{(i)} + h_{i+1} \dots, u_k^{(i)} u_i^{(i)} + h_k, x_{k+1}, \dots, x_N).\end{aligned}$$

It is convenient to write the coordinates of U_i as

$$\left(\frac{x_1 - h_1}{x_i - h_i}, \dots, \frac{x_{i-1} - h_{i-1}}{x_i - h_i}, x_i - h_i, \frac{x_{i+1} - h_{i+1}}{x_i - h_i}, \dots, \frac{x_k - h_k}{x_i - h_i}, x_{k+1}, \dots x_N \right).$$

The exceptional divisor E is written as $u_i = 0$ in U_i and each point in the center of blowup corresponds to a subvariety isomorphic to $\mathbb{P}^{k-1}$: $(x_1 - h_1 : \dots : x_{k-1} - h_k)$. Hence E is locally a direct product $V \times \mathbb{P}^{k-1}$. We called such $\mathbb{P}^{k-1}$ a fiber of the exceptional divisor.

Let us give a simple example in $\mathbb{P}^3$.
We have the homogeneous coordinates of $\mathbb{P}^3 = (X_0 : X_1 : X_2 : X_3)$ and consider one chart $X_0 \neq 0$ give by the affine coordinates $(x_1, x_2, x_3) = (X_1/X_0, X_2/X_0, X_3/X_0)$
The blow-up of the **line** $x_1 = x_2 = 0$ is given by:

$$\mathbb{P}^3 \leftarrow (x_1, \frac{x_2}{x_1}, x_3) \cup (\frac{x_1}{x_2}, x_2, x_3) \equiv (x_1, \zeta, x_3) \cup (\frac{1}{\zeta}, x_2, x_3)$$

and the exceptional divizor is nothing but the *plane* $\mathbb{C} \times \mathbb{P}^1$.
Also blow up of the point $(x_1, x_2, x_3) = (0, 0, 0)$ is given by:

$$\mathbb{P}^3 \leftarrow (x_1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \cup (\frac{x_1}{x_2}, x_2, \frac{x_3}{x_2}) \cup (\frac{x_1}{x_3}, \frac{x_2}{x_3}, x_3)$$

which are exactly the three charts of $\mathbb{P}^2$.
If we consider the Cremona transformation of $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$

$$\sigma(X : Y : Z : T) = (YZT : XZT : XYT : XYZ)$$

then the exceptional divizor corresponding to blow up at $(1 : 0 : 0 : 0)$ is $E_0 \cong \mathbb{P}^2$

$$(y, z, t) \leftarrow (y, z/y, t/y) \cup (y/z, z, t/z) \cup (y/t, z/t, t) \equiv (u_1, v_1, w_1) \cup (u_2, v_2, w_2) \cup (u_3, v_3, w_3)$$

where we consider the affine variable $y = Y/X, z = Z/X, t = T/X$ and E_0 is given by $(u_1 = 0) \cup (v_2 = 0) \cup (w_3 = 0)$ So in the first chart $y = u_1, z = u_1 v_1, t = u_1 w_1$ and accordingly the image of E_0 through σ is

$$\sigma_*(E_0) \equiv \sigma|_{u_1=0} = u_1^2(u_1 v_1 w_1 : v_1 w_1 : w_1 : v_1)|_{u_1=0} = (0 : v_1 w_1 : w_1 : v1) \cong$$

$$\cong (w_1, w_1/v_1, 1) \cup (v_1, 1, v_1/w_1) \cup (1, 1/v_1, 1/w_1)$$

which is again a plane isomorphic to $\mathbb{P}^2$ (is the opposite face of the E_0 blow-up point of the $\mathbb{P}^3$ tetrahedron). Now if we blow up ALL the vertices of the $\mathbb{P}^3$ tetrahedron we get the following

$$\sigma_*(E_0) = H - E_1 - E2 - E_3$$

$$\sigma_*(E_1) = H - E_0 - E2 - E_3$$

$$\sigma_*(E_2) = H - E_1 - E0 - E_3$$

$$\sigma_*(E_3) = H - E_1 - E2 - E_0$$

where H is the triangular "face" ($\mathbb{P}^2$) opposite to E_i.

Each of the two mappings can be lifted to a pseudo-automorphism on a rational projective variety $\mathcal{X}$ obtained by successive 16 blow-ups from $(\mathbb{P}^1)^4$. First mapping:

$$
\begin{array}{ll}
C_1 : q_1^{-1} = p_1^{-1} = 0 & U_1 : (u_1, v_1, q_2, p_2) = (q_1^{-1}, q_1 p_1^{-1}, q_2, p_2) \\
C_2 : u_1 = v_1 + 1 = 0 & U_2 : (u_2, v_2, q_2, p_2) = (u_1, u_1^{-1}(v_1 + 1), q_2, p_2) \\
C_3 : u_2 = v_2 + b^{(1)} = 0 & U_3 : (u_3, v_3, q_2, p_2) = (u_2, u_2^{-1}(v_2 + b^{(1)}), q_2, p_2) \\
C_4 : u_3 = v_3 + (b^{(1)})^2 + a_0^{(1)} = 0 & \\
\quad U_4 : (u_4, v_4, q_2, p_2) = (u_3, u_3^{-1}(v_3 + (b^{(1)})^2 + a_0^{(1)}), q_2, p_2) & \\
C_5 : q_1^{-1} = p_1 = 0 & U_5 : (u_5, v_5, q_2, p_2) = (q_1^{-1}, q_1 p_1, q_2, p_2) \\
C_6 : u_5 = v_5 - a_1^{(1)} = 0 & U_6 : (u_6, v_6, q_2, p_2) = (u_5, u_5^{-1}(v_5 - a_1^{(1)}), q_2, p_2) \\
C_7 : q_1 = p_1^{-1} = 0 & U_7 (v_7, u_7, q_2, p_2) = (q_1 p_1, p_1^{-1}, q_2, p_2) \\
C_8 : u_7 = v_7 + a_2^{(1)} = 0 & U_8 : (v_8, u_8, q_2, p_2) = (u_7^{-1}(u_7 + a_2^{(1)}), u_7, q_2, p_2)
\end{array}
$$

$$
\begin{array}{ll}
C_9 : p_2^{-1} = q_2^{-1} = 0 & U_9 : (q_1, p_1, u_9, v_9) = (q_1, p_1, q_2^{-1}, p_2^{-1} q_2) \\
C_{10} : u_9 = v_9 + 1 = 0 & U_{10} : (q_1, p_1, u_{10}, v_{10}) = (q_1, p_1, u_9, u_9^{-1}(v_9 + 1)) \\
C_{11} : u_{10} = v_{10} + b^{(2)} = 0 & U_{11} : (q_1, p_1, u_{11}, v_{11}) = (q_1, p_1, u_{10}, u_{10}^{-1}(v_{10} + b^{(2)})) \\
C_{12} : u_{11} = v_{11} + (b^{(2)})^2 + a_0^{(2)} = 0 & \\
\quad U_{12} : (q_1, p_1, u_{12}, v_{12}) = (q_1, p_1, u_{11}, u_{11}^{-1}(v_{11} + (b^{(2)})^2 + a_0^{(2)})) & \\
C_{13} : p_2 = q_2^{-1} = 0 & U_{13} : (q_1, p_1, u_{13}, v_{13}) = (q_1, p_1, q_2^{-1}, p_2 q_2) \\
C_{14} : u_{13} = v_{13} - a_1^{(2)} = 0 & U_{14} : (q_1, p_1, u_{14}, v_{14}) = (q_1, p_1, u_{13}, u_{13}^{-1}(v_{13} - a_1^{(2)})) \\
C_{15} : p_2^{-1} = q_2 = 0 & U_{15} : (q_1, p_1, v_{15}, u_{15}) = (q_1, p_1, p_2 q_2, p_2^{-1}) \\
C_{16} : u_{15} = v_{15} + a_2^{(2)} = 0 & U_{16} : (q_1, p_1, v_{16}, u_{16}) = (q_1, p_1, u_{15}^{-1}(v_{15} + a_2^{(2)}), u_{15})
\end{array}
$$

Second mapping:

$$
\begin{array}{ll}
C_1 : q_2^{-1} = p_1^{-1} = 0 & U_1 : (q_1, v_1, u_1, p_2) = (q_1, q_2 p_1^{-1}, q_2^{-1}, p_2) \\
C_2 : u_1 = v_1 + 1 = 0 & U_2 : (q_1, v_2, u_2, p_2) = (q_1, u_1^{-1}(v_1 + 1), u_1, p_2) \\
C_3 : u_2 = q_1 + p_2 - b_1 = 0 & U_3 : (q_1, v_2, u_3, v_3) = (q_1, v_2, u_2, u_2^{-1}(q_1 + p_2 - b_1) \\
C_4 : u_3 = v_3 + a_0 = 0 & U_4 : (q_1, v_2, u_4, v_4) = (q_1, v_2, u_3, u_3^{-1}(v_3 + a_0)) \\
C_5 : q_2^{-1} = p_2 = 0 & U_5 : (q_1, p_1, u_5, v_5) = (q_1, p_1, q_2^{-1}, p_2 q_2) \\
C_6 : u_5 = v_5 + a_2 = 0 & U_6 : (q_1, p_1, u_5, v_5) = (q_1, p_1, u_5, u_5^{-1}(v_5 + a_2)) \\
C_7 : q_1 = p_1^{-1} = 0 & U_7 : (v_7, u_7, q_2, p_2) = (q_1 p_1, p_1^{-1}, q_2, p_2) \\
C_8 : u_7 = v_7 - a_4 = 0 & U_8 : (v_8, u_8, q_2, p_2) = (u_7^{-1}(v_7 - a_4), u_7, q_2, p_2)
\end{array}
$$

$$
\begin{array}{ll}
C_9 : q_1^{-1} = p_2^{-1} = 0 & U_9 : (u_9, p_1, q_2, v_9) = (q_1^{-1}, p_1, q_2, q_1 p_2^{-1}) \\
C_{10} : u_9 = v_9 + 1 = 0 & U_{10} : (u_{10}, p_1, q_2, v_{10}) = (u_9, p_1, q_2, u_9^{-1}(v_9 + 1)) \\
C_{11} : u_{10} = q_2 + p_1 - b_2 = 0 & U_{11} : (u_{11}, v_{11}, q_2, v_{10}) = (u_{10}, u_{10}^{-1}(q_2 + p_1 - b_2), q_2, v_{10}) \\
C_{12} : u_{11} = v_{11} + a_3 = 0 & U_{12} : (u_{12}, v_{12}, q_2, v_{10}) = (u_{11}, u_{11}^{-1}(v_{11} + a_3), q_2, v_{10}) \\
C_{13} : q_1^{-1} = p_1 = 0 & U_{13} : (u_{13}, v_{13}, q_2, p_2) = (q_1^{-1}, q_1 p_1, q_2, p_2) \\
C_{14} : u_{13} = v_{13} + a_5 = 0 & U_{14} : (u_{14}, v_{14}, q_2, p_2) = (u_{13}, p_2, q_2, u_{13}^{-1}(v_{13} + a_5)) \\
C_{15} : p_2^{-1} = q_2 = 0 & U_{15} : (q_1, p_1, v_{15}, u_{15}) = (q_1, p_1, p_2 q_2, p_2^{-1}) \\
C_{16} : u_{15} = v_{15} - a_1 = 0 & U_{16} : (q_1, p_1, v_{16}, u_{16}) = (q_1, p_1, u_{15}^{-1}(v_{15} - a_1), u_{15})
\end{array}
$$

Theorem

The push-forward action of φ on $H^2(\mathcal{X},\mathbb{Z})$ is as follows:
Case $A_2^{(1)}+A_2^{(1)}$:

$$\begin{array}{l}
H_{q_1}\mapsto H_{p_2},\quad H_{p_1}\mapsto 2H_{q_2}+H_{p_2}-E_9-E_{10}-E_{13}-E_{14}\\
H_{q_2}\mapsto H_{p_1},\quad H_{p_2}\mapsto H_{q_1}+2H_{p_1}-E_1-E_2-E_5-E_6\\
E_1\mapsto H_{p_2}-E_{10},\quad E_2\mapsto H_{p_2}-E_9,\quad E_3\mapsto E_{15},\quad E_4\mapsto E_{16},\\
E_5\mapsto E_{11},\quad E_6\mapsto E_{12},\quad E_7\mapsto H_{p_2}-E_{14},\quad E_8\mapsto H_{p_2}-E_{13},\\
E_9\mapsto H_{p_1}-E_2,\quad E_{10}\mapsto H_{p_1}-E_1,\quad E_{11}\mapsto E_7,\quad E_{12}\mapsto E_8,\\
E_{13}\mapsto E_3,\quad E_{14}\mapsto E_4,\quad E_{15}\mapsto H_{p_1}-E_6,\quad E_{16}\mapsto H_{p_1}-E_5
\end{array}\tag{16}$$

Case: $A_5^{(1)}$:

$$\begin{array}{l}
H_{q_1}\mapsto H_{p_2},\quad H_{p_1}\mapsto H_{p_1}+H_{q_2}+H_{p_2}-E_1-E_2-E_5-E_6\\
H_{q_2}\mapsto H_{p_1},\quad H_{p_2}\mapsto H_{q_1}+H_{p_1}+H_{p_2}-E_9-E_{10}-E_{13}-E_{14}\\
E_1\mapsto H_{p_1}-E_2,\quad E_2\mapsto H_{p_1}-E_1,\quad E_3\mapsto E_7,\quad E_4\mapsto E_8,\\
E_5\mapsto E_3,\quad E_6\mapsto E_4,\quad E_7\mapsto H_{p_2}-E_6,\quad E_8\mapsto H_{p_2}-E_5,\\
E_9\mapsto H_{p_2}-E_{10},\quad E_{10}\mapsto H_{p_2}-E_9,\quad E_{11}\mapsto E_{15},\quad E_{12}\mapsto E_{16},\\
E_{13}\mapsto E_{11},\quad E_{14}\mapsto E_{12},\quad E_{15}\mapsto H_{p_1}-E_{14},\quad E_{16}\mapsto H_{p_1}-E_{13}
\end{array}\tag{17}$$

Theorem

For Case $A_2^{(1)} + A_2^{(1)}$, the linear system of the anticanonical divisor class $\delta = 2\sum_{i=1}^{2}(H_{q_i} + H_{p_i}) - \sum_{i=1}^{16} E_i$ is given by

$$(\alpha_0 + \alpha_1 I_1)(\beta_0 + \beta_1 I_2) = 0 \tag{18}$$

for any $(\alpha_0 : \alpha_1), (\beta_0 : \beta_1) \in \mathbb{P}^1$, where I_i are given by

$$\begin{aligned} I_1 &= q_1 p_1(q_1 + p_1 - b) - a(q_1 + p_1) \\ I_2 &= q_2 p_2(q_2 + p_2 - b) - a(q_2 + p_2). \end{aligned} \tag{19}$$

and fibers $\alpha_0 + \alpha_1 I_1 = 0$ and $\alpha_0 + \alpha_1 I_2 = 1$ are mapped to each other, while for Case $A_5^{(1)}$, the linear system is given by

$$\alpha_0 + \alpha_1 I_1 + \alpha_2 I_2 = 0, \tag{20}$$

for any $(\alpha_0 : \alpha_1 : \alpha_2) \in \mathbb{P}^2$, where I_i are given by

$$\begin{aligned} I_1 =&(q_1 p_1 - q_2 p_2)^2 + b_1 b_2(q_1 p_1 + q_2 p_2) \\ &+ b_1\left(a(p_1 + q_2) - q_1 p_1^2 - q_2^2 p_2\right) + b_2\left(a(q_1 + p_2) - q_1^2 p_1 - q_2 p_2^2\right) \\ I_2 =&(a(q_1 + p_2) + q_1 p_2(b_2 - q_2 - p_1))(a(q_2 + p_1) + q_2 p_1(b_1 - q_1 - p_2)) \end{aligned} \tag{21}$$

Discrete linearizable systems

What means *linearizable*? The system is just a disguise of a *linear discrete equation* i.e. exist dependent variable transformations which transform the system into a linear equation.
Main problem - how to detect linearisability?
Complexity growth and algebraic entropy – which shows how degree of the numerator (as a polynomial in some variables fixed by initial conditions) grows with respect to interation.
Example:

$$x_{n+1} + x_n + x_{n-1} = \frac{a}{x_n} + b$$

$x_0 = p, x_1 = q/r$ and we compute polynomial degree for the numerator or denominator (as polynomials in p, q, r) and we get:

$$1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, ..$$

which can be fitted by (n is the iteration):

$$d(n) = \frac{1}{8}(9 + 6n^2 - (-1)^n)$$

The algebraic entropy is given by

$$S = \lim_{n\to\infty} \log\left(\frac{d(n)}{n}\right)$$

We have the following results (an equivalent form of Diller-Favre theorem 2001:)

- if $d(n)$ is linear in n then the system is **linearisable** and the entropy $S = 0$
- if $d(n)$ is quadratic in n the system is *integrable* (finite number of blow-ups, affine Weyl group symmetry) preserving an elliptic fibration and again $S = 0$
- if $d(n)$ depends exponentially on n the system is **not integrable** and the entropy is $S \neq 0$, and the Gromow-Yomdin theorem saying that $S \leq h_{\mathrm{top}}$ (topological entropy)

In algebraic geometry context $d(n) = ((\phi^*)^n(H_x) \cdot H_y)$ where $\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$
Linearisable systems are complicated since they have infinite number of singularities (so an infinite number of blow ups is needed) but if we start from the singularity patterns we can blow down exceptional curves and we get the linearisation procedure.

Example (a_n arbitrary complex function)

$$x_{n+1} = y_n$$

$$y_{n+1} = a_n y_n - y_n^2/x_n$$

We blow up at the following points $E_1 : (x, y) = (0, 0), E_2 : (x, y) = (\infty, \infty)$ and we get the following singularity pattern:

$$(x/y, y) := (0, 0) \to H_x - E_1 \to H_y - E_2 \to (1/x, x/y) := (0, 0)$$

$$\text{...point} \to H_x - E_2 \to \text{curve...}$$

$$\text{...curve} \to H_y - E_1 \to \text{point...}$$

Now $H_x - E_1$ is exceptional and we can blow down with the blow down structure: $H_u = H_x, H_v = H_x + H_y - E_1 - E_2, F_1 = H_x - E_1, F_2 = H_x - E_2$ The lines corresponding to H_u and H_v are:

$$|H_u| : x - u = 0, |H_v| : x + vy = 0$$

So if $u = x, v = y/x$ our system will be linearised to

$$u_{n+1} = u_n v_n$$

$$v_{n+1} = a_n + v_n$$

Conclusions

- Singularities are essential in analysing discrete dynamical systems.
- The singularity structure may give a non-minimal elliptic surface. In order to make it minimal one has to blow down some -1 divisor classes and in the new coordinates the mappings can be solved
- linearisable systems have complicated singularities and they cannot be transformed into automorphisms.