Weakly-abelian gauge theories

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Outline

Definition

A Lie group G is called *weakly Abelian* if its Lie algebra $\mathfrak{g}$ is Abelian.

Proposition

A Lie group G is weakly Abelian iff its connected component of the identity is an Abelian Lie group, which we denote by A.

Let G be weakly Abelian and $\Gamma \stackrel{\mathrm{def.}}{=} \pi_0(G)$ be its group of components. We have an exact sequence:

$$1 \to A \stackrel{i}{\to} G \stackrel{q}{\to} \Gamma \to 1 \quad . \tag{1}$$

The conjugation action $\mathrm{Ad}_G : G \to \mathrm{Aut}(G)$ preserves A, on which it induces the *restricted adjoint action* $\mathrm{Ad}_G^A : G \to \mathrm{Aut}(A)$. The latter factors through q to the characteristic morphism $\rho : \Gamma \to \mathrm{Aut}(A)$:

$$\mathrm{Ad}_G^A = \rho \circ q \quad ,$$

which depends only on the equivalence class of the extension (1). Let $\mathrm{Ext}_\rho(\Gamma, A)$ be the group of equivalence classes of extensions (1) with characteristic morphism ρ. This is isomorphic with $H^2(\Gamma, A_\rho) = \mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A_\rho)$, where A_ρ is the Γ-module defined by ρ.

Definition

The *extension class* of G is the group cohomology class $e(G) \in H^2(\Gamma, A_\rho)$ defined by the extension sequence $1 \to A \stackrel{i}{\to} G \stackrel{q}{\to} \Gamma \to 1$.

The Lie group extension (1) gives a Lyndon-Hochschild-Serre spectral sequence in (Segal-Mitchison) cohomology of continuous groups, which in turn produces a five-term *inflation-restriction exact sequence*:

$$0 \to H^1(\Gamma, A_\rho) \stackrel{q^*}{\longrightarrow} H^1(G, A_{\mathrm{Ad}_G^A}) \stackrel{i^*}{\longrightarrow} H^1(A, A)^\Gamma \stackrel{\lambda_G}{\longrightarrow} H^2(\Gamma, A_\rho) \stackrel{q^*}{\longrightarrow} H^2(G, A_{\mathrm{Ad}_G^A}) ~, \tag{2}$$

where λ_G is the transgression morphism.

Proposition

We have:

$$e(G) = -\lambda_G(\mathrm{id}_A) ~, \tag{3}$$

where $\mathrm{id}_A \in \mathrm{Hom}(A, A) = H^1(A, A)^\Gamma$ is the identity morphism of A. In particular, we have $q^*(e(G)) = 0$.

The exponential map and reduced adjoint representation

The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ of G factors through q to the reduced adjoint representation $\overline{\rho} : \Gamma \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$:

$$\mathrm{Ad} = \overline{\rho} \circ q \quad . \tag{4}$$

Proposition

The exponential map $\exp_G : (\mathfrak{g}, +) \to A$ *of* G *is a surjective morphism of Lie groups. The Abelian group:*

$$\Lambda \stackrel{\text{def.}}{=} \ker(\exp_G) = \{\lambda \in \mathfrak{g} \mid \exp_G(\lambda) = 1_G\}$$

is a (generally non-full) lattice in $\mathfrak{g}$ *which is stable under* G *and* Γ*. The map* $C_G : \Lambda \to \pi_1(G) \stackrel{\text{def.}}{=} \pi_1(A, 1_G)$ *which sends* $\lambda \in \Lambda$ *to the homotopy class of the curve* $c_\lambda : [0, 1] \to A$ *defined through:*

$$c_\lambda(t) \stackrel{\text{def.}}{=} \exp_G(t\lambda) \quad \forall t \in [0, 1] \tag{5}$$

is an isomorphism of groups whose inverse embeds $\pi_1(G)$ *as the lattice* $\Lambda \subset \mathfrak{g}$*.*

The coefficient module and fundamental groupoid

Definition

The lattice $\Lambda \subset \mathfrak{g}$ is called the exponential lattice of G. The morphism of groups $\mathrm{Ad}_0 : G \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ obtained by corestricting Ad to Λ is called the *corestricted adjoint representation* of G. The morphism of groups $\rho_0 : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$ obtained by corestricting $\overline{\rho}$ to Λ is called the coefficient morphism of G, while the Γ-module Λ_{ρ_0} is called the coefficient module.

We have:

$$\mathrm{Ad}_0 = \rho_0 \circ q \quad .$$

The *coefficient crossed module* $\mathcal{X}_0(G) \stackrel{\mathrm{def.}}{=} (\Lambda, \Gamma, \mathbb{1}_\Gamma, \rho_0)$ is *algebraically* weakly-equivalent with the *exponential crossed module* $\mathcal{X}_1(G) \stackrel{\mathrm{def.}}{=} (\mathfrak{g}, G, \exp_G, \mathrm{Ad})$.

Proposition

The crossed module defined by $\Pi_1(G)$ is isomorphic with the exponential crossed module $\mathcal{X}_1(G)$ and hence the fundamental 2-group $\Pi_1(G)$ is isomorphic with the 2-group $\mathrm{X}_1(G) = G \,/\!\!/_{\exp_G} \mathfrak{g}$ defined by $\mathcal{X}_1(G)$.

The obstruction class of G

Let $\xi(G) \in H^3(\Gamma, \Lambda_{\rho_0})$ be the Taylor obstruction class of G, which vanishes iff G admits a proper universal covering *group*. Given a topological group H and a morphism of topological groups $\alpha : H \to \Gamma$, the exponential sequence

$1 \to \Lambda \xrightarrow{j} \mathfrak{g} \xrightarrow{\exp} A \to 1$ induces a long exact sequence in group cohomology:

$$\ldots \to H^k(H, \Lambda_{\rho_0 \circ \alpha}) \xrightarrow{j_*} H^k(H, \mathfrak{g}_{\overline{\rho} \circ \alpha}) \xrightarrow{\exp_*} H^k(H, A_{\rho \circ \alpha}) \xrightarrow{\Delta_k^H} H^{k+1}(H, \Lambda_{\rho_0 \circ \alpha}) \to \ldots \quad ,$$

where Δ_k^H are the connecting morphisms. The inflation-restriction sequences of the extension (1) for group cohomology with coefficients in A and Λ fit into a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(\Gamma, A_\rho) & \longrightarrow & H^1(G, A_{\mathrm{Ad}_G^A}) & \xrightarrow{i^*} & H^1(A, A)^\Gamma & \xrightarrow{\lambda_G} & H^2(\Gamma, A_\rho) & \xrightarrow{q^*} & H^2(G, A_{\mathrm{Ad}_G^A}) \\
 & & \downarrow \Delta_1^\Gamma & & \downarrow \Delta_1^G & & \downarrow \Delta_1^A & & \downarrow \Delta_2^\Gamma & & \downarrow \Delta_2^G \\
0 & \longrightarrow & H^2(\Gamma, \Lambda_{\rho_0}) & \longrightarrow & H^2(G, \Lambda_{\mathrm{Ad}_0}) & \xrightarrow{i^*} & H^2(A, \Lambda) & \xrightarrow{\mu_G} & H^3(\Gamma, \Lambda_{\rho_0}) & \xrightarrow{q^*} & H^3(G, \Lambda_{\mathrm{Ad}_0})
\end{array} \qquad (6)$$

Let $\epsilon(G) \stackrel{\mathrm{def.}}{=} \Delta_1^A(\mathrm{id}_A) \in H^2(A, \Lambda)$ be the fundamental class of G.

Proposition

We have:

$$\xi(G) = \Delta_2^\Gamma(e(G)) = -\mu_G(\epsilon(G)) \quad .$$

In particular, we have $q^*(\xi(G)) = 0$

Relation between $\xi(G)$ and the k-invariant of BG

To any principal Γ-bundle Q on a topological space X we associate the local coefficient system $\Lambda_{\rho_0}(Q) \stackrel{\mathrm{def.}}{=} Q \times_{\rho_0} \Lambda$.

Proposition (Segal-Mitchison)

For any topological group morphism $H \xrightarrow{\alpha} \Gamma$, we have a natural isomorphism:

$$H^*(H, \Lambda_{\rho_0 \circ \alpha}) \simeq H^*(BH, \Lambda_{\rho_0}(E_\alpha \Gamma)) \quad , \tag{7}$$

where $E_\alpha \Gamma \to BH$ is the $B\alpha$-pull-back to BH of the universal bundle $E\Gamma \to B\Gamma$.

In particular, the fundamental class $\epsilon(G) \in H^2(A, \Lambda_{\rho_0})$ of G identifies with the fundamental class $\iota \in H^2(K(\Lambda, 2), \Lambda) \simeq [K(\Lambda, 2), K(\Lambda, 2)]$ of $K(\Lambda, 2)$.
The extension sequence (1) implies that the classifying space of G is an Eilenberg-MacLane fibration with fiber $BA \simeq K(\Lambda, 2)$ over the classifying space $B\Gamma \simeq K(\Gamma, 1)$ of Γ:

$$* \to BA \to BG \to B\Gamma \to * \quad . \tag{8}$$

Such fibrations are classified by an element $\kappa \in H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma))$, which is the single k-invariant of BG.

Theorem

The obstruction class $\xi(G)$ identifies with κ under the isomorphism of groups (7)*.*

The Leray-Serre spectral sequence for Λ-valued cohomology of the fibration (8) identifies with the Λ-valued Lyndon-Hochschild-Serre spectral sequence of (1). Since $H^1(K(\Lambda,2),\Lambda)=0$, the Leray-Serre spectral sequence gives a five term exact sequence:

$$0 \to H^2(B\Gamma, \Lambda_{\rho_0}(E\Gamma)) \to H^2(BG, \Lambda_{\mathrm{Ad}_0}(E\Gamma)) \to H^2(BA, \Lambda) \stackrel{\delta}{\to} H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma)) \to H^3(BG, \Lambda_{\mathrm{Ad}_0}(E\Gamma)) \qquad (9)$$

which identifies with the inflation-restriction sequence on the bottom row of (6).

Corollary

We have:

$$\kappa = -\delta(\iota)$$

where $\delta : H^2(BA,\Lambda) \to H^3(B\Gamma, \Lambda_{\rho_0}(E\Gamma))$ *is the connecting morphism of* (9).

Classification of principal bundles with weakly Abelian structure group

Let M be a d-manifold. To any principal Γ-bundle Q defined on M we associate two bundles of Abelian groups and a vector bundle, namely:

- The coefficient system $\Lambda(Q) \stackrel{\mathrm{def.}}{=} Q \times_{\rho_0} \Lambda$, where $\rho_0 : \Gamma \to \mathrm{Aut}_{\mathbb{Z}}(\Lambda)$.
- The characteristic bundle $A(Q) \stackrel{\mathrm{def.}}{=} Q \times_{\rho} A$.
- The reduced adjoint bundle $\mathfrak{g}(Q) = Q \times_{\overline{\rho}} \mathfrak{g}$.

The natural flat connection of Q induces a flat Ehresmann connection on $A(Q)$ (whose parallel transport acts through isomorphisms of groups) and a linear flat connection $\mathcal{D}$ on the vector bundle $\mathfrak{g}(Q)$. Notice that $\Lambda(Q)$ is a fiber sub-bundle of $\mathfrak{g}(Q)$ which is preserved by the parallel transport of $\mathcal{D}$.

Definition

The $\mathfrak{g}(Q)$-*valued twisted de Rham cohomology space* $H^k_{\mathcal{D}}(M, \mathfrak{g}(Q))$ is the k-th cohomology space of the *twisted de Rham complex*:

$$0 \to \Omega^0(M, \mathfrak{g}(Q)) \xrightarrow{\mathrm{d}_{\mathcal{D}}} \Omega^1(M, \mathfrak{g}(Q)) \xrightarrow{\mathrm{d}_{\mathcal{D}}} \ldots \xrightarrow{\mathrm{d}_{\mathcal{D}}} \Omega^d(M, \mathfrak{g}(Q)) \to 0 \quad .$$

Proposition

There exists a natural isomorphism of graded vector spaces:

$$H^*_{\mathcal{D}}(M, \mathfrak{g}(Q)) \simeq H^*(M, \mathcal{C}^\infty_{\mathrm{flat}}(\mathfrak{g}(Q))) = H^*(M, \mathfrak{g}(Q)_{\mathrm{disc}}) \quad .$$

The G-extension and G-obstruction class of Q

The exponential sequence $1 \to \Lambda \stackrel{j}{\to} \mathfrak{g} \stackrel{\exp}{\to} A \to 1$ induces a commutative diagram with exact rows, where δ_0 and δ are the connecting morphisms:

$$\begin{array}{ccccccccccccccccc}
\dots & \longrightarrow & H^1(M,\Lambda(Q)) & \stackrel{j_{0,*}}{\longrightarrow} & H^1_{\mathcal{D}}(M,\mathfrak{g}(Q)) & \stackrel{\exp_{0,*}}{\longrightarrow} & H^1(M,A(Q)_{\rm disc}) & \stackrel{\delta_0}{\longrightarrow} & H^2(M,\Lambda(Q)) & \stackrel{j_{0,*}}{\longrightarrow} & H^2_{\mathcal{D}}(M,\mathfrak{g}(Q)) & \stackrel{\exp_{0,*}}{\longrightarrow} & H^2(M,A(Q)_{\rm disc}) & \stackrel{\delta_0}{\longrightarrow} & H^3(M,\Lambda(Q)) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \rm id} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \rm id} & & \downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \rm id} & & \\
\dots & \longrightarrow & H^1(M,\Lambda(Q)) & \stackrel{j_*}{\longrightarrow} & 0 & \stackrel{\exp_*}{\longrightarrow} & H^1(M,\mathcal{C}^\infty(A(Q))) & \stackrel{\delta}{\longrightarrow} & H^2(M,\Lambda(Q)) & \stackrel{j_*}{\longrightarrow} & 0 & \stackrel{\exp_*}{\longrightarrow} & H^2(M,\mathcal{C}^\infty(A(Q))) & \stackrel{\delta}{\longrightarrow} & H^3(M,\Lambda(Q)) & \longrightarrow & \dots
\end{array} \qquad (10)$$

The sheaf $\mathcal{C}^\infty(\mathfrak{g}(Q))$ is acyclic, so $\delta : H^k(M,\mathcal{C}^\infty(A(Q))) \stackrel{\sim}{\to} H^{k+1}(M,\Lambda(Q))$ are isomorphisms for all $k \geq 1$ and we have $\delta_0 = \delta \circ \iota_*$, $\kappa_* \circ j_{0,*} = 0$.

Definition

The *G-extension class* and *G-obstruction class* of Q are defined through:

$$e_G(Q) \stackrel{\rm def.}{=} f^\sharp(e(G)) \in H^2(M, A(Q)_{\rm disc}) \ , \ \xi_G(Q) \stackrel{\rm def.}{=} f^\sharp(\xi(G)) \in H^3(M,\Lambda(Q)) \ ,$$

where $f : M \to B\Gamma$ is a classifying map for Q. The *smooth image* of $e_G(Q)$ is defined through:

$$e^s_G(Q) \stackrel{\rm def.}{=} \iota_*(e_G(Q)) \in H^2(M,\mathcal{C}^\infty(A(Q)) \ \ ,$$

where $\iota_* : H^2(M,A(Q)_{\rm disc}) = H^*(M,\mathcal{C}^\infty_{\rm flat}(A(Q))) \to H^2(M,\mathcal{C}^\infty(A(Q)))$ is the morphism induced by the sheaf inclusion $\mathcal{C}^\infty_{\rm flat}(A(Q)) \hookrightarrow \mathcal{C}^\infty(A(Q))$.

We have $\delta_0(e_G(Q)) = \delta(e^s_G(Q)) = \xi_G(Q)$.

Definition

A (G,q)-*lift of structure group* of Q is a pair (P,φ), where P is principal G-bundle defined on M and $\varphi: P \to Q$ is a based morphism of principal bundles above $q: G \to \Gamma$, i.e. a based isomorphism of principal Γ-bundles $\Gamma(P) \stackrel{\sim}{\longrightarrow} Q$, where $\Gamma(P) \stackrel{\mathrm{def.}}{=} P \times_q \Gamma$ is the *discrete remnant* of P.

Isomorphisms of (G,q)-lifts of structure group are defined obviously. Let $T_{G,q}(Q)$ be the set of isomorphism classes of (G,q)-lifts of Q.

Theorem

Q admits a (G,q)-lift of structure group iff $\xi_G(Q) = 0$ i.e. iff $e^s_G(Q) = 0$. In this case, $T_{G,q}(Q)$ is a torsor over $H^2(M,\Lambda(Q))$.

Definition

Suppose that Q admits a (G,q)-lift of structure group, thus $e_G(Q) \in \ker\delta_0 = \exp_{0,*}(H^2_{\mathcal{D}}(M,\mathfrak{g}(Q)))$. Then the linear and affine *characteristic lattices* of Q are the lattices in $H^2_{\mathcal{D}}(M,\mathfrak{g}(Q))$ defined through:

$$L_0(Q) \stackrel{\mathrm{def.}}{=} j_{0,*}(H^2(M,\Lambda(Q))) = \exp_{0,*}^{-1}(\{0\}) \quad , \quad L(Q) \stackrel{\mathrm{def.}}{=} \exp_{0,*}^{-1}(\{e_G(Q)\}) \quad .$$

Define:

$$Prin^0_\Gamma(M) \stackrel{\text{def.}}{=} \{Q \in Prin_\Gamma(M) \mid \xi_G(Q) = 0\} \ , \ T^{G,q}_\Gamma(M) \stackrel{\text{def.}}{=} \sqcup_{Q \in Prin^0_\Gamma(M)} T_{G,q}(Q)$$

The groupoid $Prin^0_\Gamma(M)$ acts from the left on $T^{G,q}_\Gamma(M)$.

Theorem

There exists a natural bijection:

$$\mathrm{Prin}_G(M) \xrightarrow{\sim} T^{G,q}_\Gamma(M)/Prin^0_\Gamma(M) \ .$$

Let P be a principal G-bundle defined on M.

Definition

The *discrete remnant* of P is the principal Γ-bundle $\Gamma(P) \stackrel{\mathrm{def.}}{=} P \times_q \Gamma$.

We have $\mathrm{ad}(P) = \mathfrak{g}(\Gamma(P))$. Define:

$$A(P) \stackrel{\mathrm{def.}}{=} A(\Gamma(P)) = P \times_{\mathrm{Ad}_G^A} A \quad , \quad \Lambda(P) \stackrel{\mathrm{def.}}{=} \Lambda(\Gamma(P)) = P \times_{\mathrm{Ad}_0} \Lambda$$

Notice that $\xi_G(\Gamma(P)) = 0$, hence $e_G^s(\Gamma(P)) = 0$.

Definition

The *extension class* of P is defined through:

$$e(P) \stackrel{\mathrm{def.}}{=} e_G(\Gamma(P)) \in H^2(M, \mathcal{C}^\infty_{\mathrm{flat}}(A(P))) = H^2(M, A(P)_{\mathrm{disc}})$$

The linear and affine *characteristic lattices* of P are those of $\Gamma(P)$:

$$L_0(P) \stackrel{\mathrm{def.}}{=} L_0(\Gamma(P)) = j_{0,*}(H^2(M, \Lambda(P))) = \exp_{0,*}^{-1}(\{0\}) \subset H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$$
$$L(P) \stackrel{\mathrm{def.}}{=} L(\Gamma(P)) = \exp_{0,*}^{-1}(\{e_G(P)\}) \subset H^2_{\mathcal{D}}(M, \mathrm{ad}(P)) \quad .$$

The real twisted Chern class of P

Proposition

All principal connections defined on P induce the same adjoint connection, which coincides with the distinguished flat connection $\mathcal{D}$ of $\mathrm{ad}(P) = \mathfrak{g}(\Gamma(P))$.

Proposition

The adjoint curvature $\mathcal{V}_{\mathcal{A}} \in \Omega^2(M, \mathrm{ad}(P))$ of any principal connection $\mathcal{A} \in \mathrm{Conn}(P)$ satisfies:

$$\mathrm{d}_{\mathcal{D}} \mathcal{V}_{\mathcal{A}} = 0 \quad .$$

Moreover, the $\mathrm{d}_{\mathcal{D}}$-cohomology class $\mathfrak{c} \stackrel{\mathrm{def.}}{=} [\mathcal{V}_{\mathcal{A}}]_{\mathrm{d}_{\mathcal{D}}} \in H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$ does not depend on the choice of $\mathcal{A}$ in $\mathrm{Conn}(P)$.

Definition

The twisted de Rham cohomology class $\mathfrak{c}(P) \in H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$ is called the *real twisted Chern class* of P.

Theorem

For any principal G-bundle P on M, we have $\mathfrak{c}(P) \in L(P)$. Given a principal Γ-bundle Q on M which admits (G, q)-lifts of structure group, the map:

$$T_{G,q}(Q) \ni P \to \mathfrak{c}(P) \in L(Q)$$

is a morphism of torsors over the group epimorphism $j_{0,*} : H^2(M, \Lambda(Q)) \to L_0(Q)$.

Notice that $j_{0,*}$ kills torsion, so it need not be injective. When $j_{0,*}$ is not injective, the class $\mathfrak{c}(P) \in L(Q)$ fails to classiy principal weakly-Abelian bundles.

Remark. Suppose that $e_G(Q) = 0$, so $L = L_0$ and $\xi_G(Q) = 0$. In this case, $T_{G,q}(Q)$ identifies with the Abelian group $H^2(M, \Lambda(Q))$ and (G, q)-lifts (P, φ) of Q are classified by the *integral twisted Chern class* $c(P) \in H^2(M, \Lambda(Q))$ of P, which satisfies $j_{0,*}(c(P)) = \mathfrak{c}(P)$. This occurs for example when G is a split extension of Γ by A (i.e. when $G \simeq A \rtimes_\rho \Gamma$). Then $e(G) = 0$, hence $e_G(Q) = 0$ for any principal Γ-bundle Q. In that case, any principal Γ-bundle admits (G, q)-extensions of structure group and principal G-bundles P are classified by the pair $(\Gamma(P), c(P))$. This occurs for the symplectic Abelian gauge theories which enter the formulation of $N = 1$ supergravity in four dimensions.

Recap on principal connections

Let:

- G be a Lie group with Lie algebra $\mathfrak{g}$
- $\mathrm{Ad}: G \to \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ be the adjoint representation of G.
- $p : P \to M$ a principal G-bundle with projection p on the manifold M
- $VP \subset TP$ be the vertical bundle of P.
- $\mathrm{ad}(P) \stackrel{\mathrm{def.}}{=} P \times_{\mathrm{Ad}} \mathfrak{g}$ be the adjoint bundle of P.

The space of equivariant $\mathfrak{g}$-valued forms defined on P:

$$\Omega^*(P, \mathfrak{g})^G \stackrel{\mathrm{def.}}{=} \{\eta \in \Omega^*(P, \mathfrak{g}) \,|\, r_g^*(\eta) = \mathrm{Ad}(g)^{-1}\eta\}$$

contains the subspace of horizontal forms:

$$\Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \stackrel{\mathrm{def.}}{=} \{\eta \in \Omega^*(P, \mathfrak{g})^G \,|\, \iota_X \eta = 0 \;\; \forall X \in \mathcal{C}^\infty(P, VP)\} \quad .$$

We have mutually inverse isomorphisms of graded vector spaces:

$$\Omega^*(M, \mathrm{ad}(P)) \underset{\varphi_P}{\overset{p^*}{\rightleftarrows}} \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \quad .$$

Principal connections on P form an affine space modeled on $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$:

$$\mathrm{Conn}(P) \stackrel{\mathrm{def.}}{=} \left\{\mathcal{A} \in \Omega^1(P, \mathfrak{g})^G \,|\, \iota_{X_v}\mathcal{A} = v \;\; \forall p \in P \;\; \forall v \in \mathfrak{g}\right\} \quad ,$$

where $X_v \in \mathcal{C}^\infty(P, VP)$ is the vertical vector field defined by $v \in \mathfrak{g}$.

Let $\mathrm{d}_{\mathcal{A}} : \Omega^*(P, \mathfrak{g}) \to \Omega^*(P, \mathfrak{g})$ be the covariant differential of $\mathcal{A} \in \mathrm{Conn}(P)$.

Definition

The *principal curvature* of $\mathcal{A}$ is:

$$\Omega_{\mathcal{A}} \stackrel{\mathrm{def.}}{=} \mathrm{d}_{\mathcal{A}}\mathcal{A} = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]_{\wedge} \in \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g}) \quad .$$

The *adjoint curvature of* $\mathcal{A}$ is:

$$\mathcal{V}_{\mathcal{A}} \stackrel{\mathrm{def.}}{=} \varphi_P(\Omega_{\mathcal{A}}) \in \Omega^2(M, \mathrm{ad}(P)) \quad .$$

The principal curvature satisfies the Bianchi identity:

$$\mathrm{d}_{\mathcal{A}}\Omega_{\mathcal{A}} = 0 \quad .$$

The principal and adjoint *curvature maps* $\Omega : \mathrm{Conn}(P) \to \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g})$ and $\mathcal{V} : \mathrm{Conn}(P) \to \Omega^2(M, \mathrm{ad}(P))$ are defined through:

$$\Omega(\mathcal{A}) \stackrel{\mathrm{def.}}{=} \Omega_{\mathcal{A}} \quad , \quad \mathcal{V}(\mathcal{A}) \stackrel{\mathrm{def.}}{=} \mathcal{V}_{\mathcal{A}} \quad \forall \mathcal{A} \in \mathrm{Conn}(P) \quad .$$

Recap on principal connections

We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Conn}(P) & \xrightarrow{\;\Omega\;} & \Omega^2_{\mathrm{Ad}}(P,\mathfrak{g}) \\ & \searrow^{\mathcal{V}} & \downarrow{\scriptstyle \varphi_P} \\ & & \Omega^2(M,\mathrm{ad}(P)) \end{array}$$

Let $\mathcal{D}_{\mathcal{A}} : \Gamma(M,\mathrm{ad}(P)) \to \Omega^1(M,\mathrm{ad}(P))$ be the linear connection induced by $\mathcal{A}$ on $\mathrm{ad}(P)$ and $\mathrm{d}_{\mathcal{D}_{\mathcal{A}}} f : \Omega^*(M,\mathrm{ad}(P)) \to \Omega^*(M,\mathrm{ad}(P))$ its differential. We have a commutative diagram:

$$\begin{array}{ccc} \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) & \xrightarrow{\;\mathrm{d}_{\mathcal{A}}\;} & \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) \\ \downarrow{\scriptstyle \varphi_P} & & \downarrow{\scriptstyle \varphi_P} \\ \Omega^*(M,\mathrm{ad}(P)) & \xrightarrow{\;\mathrm{d}_{\mathcal{D}_A}\;} & \Omega^*(M,\mathrm{ad}(P)) \end{array}$$

Suppose that G is a weakly Abelian Lie group.

Proposition

The following statements hold:

1. *For any* $\mathcal{A} \in \mathrm{Conn}(P)$, *we have* $\Omega_\mathcal{A} = \mathrm{d}\mathcal{A}$ *and the Bianchi identity reduces to* $\mathrm{d}\Omega_\mathcal{A} = 0$. *Thus* Ω *is an affine map with linear part:*
$$\mathrm{d}|_{\mathrm{Conn}(P)} : \mathrm{Conn}(P) \to \Omega^2_{\mathrm{Ad}}(P, \mathfrak{g}) \quad .$$
2. *We have:*
$$\mathrm{d}_\mathcal{A}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})} = \mathrm{d}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})} : \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \to \Omega^*_{\mathrm{Ad}}(P, \mathfrak{g}) \quad . \qquad (11)$$
3. *All principal connections* $\mathcal{A} \in \mathrm{Conn}(P)$ *induce the same linear connection* $\mathcal{D}_\mathcal{A}$ *on the adjoint bundle* $\mathrm{ad}(P)$ *(which we denote by* $\mathcal{D}$*) and this induced connection is flat. Moreover, the adjoint curvature satisfies:*
$$\mathrm{d}_\mathcal{D}\mathcal{V}_\mathcal{A} = 0 \quad \forall \mathcal{A} \in \mathrm{Conn}(P)$$
and $\varphi_P : (\Omega_{\mathrm{Ad}}(P, \mathfrak{g}), \mathrm{d}) \to (\Omega(M, \mathrm{ad}(P)), \mathrm{d}_\mathcal{D})$ *is an isomorphism of complexes.*
4. $\mathcal{D}$ *coincides with the flat connection induced on* $\mathrm{ad}(P) = \Gamma(P) \times_{\overline{\rho}} \mathfrak{g}$ *by the flat connection of the discrete remnant bundle* $\Gamma(P) \stackrel{\mathrm{def.}}{=} P \times_q \Gamma$.

Proposition

The twisted de Rham cohomology class of $\mathcal{V}_{\mathcal{A}}$:

$$\mathfrak{c}(P) \stackrel{\mathrm{def.}}{=} [\mathcal{V}_{\mathcal{A}}]_{\mathcal{D}} = \mathcal{V}_{\mathcal{A}} + \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M, \mathrm{ad}(P)) \in H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$$

does not depend on the choice of principal connection $\mathcal{A} \in \mathrm{Conn}(P)$. Viewing $\mathfrak{c}(P)$ as an affine space modeled on the vector space $\Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M, \mathrm{ad}(P))$, the corestricted adjoint curvature map $\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ is a surjective affine map with linear part given by:

$$\mathrm{d}_{\mathcal{D}} \circ \varphi_P|_{\Omega^1(P,\mathfrak{g})} = \varphi_P \circ \mathrm{d}|_{\Omega^1(P,\mathfrak{g})} : \Omega^1(P, \mathfrak{g}) \to \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M, \mathrm{ad}(P)) \quad .$$

Corollary

$\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ is an affine fibration with fiber at $\omega \in \mathfrak{c}(P)$ given by:

$$\mathrm{Conn}_\omega(P) \stackrel{\mathrm{def.}}{=} \{\mathcal{A} \in \mathrm{Conn}(P) \,|\, \mathcal{V}_{\mathcal{A}} = \omega\} \quad , \qquad (12)$$

which is an affine space modeled on the vector space:

$$\Omega^1_{\mathrm{Ad,cl}}(P, \mathfrak{g}) \stackrel{\mathrm{def.}}{=} \ker(\mathrm{d} : \Omega^1_{\mathrm{Ad}}(M, \mathfrak{g}) \to \Omega^2_{\mathrm{Ad}}(M, \mathfrak{g})) \stackrel{\varphi_P}{\simeq} \Omega^1_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathrm{ad}(P)) \quad .$$

The gauge group of P

Definition

The *gauge group* of P is the group $\mathrm{Aut}_b(P)$ of based automorphisms of P, whose elements are called *(global) gauge transformations* of P.

Let $\mathrm{Aut}_b(\mathrm{ad}(P))$ be the group of based automorphisms of $\mathrm{ad}(P)$.

Definition

The *adjoint representation* of $\mathrm{Aut}_b(P)$ is the linear representation induced on global sections of $\mathrm{ad}(P)$ by the morphism of groups $\mathrm{ad}_P : \mathrm{Aut}_b(P) \to \mathrm{Aut}_b(\mathrm{ad}(P))$ defined through:

$$\mathrm{ad}_P(\psi)([p,v]) \stackrel{\mathrm{def.}}{=} [\psi(p), v] \quad \forall \psi \in \mathrm{Aut}_b(P) \quad \forall p \in P \quad \forall v \in \mathfrak{g} \quad .$$

The *pullback representation* of $\mathrm{Aut}_b(P)$ is the linear representation $\mathfrak{R} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^*(P,\mathfrak{g}))$ defined through:

$$\mathfrak{R}(\psi)(\omega) \stackrel{\mathrm{def.}}{=} (\psi^{-1})^*(\omega) \quad \forall \psi \in \mathrm{Aut}_b(P) \quad \forall \omega \in \Omega^*(P,\mathfrak{g}) \quad .$$

Remark. Suppose that M is compact. Then $\mathrm{Aut}_b(P)$ is an infinite-dimensional Fréchet-Lie group whose Lie algebra identifies with $\mathcal{C}^\infty(M, \mathrm{ad}(P))$. In this case, the linear action induced by ad_P on $\mathcal{C}^\infty(M, \mathrm{Ad}_G(P))$ identifies with the adjoint representation of $\mathrm{Aut}_b(P)$ as a Lie group (hence our terminology).

The pullback representation preserves $\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})$, on which it restricts to a representation $\mathfrak{R}_{\mathrm{Ad}} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}))$.

Proposition

The following diagram commutes:

$$\begin{array}{ccc} \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) & \xrightarrow{\mathcal{R}_{\mathrm{Ad}}(\psi)} & \Omega^*_{\mathrm{Ad}}(P,\mathfrak{g}) \\ \downarrow{\scriptstyle \varphi_P} & & \downarrow{\scriptstyle \varphi_P} \\ \Omega^*(M,\mathrm{ad}(P)) & \xrightarrow{\mathrm{ad}_P(\psi)} & \Omega^*(M,\mathrm{ad}(P)) \end{array}$$

Proposition

For any $\psi \in \mathrm{Aut}_b(P)$, we have:

$$\mathrm{d}\circ \mathfrak{R}_{\mathrm{Ad}}(\psi) = \mathfrak{R}_{\mathrm{Ad}}(\psi)\circ \mathrm{d}|_{\Omega^*_{\mathrm{Ad}}(P,\mathfrak{g})} \quad , \quad \mathrm{d}_{\mathcal{D}}\circ \mathrm{ad}_P(\psi) = \mathrm{ad}_P(\psi)\circ \mathrm{d}_{\mathcal{D}} \quad .$$

Thus $\mathfrak{R}_{\mathrm{Ad}}$ and ad_P induce linear representations of the gauge group on the spaces $H^*_{\mathrm{d}}(\Omega_{\mathrm{Ad}}(P,\mathfrak{g}))$ and $H^*_{\mathrm{d}_{\mathcal{D}}}(M,\mathrm{ad}(P))$, which are equivalent through the isomorphism $\varphi_{P*} : H^*_{\mathrm{d}}(\Omega_{\mathrm{Ad}}(P,\mathfrak{g})) \xrightarrow{\sim} H^*_{\mathrm{d}_{\mathcal{D}}}(M,\mathrm{ad}(P))$ induced by φ_P.

The affine action of the gauge group on $\mathrm{Conn}(P)$

The pull-back action preserves the affine space $\mathrm{Conn}(P) \subset \Omega^1(P, \mathfrak{g})^G$, on which it restricts to an affine action $\mathfrak{R}_c : \mathrm{Aut}_b(P) \to \mathrm{Aff}(\mathrm{Conn}(P))$ with linear part:

$$\mathfrak{R}^1_{\mathrm{Ad}} \stackrel{\mathrm{def.}}{=} \mathfrak{R}_{\mathrm{Ad}}|_{\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})} : \mathrm{Aut}_b(P) \to \mathrm{Aut}(\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})) \quad .$$

Proposition

The principal and adjoint curvature maps of P are gauge-equivariant:

$$\Omega \circ \mathfrak{R}_c(\psi) = \mathfrak{R}_{\mathrm{Ad}}(\psi) \circ \Omega \quad \text{and} \quad \mathcal{V} \circ \mathfrak{R}_c(\psi) = \mathrm{ad}_P(\psi) \circ \mathcal{V} \quad \forall \psi \in \mathrm{Aut}_b(P) \quad .$$

Moreover, ad_P *preserves the affine subspace* $\mathfrak{c}(P) \subset \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-cl}}(M, \mathrm{ad}(P))$*, on which it acts through affine transformations with linear part:*

$$\mathrm{ad}_P(\psi)|_{\Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M,\mathrm{ad}(P))} : \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M, \mathrm{ad}(P)) \to \Omega^2_{\mathrm{d}_{\mathcal{D}}\text{-ex}}(M, \mathrm{ad}(P)) \ \forall \psi \in \mathrm{Aut}_b(P) \ .$$

In particular, the affine fibration $\mathcal{V} : \mathrm{Conn}(P) \to \mathfrak{c}(P)$ *is equivariant with respect to the affine actions of* $\mathrm{Aut}_b(P)$ *on* $\mathrm{Conn}(P)$ *and* $\mathfrak{c}(P)$*.*

Discrete gauge transformations

The discrete remnant $\Gamma(P) \stackrel{\text{def.}}{=} P \times_q \Gamma$ comes with a natural (G, q)-lift of structure group $\Phi_P : P \to \Gamma(P)$.

Definition

The group $\mathrm{Aut}_b(\Gamma(P))$ is called the *discrete gauge group* of P and its elements are called *discrete gauge transformations* of P.

Any $\psi \in \mathrm{Aut}_b(P)$ induces an automorphism $Q_P(\psi) \stackrel{\text{def.}}{=} \bar{\psi} \in \mathrm{Aut}_b(\Gamma(P))$ by:

$$\bar{\psi}([p, \gamma]) = [\psi(p), \gamma] \quad \forall [p, \gamma] \in \Gamma(P) \quad .$$

This fits into a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\Phi_P} & \Gamma(P) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{\psi}} \\ P & \xrightarrow{\Phi_P} & \Gamma(P) \end{array}$$

The map $Q_P : \mathrm{Aut}_b(P) \to \mathrm{Aut}_b(\Gamma(P))$ is a morphism of groups.

Definition

The discrete gauge transformation $Q_P(\psi) = \bar{\psi} \in \mathrm{Aut}_b(\Gamma(P))$ is called the *discrete remnant* of the gauge transformation $\psi \in \mathrm{Aut}_b(P)$.

Let $\mathrm{ad}_{\Gamma(P)} : \mathrm{Aut}_b(\Gamma(P)) \to \mathrm{Aut}_b(\mathrm{ad}(P))$ be the morphism of groups given by:

$$\mathrm{ad}_{\Gamma(P)}(\chi)([p,v]_{\mathrm{Ad}}) \stackrel{\text{def.}}{=} [\chi(\Phi_P(p)), v]_{\bar{\rho}} = [p, \bar{\rho}(h_\chi(p))(v)]_{\mathrm{Ad}} \ \forall p \in P \ \forall v \in \mathfrak{g}$$

with $\chi \in \mathrm{Aut}_b(\Gamma(P))$, where $\bar{\rho} : \Gamma \to \mathrm{Aut}(\mathfrak{g})$ is the reduced adjoint representation of G and we used the presentation $\mathrm{ad}(P) = \Gamma(P) \times_{\bar{\rho}} \mathfrak{g}$.

Proposition

We have $\mathrm{ad}_P = \mathrm{ad}_{\Gamma(P)} \circ Q_P$, *i.e.:*

$$\mathrm{ad}_P(\psi) = \mathrm{ad}_{\Gamma(P)}(\bar{\psi}) \ \ \forall \psi \in \mathrm{Aut}_b(P) \ \ .$$

Hence $\mathrm{ad}_P(\psi)$ *depends only on the discrete remnant of* ψ.

Let NG be the nerve of G (the nerve of the one-object groupoid defined by G):

- $N_n G = G^{\times n} \quad \forall n \geq 1 \quad , \quad N_0 G = \{1_G\}$
- face maps $\epsilon_i := \epsilon_i^n : N_n G \to N_{n-1} G$ ($n \geq 1$) and degeneracy maps $\eta^i := \eta_i^n : N_n G \to N_{n+1} G$ ($n \geq 0$) given by:

$$\epsilon_0^1(g) = \epsilon_1^1(g) = 1_G \quad , \quad \eta_0^0(1_G) = 1_G$$

$$\epsilon_i^n(g_1, \ldots, g_n) \stackrel{\text{def.}}{=} \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

$$\eta_i^n(g_1, \ldots, g_n) \stackrel{\text{def.}}{=} \begin{cases} (1_G, g_1, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_{i-1}, 1_G, g_i, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_n, 1_G) & i = n \end{cases}$$

for all $n \geq 1$.

Let $\| \ \| : \mathrm{sTop} \to \mathrm{Top}$ be the fat realization functor, where sTop is the category of simplicial spaces and maps thereof. Then $\|NG\|$ is homotopy-equivalent with $\mathrm{B}G$. Notice that the fat model $\|NG\|$ of $\mathrm{B}G$ differs up to homotopy from the Segal model (which uses the thin realization functor $| \ |$) and from the Milnor model (which uses the join construction).

Definition

The *simplicial de Rham bicomplex* $\Omega(NG)$ has components $\Omega^{p,q}(NG) \stackrel{\mathrm{def.}}{=} \Omega^q(N_p G)$ and differentials:

$$\delta' = \sum_{i=0}^{p+1} (-1)^i (\epsilon_i^{p+1})^* : \Omega^{p,q}(NG) \to \Omega^{p+1,q}(NG)$$
$$\delta'' = (-1)^p \mathrm{d}_{N_p G} : \Omega^{p,q}(NG) \to \Omega^{p,q+1}(NG) \quad .$$

Let $H^*(\Omega(NG))$ be the total cohomology of this bicomplex, which is a graded ring under the obvious operation:

$$\mathring{\wedge} : \Omega^{k_1}(N_{q_1} G) \times \Omega^{k_2}(N_{q_2} G) \to \Omega^{k_1+k_2}(N_{q_1+q_2} G) \quad .$$

$H^*(\Omega(NG))$ is computed by the spectral sequence of the vertical filtration $\Omega(NG)_q \stackrel{\mathrm{def.}}{=} \oplus_{j\geq q} \oplus_{i \geq 0} \Omega^{i,j}(NG)$. The first page is $E_1^{p,q} = H^q_{\delta'}(\Omega^{*,p}(NG)) = H^q_{\delta'}(\Omega^p(N_* G))$ with differential $\delta_1 = \delta'' : E_1^{p,q} \to E_1^{p+1,q}$, while the second page is $E_2^{p,q} = H^p_{\delta''}(H^q_{\delta'}(\Omega(NG)))$.

Theorem (Bott, Shulman, Stasheff)

There exists an isomorphism of graded rings $\zeta : H^*(\Omega(NG)) \xrightarrow{\sim} H^*(BG, \mathbb{R})$ *and isomorphisms of vector spaces:*

$$\beta_{p-q,q} : H^{p-q}(G, S^q(\mathfrak{g}^*)) \xrightarrow{\sim} H^p_{\delta'}(\Omega^q(NG)) = E_1^{q,p} \quad \forall p \geq q \quad .$$

Moreover, we have $E_1^{q,p} = 0$ *for* $p < q$*.*

Since $\delta_1|_{E_1^{q,q}} = 0$, we have epimorphisms $E_1^{q,q} \rightarrow E_2^{q,q} \rightarrow E_3^{q,q} \rightarrow \ldots$ and an edge morphism $e_q : E_1^{q,q} \rightarrow E_\infty^{q,q} \subset H^{2q}(\Omega(NG))$. Since $H^0(G, S^q(\mathfrak{g}^*)) = S^q(\mathfrak{g}^*)^G = S^q(\mathfrak{g}^*)^\Gamma$, we have $\beta_{0,q} : S^q(\mathfrak{g}^*)^\Gamma \xrightarrow{\sim} E_1^{q,q}$.

Definition

The *simplicial* and *universal* Chern-Weil morphisms of G are the morphisms of graded rings:

$$\beta \stackrel{\mathrm{def.}}{=} \oplus_{q\geq 0} e_q \circ \beta_{0,q} : S^*(\mathfrak{g}[2]^\vee)^\Gamma \xrightarrow{\sim} H^{\mathrm{even}}(\Omega(NG)) \quad .$$

and:

$$\psi \stackrel{\mathrm{def.}}{=} \zeta \circ \beta : S^*(\mathfrak{g}[2]^\vee)^\Gamma \rightarrow H^{\mathrm{even}}(BG) \quad .$$

Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the left Maurer-Cartan form of G, which is closed by the MC equation. For any $q \geq 1$ and $i = 1, \ldots, q$, let $\theta_i^{(q)} \stackrel{\mathrm{def.}}{=} (\pi_i^q)^*(\theta) \in \Omega^1(N_q G)$, where $\pi_i^q : N_q G = G^{\times q} \to G$ is the i-th projection.

Proposition

For any $T \in S^q(\mathfrak{g}^*)^\Gamma$, the form $T(\theta_1^{(q)} \stackrel{\circ}{\wedge} \ldots \stackrel{\circ}{\wedge} \theta_q^{(q)}) \in \Omega^q(N_q G)$ is δ-closed and we have:

$$\beta(T) = [T(\theta_1^{(q)} \stackrel{\circ}{\wedge} \ldots \stackrel{\circ}{\wedge} \theta_q^{(q)})]_\delta \in H^{\mathrm{even}}(\Omega(NG)) \quad .$$

Theorem (Cartan, Bott)

Suppose that G is compact. Then the following statements hold:

- $H^p(G, S^q(\mathfrak{g}^*)) = 0$ *for* $p > 0$
- *The spectral sequence E_* collapses at the first page, giving isomorphisms* $e_q : E_1^{q,q} \xrightarrow{\sim} H^{2q}(\Omega(NG))$.
- *The simplicial and universal Chern-Weil morphisms are isomorphisms of graded rings.*

The twisted simplicial de Rham bicomplex

Let $\bar{N}G \to NG$ be the simplicial universal bundle and $\bar{\mathcal{D}}$ be the simplicial flat connection on the simplicial vector bundle $\mathrm{ad}(\bar{N}G)$.

Definition

The *twisted simplicial de Rham bicomplex* $\Omega(NG, \mathrm{ad}(\bar{N}G))$ has components $\Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \stackrel{\mathrm{def.}}{=} \Omega^q(N_pG, \mathrm{ad}(\bar{N}_pG))$ and differentials:

$$\delta'_{\mathrm{ad}} = \sum_{i=0}^{p+1}(-1)^i(\epsilon_i^{p+1})^* : \Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \to \Omega^{p+1,q}(NG, \mathrm{ad}(\bar{N}G))$$

$$\delta''_{\mathrm{ad}} = (-1)^p \mathrm{d}_{\bar{\mathcal{D}}} : \Omega^{p,q}(NG, \mathrm{ad}(\bar{N}G)) \to \Omega^{p,q+1}(NG, \mathrm{ad}(\bar{N}G)) \quad .$$

Let $\delta_{\mathrm{ad}} \stackrel{\mathrm{def.}}{=} \delta'_{\mathrm{ad}} + \delta''_{\mathrm{ad}}$ and $H^*(\Omega(NG, \mathrm{ad}(\bar{N}G)))$ be the total differential and total cohomology of this bicomplex, which is a ring under the obvious operation $\mathring{\wedge} : \Omega^{k_1}(N_qG, \mathfrak{g}^{\otimes l_1}) \times \Omega^{k_2}(N_qG, \mathfrak{g}^{\otimes l_2}) \to \Omega^{k_1+k_2}(N_qG, \mathfrak{g}^{\otimes(l_1+l_2)})$.

Theorem

There exists a natural isomorphism of vector spaces:

$$\zeta_{\mathrm{ad}} : H^*(\Omega(NG, \mathrm{ad}(\bar{N}G))) \xrightarrow{\sim} H^*(BG, \mathrm{ad}(EG)_{\mathrm{disc}}) \quad .$$

which respects the cup product.

Definition

The *universal real twisted Chern class* of G is the real twisted Chern class of EG:

$$\mathfrak{c}(G) \stackrel{\text{def.}}{=} \mathfrak{c}(EG) \in H^2(BG, \mathrm{ad}(EG)_{\mathrm{disc}})$$

We have:

$$\mathfrak{c}(G) = \zeta_{\mathrm{ad}}([\mathcal{V}(G)]_{\delta_{\mathrm{ad}}})$$

where the *universal simplicial adjoint curvature* $\mathcal{V}(G) \in \Omega^2_{\delta_{\mathrm{ad}}-\mathrm{cl}}(NG, \mathrm{ad}(\bar{N}G))$ is induced by Dupont's universal simplicial connection.

Proposition

We have:

$$\mathcal{V}(G) = \theta \in \Omega^{1,1}_{\delta_{\mathrm{ad}}-\mathrm{cl}}(NG, \mathrm{ad}(PG)) = \Omega^1_{\mathrm{cl}}(G, \mathfrak{g}) \quad .$$

Moreover, for any $T \in S^q(\mathfrak{g}^*)^\Gamma$, we have:

$$\psi(T) = T(\mathfrak{c}(G) \cup \ldots \cup \mathfrak{c}(G)) \in H^{2q}(BG, \mathbb{R}) \quad .$$

Let P be a principal G-bundle on a manifold M. Recall that the Chern-Weil morphism $\psi_P : S^q(\mathfrak{g}[2]^*) \to H^{\mathrm{even}}(M, \mathbb{R})$ of P is defined through:

$$\psi_P(T) \stackrel{\mathrm{def.}}{=} [T(\mathcal{V}_{\mathcal{A}} \wedge \ldots \wedge \mathcal{V}_{\mathcal{A}})]_{\mathrm{d}} = T(\mathfrak{c}(P) \cup \ldots \cup \mathfrak{c}(P)) \quad ,$$

where $\mathcal{A} \in \mathrm{Conn}(P)$ is an arbitrary principal connection on P and the cup product includes tensorization along $\mathrm{ad}(P)$ (it is the cup product for the sheaf cohomology of $\mathcal{C}^\infty_{\mathrm{flat}}(\mathrm{ad}(P))$).

Proposition

Let $f : M \to BG$ be a classifying map for P. Then:

$$\mathfrak{c}(P) = f^\sharp(\mathfrak{c}(G)) \in H^2(M, \mathrm{ad}(P)_{\mathrm{disc}}) = H^2_{\mathcal{D}}(M, \mathrm{ad}(P))$$

and:

$$\psi_P = f^* \circ \psi \quad .$$

Relation between $\mathrm{ad}(EG)$ and $\mathfrak{g}(E\Gamma)$

Applying the universal bundle functor E to the projection morphism $q : G \to \Gamma$ gives a q-morphism of principal bundles $Eq : EG \to E\Gamma$ which covers the map $Bq : BG \to B\Gamma$. This is equivalent with a based isomorphism of principal Γ-bundles $\phi : \Gamma(EG) \xrightarrow{\sim} (Bq)^*(E\Gamma)$, i.e. a (G, q)-lift of structure group of $(Bq)^*(E\Gamma)$.

$$\begin{array}{ccc} EG & \xrightarrow{Eq} & E\Gamma \\ \downarrow & & \downarrow \\ BG & \xrightarrow{Bq} & B\Gamma \end{array}$$

Since $\mathrm{ad}(EG) \stackrel{\mathrm{def.}}{=} EG \times_{\mathrm{Ad}} \mathfrak{g} = \Gamma(EG) \times_{\overline{\rho}} \mathfrak{g}$, this gives:

$$\mathrm{ad}(EG) \simeq (Bq)^*(E\Gamma) \times_{\overline{\rho}} \mathfrak{g} = (Bq)^*(E\Gamma \times_{\overline{\rho}} \mathfrak{g}) = (Bq)^*(\mathfrak{g}(E\Gamma)) \quad .$$