Consistency conditions and fiducial 2-field models for SRRT inflation

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Definition

A two-dimensional oriented scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected, **oriented** and borderless Riemann surface (called scalar manifold)
- $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (this ensures conservation of energy)
2. $V > 0$ on $\mathcal{M}$ (this avoids technical problems but can be relaxed)

Each scalar triple defines a model of gravity coupled to scalar fields on $\mathbb{R}^4$:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},V}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|}\left[\frac{M^2}{2}R(g) - \frac{1}{2}\mathrm{Tr}_g \varphi^*(\mathcal{G}) - V\circ\varphi\right] .$$

Define the *rescaled Planck mass* $M_0 \stackrel{\mathrm{def.}}{=} \sqrt{\frac{2}{3}}M$, where M is the reduced Planck mass. Take g to describe a spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \ , \ \vec{x} = (x^1, x^2, x^3) \ , \ a(t) > 0 \ \forall t)$$

and φ to depend only on the cosmological time: $\varphi = \varphi(t)$.

The cosmological equation and geometric dynamical system

Define the *Hubble parameter* $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$ and the *rescaled Hubble function*:

$$\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0} \quad , \quad \mathcal{H}(u) \stackrel{\text{def.}}{=} \sqrt{||u||^2 + 2V(\pi(u))} \;\; \forall u \in T\mathcal{M} \quad ,$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ is the bundle projection.

Proposition

When $H > 0$, the equations of motion are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\mathcal{H}(\dot{\varphi}(t))\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0 \quad ,$$

together with the Hubble condition:

$$H(t) = \frac{1}{3M_0}\mathcal{H}(\dot{\varphi}(t)) \quad .$$

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation are called cosmological curves. The cosmological equation defines an autonomous dissipative geometric dynamical system on $T\mathcal{M}$. Any cosmological curve φ defines a *cosmological orbit* $\mathcal{O}_\varphi : I \to T\mathcal{M}$ given by $\mathcal{O}_\varphi(t) \stackrel{\text{def.}}{=} (\varphi(t), \dot{\varphi}(t))$, which describes the *state evolution* of the dynamical system.

Let $j^k(\mathcal{M})$ be the k-th jet bundle of curves in $\mathcal{M}$; notice that $j^1(\mathcal{M}) = T\mathcal{M}$.

Definition

A *classical cosmological observable* of order k is a function $f : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U}$ is an open subset of $j^k(\mathcal{M})$. Observables of order 1 are called *basic*.

Any observable of order k can be reduced to a basic observable using the cosmological equation (on-shell reduction of observables). In particular, slow roll & turn parameters of various orders can be reduced on-shell to produce basic observables. Thus local conditions on a cosmological curve which constrain these parameters can be formulated as conditions on points in $T\mathcal{M}$ (conditions on the state of the dynamical system).

Let $f_1, \ldots, f_4 : \mathcal{U} \subset T\mathcal{M} \to \mathbb{R}$ be smooth basic observables which are functionally independent for generic model parameters $(\mathcal{G}, V)$. Since $\dim T\mathcal{M} = 4$, the simultaneous conditions

$$f_1(u) = f_2(u) = f_3(u) = f_4(u) = 0 \quad (u \in \mathcal{U})$$

select a discrete set of points u in $T\mathcal{M}$ for generic $(\mathcal{G}, V)$. Hence no cosmological orbit $\mathcal{O}$ can satisfy these conditions unless $\mathcal{G}$ and V satisfy a constraint (a differential equation) which renders the model non-generic. If we require $|f_j| \ll 1$ instead, the same argument shows that $\mathcal{G}$ and V must *approximately* satisfy a differential equation.

The adiabatic and entropic equations

Let (T, N) be the positive Frenet frame of a cosmological curve $\varphi : I \to \mathcal{M}$:

$$T(t) \stackrel{\text{def.}}{=} \frac{\dot\varphi(t)}{||\dot\varphi(t)||} \quad , \quad N(t) = JT(t) \quad ,$$

where $J \in \mathrm{End}(T\mathcal{M})$ is the complex structure determined on $\mathcal{M}$ by the conformal class of $\mathcal{G}$:

$$\omega(u, v) = \mathcal{G}(Ju, v) \quad , \quad \text{where} \quad \omega \stackrel{\text{def.}}{=} \mathrm{vol}_{\mathcal{G}}$$

and let σ be an increasing proper length parameter for φ:

$$\mathrm{d}\sigma = ||\dot\varphi(t)||\mathrm{d}t \quad .$$

Projecting the cosmological equation along T and N gives respectively the *adiabatic* and *entropic* equations:

$$\ddot\sigma + \frac{1}{M_0}\mathcal{H}(\sigma, \dot\sigma)\dot\sigma + V_T(\sigma) = 0 \quad , \quad \Omega(\sigma) = \frac{V_N(\sigma)}{\dot\sigma} \quad ,$$

where

$$\mathcal{H}(\sigma, \dot\sigma) = \sqrt{\dot\sigma^2 + 2V(\sigma)} \quad ,$$

$$V_T(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(T(\sigma)) \quad , \quad V_N(\sigma) \stackrel{\text{def.}}{=} (\mathrm{d}V)(\varphi(\sigma))(N(\sigma))$$

and we defined the signed turn rate of φ through:

$$\Omega(t) \stackrel{\text{def.}}{=} -\mathcal{G}(N, \nabla_t T) \quad .$$

Definition

Consider the following functions of t associated to the cosmological curve φ:

- The first, second and third Hubble slow roll parameters:

$$\varepsilon = -\frac{\dot{H}}{H^2} \ , \ \eta_{\parallel} = -\frac{\ddot{\sigma}}{H\dot{\sigma}} \ , \ \xi = \frac{\dddot{\sigma}}{H^2\dot{\sigma}} \ .$$

- The first and second turn parameters:

$$\eta_{\perp} \stackrel{\text{def.}}{=} \frac{\Omega}{H} \ , \ \nu \stackrel{\text{def.}}{=} \frac{\dot{\eta_{\perp}}}{H\eta_{\perp}} \ .$$

- The first IR parameter κ and the conservative parameter c:

$$\kappa \stackrel{\text{def.}}{=} \frac{\dot{\sigma}^2}{2V} \ , \ c \stackrel{\text{def.}}{=} \frac{H\dot{\sigma}}{||\mathrm{d}V||} \ .$$

Remark

The *opposite relative acceleration vector* $\eta \stackrel{\text{def.}}{=} -\frac{1}{H\dot{\sigma}}\nabla_t\dot{\varphi}$ decomposes as $\eta = \eta_{\parallel} T + \eta_{\perp} N$ and we have:

$$\varepsilon = \frac{3\kappa}{1+\kappa} \ .$$

For simplicity, we take $M = 1$ i.e. $M_0 = \sqrt{\frac{2}{3}}$.

Definition

- The first, second and third slow roll conditions are the conditions $\epsilon \ll 1$, $|\eta_{||}| \ll 1$ and $|\xi| \ll 1$.
- The second order slow roll regime is defined by the joint conditions $\epsilon \ll 1$ and $|\eta_{||}| \ll 1$.
- The third order slow roll regime is defined by the joint conditions $\epsilon \ll 1$, $|\eta_{||}| \ll 1$ and $|\xi| \ll 1$.

Definition

- The rapid turn condition is the condition $|\eta_\perp| \gg 1$.
- The sustained rapid turn regime is defined by the joint conditions $|\eta_\perp| \gg 1$ and $|\nu| \ll 1$.

Proposition

Suppose that the second slow roll condition $|\eta_{||}| \ll 1$ is satisfied. Then the rapid turn condition $|\eta_\perp| \gg 1$ is equivalent with the conservative condition $c \ll 1$.

The adapted frame

Let $\mathcal{M}_0 \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid (\mathrm{d}V)(m) \neq 0\}$ be the complement of the critical locus.

Definition

The *adapted frame* of $(\mathcal{M}, \mathcal{G}, V)$ is the oriented orthonormal frame (n, τ) of $\mathcal{M}_0$ defined by the vector fields:

$$n \stackrel{\text{def.}}{=} \frac{\mathrm{grad}\, V}{||\mathrm{grad}\, V||} \quad , \quad \tau = Jn \quad .$$

Definition

The *characteristic angle* $\theta \in (-\pi, \pi]$ of φ is the angle of rotation from the adapted frame (n, τ) to the Frenet frame (T, N):

$$T = n \cos\theta + \tau \sin\theta \quad , \quad N = -n \sin\theta + \tau \cos\theta \quad .$$

The quantity $s \stackrel{\text{def.}}{=} \mathrm{sign}(\sin\theta) \in \{-1, 0, 1\}$ is called the *characteristic sign* of φ.

Proposition

We have:

$$\eta_{||} = 3 + \frac{\cos\theta}{c} \quad , \quad \eta_{\perp} = -\frac{\sin\theta}{c} \quad .$$

For any vector fields X, Y, we use the notation $V_{XY} \stackrel{\mathrm{def.}}{=} \mathrm{Hess}(V)(X,Y)$, where $\mathrm{Hess}(V) \stackrel{\mathrm{def.}}{=} \nabla \mathrm{d} V$ is the Riemannian Hessian of V.

Proposition

$$\frac{V_{TT}}{3H^2} = \frac{\Omega^2}{3H^2} + \varepsilon + \eta_{\parallel} - \frac{\xi}{3}$$
$$\frac{V_{TN}}{H^2} = \frac{\Omega}{H}\left(3 - \varepsilon - 2\eta_{\parallel} + \nu\right) \quad .$$

Theorem

Suppose that the third order slow roll conditions $\varepsilon \ll 1$, $|\eta_{\parallel}| \ll 1$ *and* $|\xi| \ll 1$ *as well as the small rate of turn condition* $|\nu| \ll 1$ *are satisfied. In this case, we have* $\cos\theta \approx -3c, \sin\theta \approx s\sqrt{1-9c^2}$ *and:*

$$V_{TN}^2 \approx 3VV_{TT}$$
$$V_{TT} \approx 9c^2 V_{nn} - 6sc\sqrt{1-9c^2}\,V_{n\tau} + (1-9c^2)V_{\tau\tau}$$
$$V_{TN} \approx -3sc\sqrt{1-9c^2}(V_{\tau\tau} - V_{nn}) - (1-18c^2)V_{n\tau} \quad .$$

These equations admit a solution c *with* $c \ll 1$ *iff:*

$$V_{n\tau}^2 V_{\tau\tau} \approx 3VV_{nn}^2$$

Corollary

The cosmological curve φ satisfies the sustained rapid turn conditions with third order slow roll at cosmological time t iff the following condition is satisfied at the point $m = \varphi(t)$ of $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} \approx 3 V V_{nn}^2 \quad .$$

Definition

The *SRRT equation* is the following condition which constrains the target space metric $\mathcal{G}$ and scalar potential V on the noncritical submanifold $\mathcal{M}_0$:

$$V_{n\tau}^2 V_{\tau\tau} = 3 V V_{nn}^2$$

A metric $\mathcal{G}$ on $\mathcal{M}_0$ which satisfies this equation for a fixed scalar potential V is called an *SRRT metric relative to* V.

The SRRT equation can be written as a nonlinear differential equation for the pair $(\mathcal{G}, V)$ on $\mathcal{M}_0$. When $\mathcal{G}$ is fixed, it can be viewed as a nonlinear second order PDE for V. When V is fixed, it can be viewed as a nonlinear first order PDE for $\mathcal{G}$.

Let $S \stackrel{\mathrm{def.}}{=} \mathrm{Sym}^2(T^*\mathcal{M})$ and $S_+ \subset S$ be the fiber sub-bundle consisting of positive-definite tensors. When V is fixed, the SRRT equation has the form:

$$\mathcal{F}(j^1(\mathcal{G})) = 0 \quad ,$$

where $\mathcal{F} : j^1(S_+) \to \mathbb{R}$ is a smooth function which depends on V.
Let $L = \det T^*\mathcal{M} = \wedge^2 T^*\mathcal{M}$ be the real determinant line bundle of $\mathcal{M}$ and L_+ be its sub-bundle of positive vectors. Fixing the complex structure J determined by $\mathcal{G}$, the map $\mathcal{G} \to \omega$ gives an isomorphism of fiber bundles $S_+ \stackrel{\sim}{\to} L_+$ which induces an isomorphism $j^1(S_+) \stackrel{\sim}{\to} j^1(L_+)$. Use this to transport $\mathcal{F}$ to a function $F := F_V^J : j^1(L_+) \to \mathbb{R}$. Then the SRRT equation becomes:

$$F(j^1(\omega)) = 0 \quad .$$

This is a contact Hamilton-Jacobi equation for $\omega \in \Gamma(L_+)$ relative to the Cartan contact structure of $j^1(L_+)$. F restricts to a cubic polynomial function on the fibers of the natural projection $j^1(L_+) \to L_+$.

In local isothermal coordinates (U, x^1, x^2) on $\mathcal{M}$ relative to J, we have:

$$\mathrm{d}s^2_{\mathcal{G}} = e^{2\phi}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \quad , \quad \omega = e^{2\phi}\mathrm{d}x^1 \wedge \mathrm{d}x^2$$

and one can write the contact HJ equation as a nonlinear first order PDE for the conformal exponent ϕ, which is cubic in the partial derivatives $\partial_1\phi$ and $\partial_2\phi$. A change of local isothermal coordinates corresponds to a contact transformation.

Let $\mathcal{G}_0$ be the locally-defined flat metric with squared line element $\mathrm{d}s_0^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2$ and define the *modified Euclidean gradient* of V through:

$$\mathrm{grad}_0^J V \stackrel{\mathrm{def.}}{=} J\mathrm{grad}_0 V \quad ,$$

where $\mathrm{grad}_0 V = \mathrm{grad}_{\mathcal{G}_0} V = \partial_1 V \partial_1 + \partial_2 V \partial_2$ is the ordinary Euclidean gradient. Let $\cdot$ denote the Euclidean scalar product defined by $\mathcal{G}_0$, thus $\partial_i \cdot \partial_j = \delta_{ij}$. Let:

$$H_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, \mathrm{grad}_0 V) = \partial_i \partial_j V \partial_i V \partial_j V \quad ,$$
$$\tilde{H}_0 = \mathrm{Hess}_0(V)(\mathrm{grad}_0 V, J\mathrm{grad}_0 V) = -\partial_i \partial_j V \partial_i V \varepsilon_{jk} \partial_k V \quad .$$

Let $U \subset \mathcal{M}_0$ and $U_0 \subset \mathbb{R}^2$ be the image of U in the isothermal chart (U, x^1, x^2). The isothermal Liouville coordinates $(U, x^1, x^2, u, p_1, p_2)$ induce an isomorphism of fiber bundles $j^1(L_+)|_U \simeq U_0 \times \mathbb{R} \times \mathbb{R}^2$. Consider the smooth functions $A, B : U_0 \times \mathbb{R}^2 \to \mathbb{R}$ defined through:

$$A(x,p) \stackrel{\mathrm{def.}}{=} (\partial_i V)(x) p_i \quad , \quad B(x,p) \stackrel{\mathrm{def.}}{=} -\epsilon_{ij} (\partial_j V)(x) p_i \quad .$$

The linear transformation $\mathbb{R}^2 \ni (p_1, p_2) \to (A(x), B(x)) \in \mathbb{R}^2$ is nondegenerate for $x \in U_0$, with inverse:

$$p_1 = \frac{\partial_1 V A - \partial_2 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad , \quad p_2 = \frac{\partial_2 V A + \partial_1 V B}{(\partial_1 V)^2 + (\partial_2 V)^2} \quad .$$

Theorem

In isothermal Liouville coordinates (x^1, x^2, u, p_1, p_2) on $j^1(L_+)|_U$, the contact Hamiltonian is given by the smooth function $F : U_0 \times \mathbb{R}^3 \to \mathbb{R}$ given by:

$$F(x,u,p) \overset{\text{def.}}{=} -[B(x) - \tilde{H}_0(x)]^2[A(x,p) + (\Delta_0 V)(x) - H_0(x)] - 3e^{2u}V[A(x,p) - H_0(x)]^2$$

and the contact Hamilton-Jacobi equation takes the form:

$$F(x_1, x_2, \phi, \partial_1\phi, \partial_2\phi) = 0 \quad .$$

Remark

- The contact HJ equation can be solved *locally* through the method of characteristics.
- The contact Hamiltonian is proper in the sense of Crandall & Lyons, i.e. is nondecreasing in u. Hence the Dirichlet problem can be approached *globally* using the theory of viscosity solutions.

We have:

$$-F = AB^2 - 3Ve^{2u}A^2 + (\Delta_0 V - H_0)B^2 - 2\tilde{H}_0 AB + (6Ve^{2u}H_0 + \tilde{H}_0^2)A + 2\tilde{H}_0(H_0 - \Delta_0 V)B - F_0 \quad ,$$

where:

$$F_0 = -\tilde{H}_0^2[(\Delta_0 V) - H_0] + 3Ve^{2u}H_0^2 \quad .$$

Define:

$$P_1 \overset{\text{def.}}{=} A - H_0 \quad , \quad P_2 = B - \tilde{H}_0 \quad ,$$

The momentum curve is the curve $C_{x,u}$ defined by the condition $F(x,u,p)=0$ in the p-plane. This curve passes through the origin of P-plane, i.e. through the point with coordinates:

$$p_1 \quad := \quad p_{01} \overset{\mathrm{def.}}{=} -\frac{\mathrm{grad}\, V\cdot(-H_0,\tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_1 V H_0 - \partial_2 V\tilde{H}_0}{(\partial_1 V)^2+(\partial_2 V)^2}$$

$$p_2 \quad := \quad p_{02} \overset{\mathrm{def.}}{=} \frac{\mathrm{grad}_J V\cdot(-H_0,\tilde{H}_0)}{||\mathrm{d}V||^2} = \frac{\partial_2 V H_0 + \partial_1 V\tilde{H}_0}{(\partial_1 V)^2+(\partial_2 V)^2}$$

in the p-plane. The singular points of the momentum curve coincide with the characteristic points of the contact HJ equation.

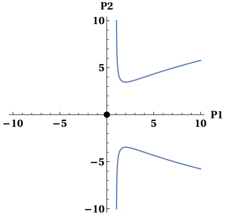
Proposition

The origin of the P-plane is the only singular point of the momentum curve. When $(\Delta_0 V)(x)=0$, the curve is reducible and F factorizes as:

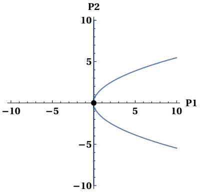
$$F = P_1(P_2^2 - 3Ve^{2u}P_1) \quad .$$

The curve is symmetric under reflection in the P_1-axis. When $(\Delta_0 V)(x)>0$, it is connected and contained in the half-space $P_1 \geq -(\Delta V)(x)$, being the union of two embedded curves which intersect each other at the origin of the P-plane. When $(\Delta_0 V)(x)<0$, it has three connected components, namely the origin of the (P_1,P_2)-plane (which is its only singular point) and two connected components which are nonsingular and contained in the half-space $P_1 > -(\Delta_0 V)(x)$.

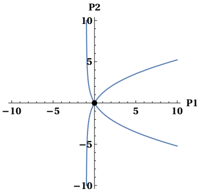
The momentum curve



() $(\Delta_0 V)(x) = -1.$

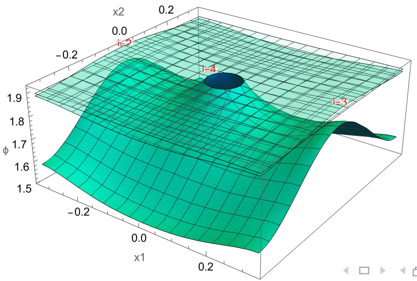


() $(\Delta_0 V)(x) = 0.$



() $(\Delta_0 V)(x) = 1.$

Figure: The momentum curve for $V(x)e^{2u(x)} = 1$ in the cases $(\Delta_0 V)(x) = -1, 0, 1$. The singular point of the curve is shown as a black dot.



Let $c \in U_0$ be an isolated critical point of V and λ_1, λ_2 be the principal values of $\mathrm{Hess}(V)(c)$. In principal isothermal coordinates centered at c, we have:

$$V(x) = V(c) + \frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2) + \mathcal{O}(||x||_0^3) \ .$$

Consider the following homogeneous polynomial functions of degree two in the variables x_1 and x_2, where $k \in \mathbb{Z}_{>0}$:

$$s_k(x) \stackrel{\mathrm{def.}}{=} \lambda_1^k x_1^2 + \lambda_2^k x_2^2 \ .$$

Proposition

We have:

$$F(x,u,p) = -\frac{a_1(x,u)x^1 p_1 + a_2(x)x^2 p_2 - b(x,u)}{s_2(x)^3} + \mathcal{O}(||x||_0^2) \ ,$$

where a_i and b are homogeneous polynomial functions of degree six in x_1 and x_2 (whose coefficients depend on u) given by:

$$a_i(x,u) = \lambda_i s_2(x) \left[t_i(x) + 6V(c)e^{2u} s_2(x) s_3(x)\right]$$

with:

$$t_1(x) = \lambda_1 \lambda_2^2 (\lambda_1 - \lambda_2) x_2^2 [s_2(x) - 3\lambda_2 s_1(x)]$$
$$t_2(x) = \lambda_2 \lambda_1^2 (\lambda_2 - \lambda_1) x_1^2 [s_2(x) - 3\lambda_1 s_1(x)] \ .$$

and:

$$b(x,u) = -\lambda_1^3 \lambda_2^3 (\lambda_1 - \lambda_2)^2 x_1^2 x_2^2 s_1(x) + 3V(c)e^{2u} s_2(x) s_3(x)^2 \ .$$

Corollary

The contact HJ equation is approximated to first order in $||x||_0$ by the following quasilinear first order PDE:

$$a_1(x,\phi)x^1\partial_1\phi + a_2(x,\phi)x^2\partial_2\phi = b(x,\phi) \quad . \tag{1}$$

This quasilinear PDE can be studied by the Lagrange-Charpit method. Its scale-invariant solutions can be studied by reduction to a nonlinear ODE for a function defined on the unit circle.

Proposition

Suppose that ϕ satisfies the quasilinear equation (1) and that we have $\varphi(x) \gg 1$. Then ϕ is an approximate solution of the following linear first order PDE:

$$2s_2(x)\lambda_i x^i \partial_i\phi = s_3(x) \quad , \tag{2}$$

which it satisfies up to corrections of order $\mathcal{O}\left(\frac{e^{-2\phi}}{3V(c)}\right)$.

Solutions which blow up at an isolated critical point

Consider the polar coordinate system (r, θ) defined though:

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad . \tag{3}$$

Proposition

Suppose that $\lambda_1 \neq \lambda_2$. Then the general smooth solution of the linear equation (2) *is:*

$$\phi(r,\theta) = \phi_0(\theta) + Q_0\left(\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\log r + \frac{1}{\lambda_1}\log|\cos\theta| - \frac{1}{\lambda_2}\log|\sin\theta|\right) \quad , \tag{4}$$

where:

$$\phi_0(\theta) = \frac{1}{4}\log(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta) - \frac{1}{2}\,\frac{\lambda_2\log|\cos\theta| - \lambda_1\log|\sin\theta|}{\lambda_2-\lambda_1} \tag{5}$$

and Q_0 is an arbitrary smooth function of a single variable.

Proposition

Suppose that $\lambda_1 = \lambda_2 := \lambda$. Then the linear equation (2) *reduces to:*

$$x^i\partial_i\phi = \frac{1}{2} \quad , \tag{6}$$

whose general solution is:

$$\phi(r,\theta) = \frac{1}{2}\log r + Q_0(\theta) \quad , \tag{7}$$

where $Q_0 \in \mathcal{C}^\infty(\mathrm{S}^1)$ is an arbitrary smooth function.

Suppose that $\lambda_1 \neq \lambda_2$. The general solution (4) reads:

$$\phi(r,\theta) = \phi_0(\theta) + Q\left(\log r + \frac{\lambda_2 \log|\cos\theta| - \lambda_1 \log|\sin\theta|}{\lambda_2 - \lambda_1}\right)$$

and satisfies $\lim_{r\to 0}\phi(r,\theta) = +\infty$ iff $\lim_{w\to -\infty} Q(w) = +\infty$. In this case, we have:

$$\phi \approx Q(\log r) \quad \text{for} \quad r \ll 1 \quad ,$$

so ϕ is rotationally-invariant near c. The corresponding SRRT metric is asymptotically rotationally-invariant at c, with Gaussian curvature:

$$K \approx -e^{-2\phi}\Delta\phi \approx -e^{-2Q(\log r)} Q''(\log r) \quad \text{for} \quad r \ll 1 \quad .$$

Requiring $K = K_c$ for some constant K_c gives:

$$e^{-2Q(w)} Q''(w) = K_c \ .$$

Also require that $\mathcal{G}$ is geodesically complete at c. For $K_c = 0$, we can take $Q(w) = -w$, which gives $\phi(r,\theta) \approx_{r\ll 1} -\log r$ and:

$$\mathrm{d}s^2 \approx_{r\ll 1} \frac{1}{r^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) = \mathrm{d}\rho^2 + \mathrm{d}\theta^2 \quad , \quad \text{where} \quad \rho \overset{\text{def.}}{=} \log r \quad .$$

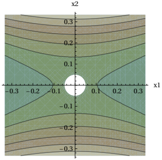
so $\mathcal{G}$ asymptotes at c to the metric on a flat cylinder. For $K_c = -1$, the SRRT metric $\mathcal{G}$ asymptotes to the hyperbolic cusp metric at c:

$$\mathrm{d}s^2 \approx \frac{1}{(r\log r)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) \quad \text{for} \quad r \ll 1 \quad . \tag{8}$$

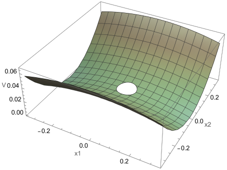
A natural Cauchy problem

Consider a circle $C_R \subset U_0$ of radius $R < 1$ centered at $0 \in U_0$ and the b.c.:

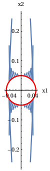
$$\phi|_{C_R} = -\log[R \log(1/R)] \ .$$



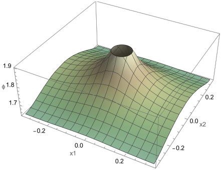
() Contour plot of the potential.



() 3D plot of the potential.

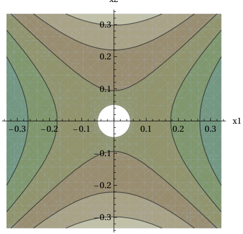


() Projected characteristic curves.

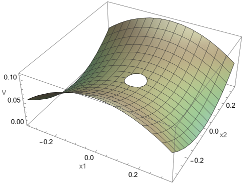


() Solutions of the Dirichlet problem for the viscosity perturbation with $\mathfrak{c} = e^{-7}$.

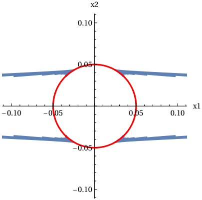
Figure: The potential, projected characteristics and a viscosity approximant of the solution of the Dirichlet problem for the contact Hamilton-Jacobi equation for $V_c = 1/90$ and $\lambda_1 = -1/5$, $\lambda_2 = 1$ with $R = 1/20$.



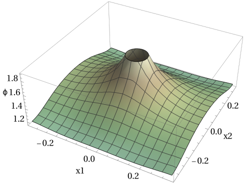
() Contour plot of the potential.



() 3D plot of the potential.



() Some characteristic curves projected on the (x_1, x_2)-plane.



() Solution of the Dirichlet problem for the viscosity perturbation with $\mathfrak{c} = e^{-8}$.

Figure: The potential, projected characteristics and a viscosity approximant of the solution of the Dirichlet problem for $V_c = 1/18$ and $\lambda_1 = -1$, $\lambda_2 = 1$ with $R = 1/20$.