Conformal Einstein's equations

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Einstein's equations

Choquet-Bruhat 52, Choquet-Bruhat, Geroch 69, Friedrich 83-86

Evolution in Einstein's equations

Lorentian manifold $(M, g_{\mu\nu})$ dimension d signature $(- + \ldots +)$

Einstein field equations (without matter)

$$0 = Q_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu}, \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1}$$

where $\Lambda \in \mathbb{R}$ is cosmological constant, $R_{\mu\nu}$, $G_{\mu\nu}$ is the Ricci tensor and Einstein tensor respectively, $R = R^{\mu}_{\mu}$. Lagrangean $\sqrt{g}R$.

Local evolution (Cauchy problem)

1. Existence and uniqueness for the development from initial data at spacelike hypersurface $\Sigma \subset M$ (Cauchy surface), local in time.
2. Continuous dependence on the data (typically in some Sobolev norm)
3. Finite speed of propagation (dependence of solution only on the part of Σ which can be achieved by causal curve)

Well-posedness of evolution equations.

Geometric well-posedness

1. **Evolution in general relativity is not unique due to invariance under diffeomorphisms.** From a solution with given initial data, we can construct another one with the same initial data by applying a diffeomorphism identical around the initial surface (Einstein's hole argument).
2. **Dependences among equations**. Bianchi indenties

$$\nabla^\mu Q_{\mu\nu} = 0.$$

Initial data on Σ have constraints $Q_{\mu i} n^\mu|_\Sigma = 0$ and $Q_{\mu\nu} n^\mu n^\nu|_\Sigma = 0$.

Geometric well-posedness

We need to consider some conditions on the coordinate system (gauge fixing) such that every solution can be transform to this coordinate system by a unique diffeomorphism preserving Σ. **Existence and uniqueness up to diffeomorphism (or in specific gauge).**

Choquet-Bruhat method

The harmonic gauge (de Donder gauge) $F_\mu = 0$

$$F_\mu = \Box_s x_\mu \text{ scalar wave operator .}$$

The gauge fixed Ricci tensor

$$E_{\mu\nu} = -\frac{1}{2} \boxdot g_{\mu\nu} + \dots, \quad \boxdot = g^{\mu\nu}\partial_\mu\partial_\nu.$$

where ... denotes nonlinear terms which however depend only on up to first derivative.

Quasi-linear wave equation

Equation of the form $\boxdot_{g(u)} u + F(D^1 u) = 0$ is **well-posed**.

The idea is to use gauge fixed $Q^f_{\mu\nu} = 0$ for the evolution

$$E_{\mu\nu} = \lambda g_{\mu\nu} \quad (2)$$

Are solutions to this system also solutions to Einstein's equations?

Relation of $Q^f_{\mu\nu} = 0$ to Einstein's equations

We decompose Ricci tensor into gauge fixed part $E_{\mu\nu}$ and the rest

$$R_{\mu\nu} = E_{\mu\nu} + \frac{1}{2}\left(\nabla_\mu F_\nu + \nabla_\nu F_\mu\right), \quad E_{\mu\nu} = -\frac{1}{2}\boxdot g_{\mu\nu} + \ldots,$$

Bianchi identities

$$0 = \nabla^\mu Q_{\mu\nu} = \nabla^\mu \left(E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E\right) + \left(\frac{1}{2}\boxdot + \ldots\right) F_\nu$$

so $Q^f_{\mu\nu} = 0$ implies linear wave equation for F_μ.

On a Cauchy surface Σ

1. $F_\mu|_\Sigma = 0$ is a choice of coordinate system,
2. $n^\nu \nabla_\nu F_\mu|_\Sigma = 0$ due to contraints on the initial surface

Uniqueness of solution to the wave equation shows $F_\nu = 0$ and

$$E_{\mu\nu} = \lambda g_{\mu\nu},\ F_\mu = 0 \Longrightarrow Q_{\mu\nu} = 0.$$

Geometric uniqueness, local existence by gluing technique.

Global analysis

1. Choquet-Bruhat method gives only information about short time behaviour (local in time)
2. It is important to understand long time asymptotics
   1. Gravitational radiation is describe by field far from the source
   2. Does solution without singularities stay so under small perturbations? (Stability of solutions).

Penrose compactification of M, asymptotically simple solutions

Manifold $M \subset \overline{M}$ with a smooth Lorentzian metric $\overline{g}_{\mu\nu}$ such that on M

$$\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega \in C^\infty(\overline{M}).$$

The conformal boundary $\partial M \subset \{\Omega = 0\}$.

Causal structure preserved by conformal rescaling.

Global analysis

Boundary is an umbillic hypersurface:

1. ∂M null for $\Lambda = 0$,
2. ∂M timelike for $\Lambda < 0$,
3. ∂M spacelike for $\Lambda > 0$ (we will call M an asymptotically de Sitter spacetime).

In the case of $\Lambda = 0$ possible conical points on $\{\Omega = 0\}$.

Description of asymptotic infinity in terms of conformally rescaled metric

1. Condition for asymptotic simplicity: is it stable under perturbations?
2. Radiation: what is description of initial data on the boundary?

This talk devoted to $\Lambda > 0$ case.

Conformal method of Friedrich

Friedrich's approach

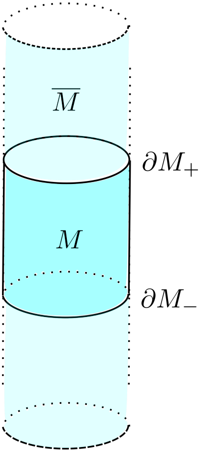
Find a set of equations such that

1. they are more general: every solution to Einstein's equations is also a solution to these equations,
2. they are conformally invariant,
3. the system is hyperbolic (after imposing suitable gauge),
4. the scale factor Ω and properties of being conformal to Einsteinian metric propagate by hyperbolic equations too.

Conformal method of Friedrich

1. Stability of asymptotically dS spaces follows from well-posedness,
2. Understanding of the initial data on conformal boundary
3. Friedrich's system invented for $3+1$. It evolves many variables and the idea is based on specific for 4 dimension propety of Bianchi identity.

Can one extend it to higher dimensions? What is geometric origin of such systems?



Conformal boundary (Starobinsky expansion)

Coordinate system ($\Sigma = \partial M$ spacelike, $M = \Sigma \times [0, \epsilon)$)

$$\bar{g} = \Omega^2 g = -d\Omega^2 + \tilde{g}_{ij}(\Omega, x) dx^i dx^j \tag{3}$$

where $\tilde{g}_{ij}$ is a Ω dependent family of metrics on Σ.

The freedom in the choice is parametrized by diffeomorphisms and conformal transformations on Σ.

- Equations second order in Ω. We can apply Frobenius method for expansion in the series around regular singular point $\Omega = 0$.
- There are subtleties related to dependences among equations. We only use part of the equations to determine the metric. Vanishing of the whole $Q_{\mu\nu}$ follows from Bianchi identities.

Conformal boundary (Starobinsky expansion)

Expansion $\tilde{g} = \sum_{n=0}^{\infty} \Omega^n \tilde{g}^n$, dimension of Σ, $d' = d - 1$

1. $\tilde{g}_{ij}^{(0)}$ freely specified, $\tilde{g}_{ij}^{(n)}$ determined by $Q_{ij} = 0$ equation

$$\tilde{R}_{ij}^{(2n)} + \ldots + \mathrm{tm}\, \tilde{g}_{ij}^{(2n+2)}, \quad n \leq \frac{d'}{2} - 1 \tag{3}$$

   Every odd term vanishes, until...

2. **for d' odd**: At order $n = d' - 1$, $\tilde{g}_{ij}^{(d'-1)}$ free up to conditions

$$\nabla^i \tilde{g}_{ij}^{(d'-1)} = 0, \quad \mathrm{tr}\, \tilde{g}^{(d'-1)} = 0. \tag{4}$$

   Holographic stress energy tensor

3. **For d' even**: At order $n = d' - 2$, there is an obstruction for smooth extension. In general, the metric cannot be smooth (but it admits a polylogarithmic expansion). In this case $H_{ij} = Q_{ij}^{(d'-2)}$ is independent of the rest of the expansion.
4. The rest of the expansion is determined. Vanishing of the whole $Q_{\mu\nu}$ follows from Bianchi identities.

Generalization of the Friedrich system

1. It suggests that only in even dimension d (d' odd) we should be able to evolve the equations through the boundary.
2. It needs to involve at least d derivative of the metric. because part of the data on the boundary is hidden in the holographic stress energy tensor.

In order to find such system we need to better understand conformal geometry.

Conformal geometry

Hansen, Schouten 24, Fefferman, Graham 85, Bailey, Eastwood, Graham 01, Gover, Hirachi...

Conformal geometry

(Pseudo-)riemannian geometry

1. There exists a distinguished connection (Levi-Civita)
2. There exists a normal coordinate construction around $x \in M$ with residual symmetry of orthogonal group.
3. Polynomial invariants can be written as contraction of

$$g^{\mu\nu}, \quad \nabla_{\mu_1} \cdots \nabla_{\mu_n} R_{\mu\nu\rho\sigma} \tag{5}$$

   so-called even Weyl invariants.
4. With a choice of orientation additionally contractions $\sqrt{g}\epsilon_{\mu_1 \ldots \mu_n}$ (odd Weyl invariants).
5. It is useful to introduce weight ω for Riemannian invariant K

$$K^{\nu_1 \ldots \nu_{n_U}}_{\mu_1 \ldots \mu_{n_L}}[e^{2c} g] = e^{(\omega + n_L - n_U)c} K^{\nu_1 \ldots \nu_{n_U}}_{\mu_1 \ldots \mu_{n_L}}[g]. \tag{6}$$

   Space of invariants of given weight is finite dimensional and $\omega \leq 0$

$$\omega(g_{\mu\nu}) = 0, \quad \omega(\nabla_{\mu_1} \cdots \nabla_{\mu_n} R_{\mu\nu\rho\sigma}) = -(n+2). \tag{7}$$

Conformal geometry

Conformal class of the metric

g and g' are conformally related if $g_{\mu\nu} = e^{2\phi} g'_{\mu\nu}$ for $\phi \in C^\infty(M)$.

Interesting problem is to classify conformal invariants on M of weight ω

$$K[e^{2\phi} g] = e^{(\omega + n_L - n_U)\phi} K[g], \quad \phi \in C^\infty(M). \tag{5}$$

They need to be also Riemannian invariants so $\omega \leq 0$.

1. There is no distinguished connection.
2. There exists normal coordinate construction by conformal geodesics, but the residual group is not semi-simple
3. Problem of construction and classification of invariants is hard.

Examples: Weyl tensor, contraction of multiple Weyl tensor, Bach tensor only in 4 dimensions.

Fefferman-Graham construction

1. We can associate Starobinsky expansion to the metric (now M plays a role of the boundary). Conformal transformations of M corresponds to diffeomorphisms of the bulk.
2. It is useful to work with one more dimension $\mathbf{M} = \overset{t}{\mathbb{R}^*} \times M \times \overset{\rho}{\mathbb{R}}$

   $$\mathbf{g} = 2\rho dt^2 + 2td\rho dt + t^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \rho = \Omega^2, \quad \tilde{g}_{\mu\nu}|_{t=1,\rho=0} = g_{\mu\nu},$$

   which satisfies Einstein's equation $\mathbf{R}_{IJ} = O_+(\rho^{d/2-1})$.
3. $\mathbf{T}^I := t\partial_0$ is a conformal Killing $\boldsymbol{\nabla}_I \mathbf{T}_J = \mathbf{g}_{IJ}$.
4. Diffeomorphisms corresponding to conformal transformations preserves $\mathcal{N} = \{\rho = 0\}$ surface and there $(t, x^\mu) \to (t + \phi, x^\mu)$
5. Cross sections ι of the projection from $\mathcal{N}$

   $$\pi \colon \mathcal{N} \ni (t, x) \to x \in M \tag{6}$$

   are in one to one correspondence with the choice of metrics in conformal class (bundle of scales $g = \iota^* \mathbf{g}$).

Fefferman-Graham construction

1. Consider a Riemannian invariant $\mathbf{K}_{I_1 \ldots I_n}$ of the ambient metric such that $\mathcal{L}_{\mathbf{T}}\mathbf{K} = (\omega + n)\mathbf{K}$. Suppose that every contraction of $\mathbf{K}$ with $\mathbf{T}$ vanishes on $\mathcal{N}$
2. We define $K_{\mu_1 \ldots \mu_n}$ by first restricting $\mathbf{K}$ to $\mathcal{N}$ and then pulling back by ι. The result is a conformal invariant of weight ω.
3. In particular every scalar Riemannian invariant of the ambient metric provides by this construction a conformal invariant.

Bailey-Eastwood-Graham

These are all (even) conformal scalar invariants for fixed weight.

In the case of higher valence tensors such results are not known.

Remark

Vector $\mathbf{T}$ is preserved by diffeomorphisms implementing gauge transformations, thus it can also be used in the construction.

Fefferman-Graham obstruction tensor

For d even, Fefferman-Graham obstruction tensor

$$H_{\mu\nu} = \tilde{S}_{\mu\nu}^{\left[\frac{d}{2}-1\right]}, \qquad [n] = (2n) \tag{6}$$

is well-defined (here $\tilde{S}_{\mu\nu}$ is the Ricci tensor for the ambient space $t = 1$).

Geometric definition

Previous construction applied to $\mathbf{K}_{IJ} = \boldsymbol{\rho}^{1-\frac{d}{2}}\mathbf{R}_{IJ}$ where

$$\boldsymbol{\rho} := \frac{1}{2}\mathbf{T}_I\mathbf{T}^I = \rho t^2 \tag{7}$$

We notice that $\mathbf{R}_{IJ} = O(\boldsymbol{\rho}^{d/2-1})$ and $\mathbf{T}^I\mathbf{R}_{IJ} = 0$.

Obstruction tensor is a conformal invariant of weight $-d$.

Fefferman-Graham obstruction tensor

1. $H_{\mu\nu}[e^{2\phi}g] = e^{(2-d)\phi}H_{\mu\nu}[g]$ (weight $-d$)
2. It is traceless and divergence-free

$$\operatorname{tr} H = 0, \quad \nabla^{\mu} H_{\mu\nu} = 0 \tag{6}$$

3. It has a Lagrangean $\mathcal{L} = \sqrt{\tilde{g}}^{\left[\frac{d}{2}\right]}$ (holograpohic volume).
4. If g satisfies Einstein's equations then

$$2\rho dt^2 + 2td\rho dt + t^2(1+\lambda\rho)^2 g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{7}$$

is Ricci flat. This means that

$$Q_{\mu\nu} = 0 \Longrightarrow H_{\mu\nu} = 0. \tag{8}$$

Anderson's proposition (dimension $d \geq 4$ even)

Friedrich's approach with $H_{\mu\nu} = 0$

Find a set of equations such that

1. they are more general: every solution to Einstein's equations is also a solution to these equations,

$$Q_{\mu\nu} = 0 \Longrightarrow H_{\mu\nu} = 0 \tag{9}$$

2. they are conformally invariant,

$$H_{\mu\nu}[e^{2\phi} g] = 0 \Longleftrightarrow H_{\mu\nu}[g] = 0 \tag{10}$$

3. the system is hyperbolic (after imposing suitable gauge), ? There are not only diffeomorphisms but also conformal transformations.
4. the scale factor Ω and properties of being conformal to Einsteinian metric propagate by hyperbolic equations too. ?

Anderson-Fefferman-Graham (AFG) equations

Günther '70, Anderson, Anderson-Chruściel '05, WK' 23

Fefferman-Graham obstruction tensor

We denote $\tilde{S}_{\mu\nu}$, $\tilde{S}_{\mu\infty}$ and $\tilde{S}_{\infty\infty}$ components of ambient Ricci tensor.

1. Recursive determination of $\tilde{g}^{[n]}_{\mu\nu}$

$$0 = \tilde{S}^{[n]}_{\mu\nu} = \tilde{R}^{[n]}_{\mu\nu} + \ldots + \mathrm{tm}\, \tilde{g}^{[n+1]}_{\mu\nu}, \quad n \leq \frac{d}{2} - 2 \qquad (11)$$

2. For $n = 0$ we have $\tilde{g}^{[1]}_{\mu\nu} = -2P_{\mu\nu}$ where the Schouten tensor

$$P_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} + \frac{R}{2(d-1)} g_{\mu\nu}\right) \qquad (12)$$

3. For $n = 1$, Bach tensor $B_{\mu\nu}$ appears ($H^{d=4}_{\mu\nu} = B_{\mu\nu}$)

$$\tilde{g}^{[2]}_{\mu\nu} = \frac{1}{d-4} B_{\mu\nu} + (\ldots)\, g_{\mu\nu}, \quad B_{\mu\nu} = \Box P_{\mu\nu} - \nabla_\mu \nabla_\nu P + \ldots \qquad (13)$$

4. In general, $H_{\mu\nu}$ depends on the d-derivatives of the metric (growing complexity).

Fefferman-Graham obstruction tensor

To simplify notation we introduce ∂_∞^{-1} the integration in $\rho = \Omega^2$ from $\rho = 0$ (inverse to ∂_∞)

1. We can simplify computation using remaining parts of the Ricci tensor
$$\tilde{A}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu \tilde{G}_\nu + \tilde{\nabla}_\nu \tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma},$$
where we introduce functions
$$\tilde{\gamma} = \partial_\infty^{-1}\tilde{S}_{\infty\infty}, \quad \tilde{G}_\mu = 2\partial_\infty^{-1}\tilde{S}_{\mu\infty} - \partial_\mu\partial_\infty^{-1}\tilde{\gamma}, \tag{11}$$
2. Equations $\tilde{A}^{[n]}_{\mu\nu} = 0$ for $n \geq 2$
$$0 = \Box\tilde{g}^{[n]}_{\mu\nu} + \ldots + (n+1)\left(n - \frac{d}{2} + 1\right)\tilde{g}^{[n+1]}_{\mu\nu} \tag{12}$$
3. Recursive computation gives
$$H_{\mu\nu} = c\Box^{\frac{d}{2}-2}\left(\Box P_{\mu\nu} - \nabla_\mu\nabla_\nu P\right) + \ldots \tag{13}$$

Anderson-Fefferman-Graham (AFG) equations

1. Initial data: $D^{d-1}g_{\mu\nu}|_\Sigma$.
2. Gauge freedom: diffeomorphisms and conformal transformations. Constraints

$$H_{\mu i}n^\mu|_\Sigma = 0, \quad H_{\mu\nu}n^\mu n^\nu|_\Sigma = 0, \quad H^\mu_\mu|_\Sigma = 0. \tag{14}$$

3. They are higher derivatives equations. Are they of hyperbolic type in some gauge?
4. They are related to Einstein's equations in the ambient space. Can we apply Choquet-Bruhat method (corresponding decomposition of the FG tensor)?
5. Why the property of being solution to Einstein's equation propagates from initial Cauchy surface?

Gauge fixing for AFG equations

Gauge fixing conditions (diffeomorphisms and conformal transformations)

$$F_\mu = \Box_s x_\mu = 0, \quad P = \frac{1}{2(n-1)} R = 0 \tag{15}$$

Gauge fixed obstruction tensor

$$H_{\mu\nu} = c\Box^{\frac{d}{2}-2}\left(\Box P_{\mu\nu} - \nabla_\mu \nabla_\nu P\right) + \ldots = -\frac{1}{2} \boxdot^{\frac{d}{2}} g_{\mu\nu} + \ldots \tag{16}$$

Problems with hyperbolicity in the case of multiple characteristics (Ivrii)

$$\boxdot_{g(u)}^{N+1} u + F(D^{2N+1}u) = 0, \quad D^k u \text{ k-th jets}$$

is in general not well-posed for $N > 0$.

1. Weakly hyperbolic (we can compute all time derivatives, convergent series for analytic initial data).
2. One needs to control many lower order terms (Levi conditions).

Generalized quasi-linear wave equation (recursive)

Consider a system for $u^{[k]}$ fields $0 \leq n \leq N$,

$$\Box u^{[n]} + F^n(D^1 u^{[n]}, D^2 u^{[k]}_{k<n}) + c_n u^{[n+1]} = 0,$$

where $\Box = g^{\mu\nu}(u^{[0]})\partial_\mu \partial_\nu$ and $c_n \neq 0$, $n < N$ and $c_N = 0$.

1. It is not a system of quasi-linear wave equations.
2. It is well-posed in smooth category (proof by introducing many auxiliary (derivative) variables $D^{N-k} u^{[k]}$)
3. Leray hyperbolicity (shift in Sobolev spaces $u^{[n]} \in \mathcal{H}^{s-n}$).

Recursive elimination of $u^{[k]}$ for $k \geq 1$ in terms of $u = u^{[0]}$ gives

$$0 = \Box^{N+1} u + \dots, \ 2N + 2 \text{ order equation.}$$

This equation is equivalent to the system, thus also well-posed.

Gauge fixing for AFG equation

The gauge fixed tensors $\tilde{A}$ tensors ($\tilde{E}^{[n]}$)

$$\tilde{E}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{2}(\tilde{\nabla}_\mu \tilde{G}_\nu + \tilde{\nabla}_\nu \tilde{G}_\mu) - \tilde{g}_{\mu\nu}\tilde{\gamma},$$

where the gauge fixing functions (∂_∞^{-1} integration in ρ)

$$\tilde{\gamma} = -\frac{1}{2}\tilde{g}^{[0]\xi\chi}\tilde{g}^{[1]}_{\xi\chi} + \partial_\infty^{-1}\tilde{S}_{\infty\infty}, \quad \tilde{G}_\mu = F_\mu + 2\partial_\infty^{-1}\tilde{S}_{\mu\infty} - \partial_\mu\partial_\infty^{-1}\tilde{\gamma},$$

Equivalence

$$\left\{ \begin{array}{l} \tilde{E}_{\mu\nu} = O(\rho^{d/2}), \\ \tilde{\gamma} = O(\rho^{d/2-1}), \ \tilde{G}_\mu = O(\rho^{d/2}) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} H_{\mu\nu} = 0, \\ R = 0, \ F_\mu = 0 \end{array} \right\}$$

If $\tilde{E}_{\mu\nu} = O(\rho^{d/2})$ then $\tilde{\gamma}^{[0]} \propto R$ and $\tilde{G}^{[0]}_\mu = F_\mu = \Box_s x_\mu$.

The AFG equation is well-posed WK' 23

1. The equation $\tilde{E}_{\mu\nu} = O(\rho^{d/2})$ (recursive, generalized hyperbolic system)
$$\tilde{E}^{[n]}_{\mu\nu} = -\frac{1}{2}\left[\Box_{\tilde{g}}\tilde{g}_{\mu\nu}\right]^{[n]} + \ldots + c_n \tilde{g}^{[n+1]}_{\mu\nu},$$
for $\tilde{g}^{[k]}_{\mu\nu}$ for $k = 0, \ldots, \frac{d}{2} - 1$ (where $c_{d/2-1} = 0$).
2. Bianchi identity gives recursive, generalized hyperbolic equations for the gauge functions
$$-\frac{1}{2}\Box_{\tilde{g}}\tilde{\gamma} + \ldots = O(\rho^{d/2-1}), \quad -\frac{1}{2}\Box_{\tilde{g}}\tilde{G}_{\mu} + \ldots = O(\rho^{d/2}).$$
3. Vanishing of the initial condition for this system follows from vanishing of R and F_μ to sufficient order on Σ and constraints on Σ.

Well-posedness in this gauge follows from standard gluing technique in the same way as for case of the Einstein's equations.

Almost Einstein structure

Spacetime M is conformal to a solution of Einstein's equations (for $\Omega \neq 0$) if

$$\mathrm{tf}(\nabla_\mu \nabla_\nu \Omega + P_{\mu\nu}\Omega) = 0, \tag{17}$$

where tf is the traceless part. The condition (**??**) defines almost Einstein structure (Ω arbitrary not vanishing identically).

Almost Einstein (Nurowski-Gover, Gover, Graham-Willse)

Existence of the covariantly constant covector $\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$,

$$\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1}).$$

Relation to (**??**): $\Omega = \boldsymbol{\sigma}|_{t=1}^{[0]}$. Moreover, $\mathbf{I}_I \mathbf{I}^I \propto \Lambda + O(\rho^{d/2-1})$.

Reminder: Killing equation propagation for Einstein's vacuum gravity

$$(\Box + \ldots)X_\mu = 0 \Longrightarrow (\Box + \ldots)\nabla_{(\mu} X_{\nu)} = 0.$$

Propagation of (almost) Einstein structure

Propagation ($\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$, $\boldsymbol{\sigma} = t\tilde{\sigma}$)

$$\Box\boldsymbol{\sigma} = O(\rho^{d/2+1}), \Longleftrightarrow [\Box_{\tilde{g}}\tilde{\sigma}]^{[n]} + \ldots + c_n \tilde{\sigma}^{[n+1]} = 0, \; n \leq \frac{d}{2}.$$

Well-posed generalized hyperbolic system.

- If $\Box\boldsymbol{\sigma} = O(\rho^{d/2+1})$ and $\mathbf{R}_{IJ} = O(\rho^\infty)$ then

$$(\Box + \ldots)\, \boldsymbol{\nabla}_I \mathbf{I}_J = \boldsymbol{\nabla}_I \boldsymbol{\nabla}_J \Box\boldsymbol{\sigma} + O(\rho^\infty) = O(\rho^{d/2-1}),$$

- It is a well-posed system for $\tilde{K}^{[n]}_{IJ} = \boldsymbol{\nabla}_I \mathbf{I}_J|^{[n]}_{t=1}$ for $n \leq \frac{d}{2} - 2$. If the initial data vanish, then $\boldsymbol{\nabla}_I \mathbf{I}_J = O(\rho^{d/2-1})$ everywhere.
- The initial conditions reduce to

$$D^{d-1}\,\mathrm{tf}(\nabla_\mu \nabla_\nu \Omega + P_{\mu\nu}\Omega)|_\Sigma = 0$$

thanks to recursive structure of the propagation equation.

Propagation of (almost) Einstein structure

Propagation ($\mathbf{I}_I = \partial_I \boldsymbol{\sigma}$, $\boldsymbol{\sigma} = t\tilde{\sigma}$)

$$\Box\boldsymbol{\sigma} = O(\rho^{d/2+1}), \Longleftrightarrow [\Box_{\tilde{g}}\tilde{\sigma}]^{[n]} + \ldots + c_n\tilde{\sigma}^{[n+1]} = 0,\ n \leq \frac{d}{2}.$$

Well-posed generalized hyperbolic system.

- If $H_{\mu\nu} = 0$, then in Lorentzian setting, there exists a non-unique Ricci flat extension $\mathbf{R}_{IJ} = O(\rho^{\infty})$,
- The equation $\Box\boldsymbol{\sigma} = O(\rho^{d/2+1})$ depends on the extension of the metric of order $O(\rho^{d/2})$. It is equivalent to supercritical GJMS equation $P_{d+1}\Omega = 0$.
- Solution $\boldsymbol{\sigma} + O(\rho^{d/2+1})$ may depend on the extension of the metric. However, if it defines an almost Einstein structure, then this dependence disappears.
- Well-posedness ensures propagation of almost Einstein structures.

Some applications

Wald, Zoupas; Bac, WK, Lewandowski, Broda 23

Symplectic potentials

(Covariant) variational bi-complex

1. Variational identity (symplectic form $\omega = \delta\Theta$)

$$\delta\mathcal{L} = \mathrm{E}\,\delta g + \mathrm{d}\Theta, \quad \mathcal{L} \in \Lambda^{d,0},\ \Theta \in \Lambda^{d-1,1} \tag{18}$$

   where $\Lambda^{k,l}$ are natural k-forms, l-linear and antisymmetric in δg.

Acyclicity of the bicomplex (Gilkey, Wald,...):

1. If $\kappa \in \Lambda^{d,0}$

$$\mathrm{E}_g(\kappa) = 0 \iff \kappa = \mathrm{d}\xi + \text{characteristic classes} \tag{19}$$

   The characteristic classes are Pontrygin forms and Euler class. In terms of densities first is odd second even.

2. For $(n,k) \neq (d,0)$ the d complex is acyclic

$$\mathrm{d}\kappa = 0 \Rightarrow \kappa = \mathrm{d}\xi, \quad \kappa \in \Lambda^{n,k} \tag{20}$$

It holds also in the presence of additional tensor fields (then additionally $\mathrm{E}_\phi = 0$ in the first part).

Symplectic potentials

Ambiguities:

1. in Lagrangean

$$\mathcal{L}' = \mathcal{L} + \eta, \quad \mathrm{E}_g(\eta) = 0, \quad \Theta' = \Theta + \Theta_\eta \tag{18}$$

   Thus η is a sum of total divergence and the Euler class (even case).

2. in symplectic potential

$$\Theta' = \Theta + \mathrm{d}\kappa. \tag{19}$$

   There exists a canonical Θ_{can} for every choice of torsion-free connection (integration by parts).

3. Symplectic form $\delta\Theta$ unique up to exact forms.

The presymplectic potential is used for constructing Noether currents for symmteries $j \in \Lambda^{d-1,0}$. In the case of charges associated to local gauge transformations by acyclicity $j = \mathrm{d}\kappa$, where $\kappa \in \Lambda^{d-2,0}$. These objects are used for defining charges in the asmptotic regions.

Problems: Ambiguities in Θ singular behaviour at ∂M.

Symplectic potentials

1. For Einstein's gravity

$$\mathcal{L} = \sqrt{g}R, \quad \Theta_{can,GR} = *(\tilde{g}^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu} - \tilde{g}^{\rho\nu}\delta\Gamma^{\mu}_{\mu\nu}) \tag{18}$$

   Singular behaviour at conformal boundary.
2. Can we correct it such that the limit is well-defined? Usually done in specific Starobinsky gauge by substraction method.
3. However, now we have an envelope AFG theory

$$\mathcal{L}_1 = \sqrt{\tilde{g}}^{\left[\frac{d}{2}\right]}, \tag{19}$$

   There is no distinguished connection, but there exists Θ_1 such that for solutions of Einstein's equations

$$\Theta_1 = \Lambda^{\frac{d}{2}-1}\Theta_{can,GR} \tag{20}$$

4. We look for correction of $\mathcal{L}_1$ and Θ_1.

Symplectic potentials

1. Suppose that we have conformally invariant $\mathcal{L}_2$ and Θ_2, then the limit of Θ_2 exists and $\Theta_2 = \Theta_1 + \ldots$ have the same symplectic form.
2. On the boundary, conformally invariant presymplectic potential is unique (on Einstein's solutions) $\propto \tau^{ij} \delta g_{ij}$.
3. Alexakis decomposition of global conformal invariants

$$\mathcal{L}_1 = \text{conf} + \text{Euler} + \mathrm{d}\,(\ldots) \tag{18}$$

   The last two parts have trivial Euler-Lagrange equations
4. In the case of Θ such decomposition is not known.

Bac, WK, Lewandowski, Broda '23

In dimension 4 we can write $\mathcal{L}_2$ as Lagrangean for Yang-Mills theory for conformal Cartan connection. The standard YM sympletic potential turns out to be conformally invariant.

Conjecture: Θ_2 can be also defined for higher even dimensions.

Summary

- AFG equations can be used as generalization of Friedrich's method to any even dimension.
- They provide nice geometric framework: propagation of almost Einstein structures, symplectic currents
- Useful for understanding initial data on the conformal boundary.

Outlook:

- Better understanding what happens at the boundary.
- Extension of the symplectic potential method to higher even dimensions.
- Can one add (conformal) matter?

Thank you!